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Two-fold automorphisms of graphs

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Abstract

In this paper we shall present a natural generalisation of the notion of automorphism of a graph or digraph G, namely a two-fold automorphism. This is a pair (α, β) of permutations of the vertex set $\mathrm{V}(G)$ which acts on ordered pairs of vertices of G in the natural way. The action of (α, β) on all such ordered pairs gives a graph which is two-fold isomorphic to G. When $\alpha = \beta$ the two-fold automorphism is just a usual automorphism. Our main results concern those graphs which admit a two-fold automorphism with $\alpha \neq \beta$.

1 Introduction and Background

In this paper, the terms *graph* and *digraph* refer to special cases of what we call (*mixed*) *oriented graphs*. A (*mixed*) *oriented graph* is a finite set $\mathrm{V}(G)$ whose elements are called vertices together with a set of distinct ordered pairs of (not necessarily distinct) elements of $\mathrm{V}(G)$. These ordered pairs are called *arcs*. When referring to an arc (u, v) we say that u is *adjacent to* v and v is *adjacent from* u. If G contains the arc (u, v), sometimes denoted by $u \longrightarrow v$, and also the opposite arc (v, u), then we say that (u, v) is *self-paired*. We distinguish between two special types of oriented graphs. A (*simple*) *graph* is an oriented graph which contains no arc of the form (u, u) and in which every arc is self-paired. The pair of self-paired arcs (u, v) and (v, u) are together called an edge and are treated as the unordered pair $\{u, v\}$. In other words, a *graph* is undirected and does not have loops or multiple edges. The set of vertices adjacent to a given vertex v in a graph is denoted by $\mathrm{N}(v)$. A *digraph* is an oriented graph in which no arc of the form (u, v), $u \neq v$ is self-paired, but loops, that is, arcs of the form (u, u) are allowed. In this paper we shall be dealing mostly with graphs, but considering every edge to be a union of a pair of self-paired arcs is necessary for a clear exposition of the ideas being presented. Otherwise, any undefined terms which we use are standard and can be found in any graph theory textbook such as [1].

The construction of orbital graphs and digraphs is well known (see [5] for example). Let Γ be a transitive group of permutations of V and consider the induced action of Γ on the set $V \times V$ of ordered pairs of elements of V. The arc (u, v) and its orbit under the action of Γ on $V \times V$ together with the vertices of V form what is called an orbital of Γ. Γ acts as a transitive group of automorphisms of the orbital. If this orbital is disconnected then the connected components of the orbital are isomorphic.

In [4] we studied the following construction. Let Γ be a subgroup of $S_V \times S_V$, where S_V is the symmetric group acting on V. Any element (α, β) of Γ acts on any element of V $\times$V by $(\alpha, \beta)(u, v) = (\alpha(u), \beta(v))$. A two-fold orbital is then an orbit of some pair (u, v) under the action of Γ. A two-fold orbital is generally a mixed oriented graph which may contain loops even if $u \neq v$. If it is a graph we call it a two-fold orbital graph (TOG) and if it is a digraph we call it a two-fold orbital digraph (TOD). If a two-fold orbital graph or digraph is disconnected then the components are not necessarily isomorphic. The action of the pair (α, β) is not as well-behaved as that of the usual automorphisms. In [4] we showed that some interesting results can however be obtained about disconnected two-fold orbitals, hence suggesting that these graphs and digraphs were worth further investigation.

We need a few more definitions here. Two graphs or digraphs G and H are said to be *two-fold isomorphic* or *TF-isomorphic* if there exist bijections α and β from V(G) to V(H) such that (u, v) is an arc of G if and only if $(\alpha(u), \beta(v))$ is an arc of H. A pair of non-isomorphic graphs may be two-fold isomorphic. For example, the Petersen graph shown in Figure 1 is TF-isomorphic to the second graph shown in the same figure, with α mapping $1, 2, \ldots, 10$ to $9', 4', 3', 2', 7', 6', 5', 8', 1', 10'$, respectively, and β mapping each $i' \in \{1, 2, \ldots, 10\}$ to i'.

A TF-isomorphism from G to itself is called a *TF-automorphism.* The set of all TF-automorphisms on G is a group and is denoted by $\mathrm{Aut}^{\mathbf{TF}} G$. We shall be mainly interested in (α, β) with $\alpha \neq \beta$. We will call these *non-trivial* TF-automorphisms or TF-isomorphisms. Of course, a graph is a TOG or a union of TOGs if and only if it admits a TF-automorphism.

Figure 2 shows the graph $G = C_6$ having a non-trivial automorphism (α, β) where $\alpha = (1\ 2\ 3)(4)(5)(6)$ and $\beta = (1)(2)(3)(4\ 5\ 6)$, but neither α nor β is an automorphism of G.

As far as we know, the only other papers apart from [4] in which two-fold isomorphisms of graphs or digraphs have been studied are Zelinka's [8, 9]. In [9], Zelinka calls the two-fold isomorphisms isotopies, inspired by the studies of isotopies in groupoids where two groupoids G_1 and G_2 are said to be isotopic if there exists three bijections

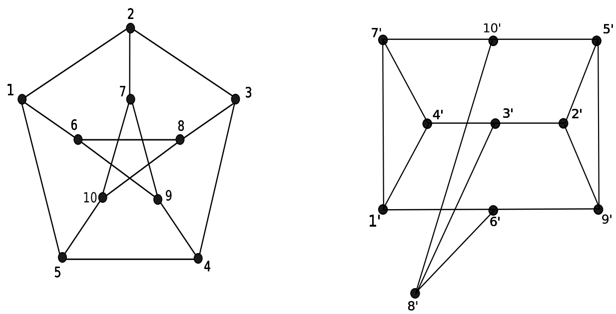

Figure 1: A pair of TF-isomorphic but not isomorphic graphs.

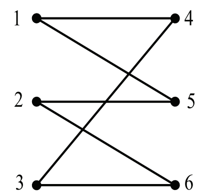

Figure 2: The graph C_6.

$\phi,\ \psi,\ \chi$ from G_1 to G_2 such that for any three elements $a,\ b,\ c\ \in G_1$, the equality $ab = c$ is equivalent to the equality $\phi(a)\psi(b) = \chi(c)$ in G_2.

The next definition that we require is quite well known [2, 6, 7]. The *canonical double cover (CDC)* of a graph or digraph G, also called its *duplex*, is the graph or digraph whose vertex set is $V(\mathrm{G}) \times \{0, 1\}$ and in which there is an arc from (u, i) to (v, j) if and only if $i \neq j$ and there is an arc from u to v in G. The canonical double cover of G is often described as the direct or categorical product $\mathrm{G}\times K_2$ (see [3], for example), and is sometimes also referred to as the bipartite double cover of the graph G denoted by $\mathbf{B}(G)$. If G is bipartite then it is well known that $\mathbf{B}(G)$ is a disconnected graph with two components each isomorphic to G. This fact will be exploited in Section 3.1.

The main connection between TF-isomorphisms and CDCs is the following result proved in [6].

Theorem 1.1. *Let G and H be either both graphs or both digraphs. Then G and H are TF-isomorphic if and only if they have isomorphic double covers.*

Using TF-isomorphisms and canonical double covers we were able to show, in [4], that although the components of a disconnected TOG need not be isomorphic, something interesting can still be said about their structure.

Theorem 1.2. *Let G be a disconnected TOG without isolated vertices. Let its components be $G_1,\ G_2,\ \cdots,\ G_k$ such that*

$$|V(G_1)| \geq |V(G_2)| \geq\ \cdots \geq |V(G_k)|$$

Then each G_i is still a TOG and moreover

(i) If $|V(G_1)| = |V(G_k)|$ then the components G_i are pairwise TF-isomorphic ;

(ii) Otherwise there is some index $r \in \{1, \cdots, k-1\}$ such that $G_1, \cdots, G_r$ are isomorphic, $G_{r+1}, \cdots, G_k$ are TF-isomorphic and G_1 is isomorphic to the canonical double cover of each G_i, $i \in \{r+1, \cdots, k\}$.

On reflection, the close link between CDCs and TF-isomorphisms or TF-automorphisms should not be surprising because we can describe a TF-automorphism or a TF-automorphism using the following construction. Let G be a graph and consider the CDC of G denoted by $\mathbf{B}(G)$. For convenience we shall denote vertices $(u, 0)$ of $\mathbf{B}(G)$ simply by u, and we call them the *left vertices*, and we shall denote the vertices $(u, 1)$ by u' and call them the *right vertices*. Now, change every edge of $\mathbf{B}(G)$ into an arc from the corresponding left vertex to the corresponding right vertex, and denote this graph by $\vec{\mathbf{B}}(G)$. An example is illustrated in Figure 3. At this stage of the construction we have merely split up each edge into its two arcs and placing these in a way in which we can easily describe the action of a pair of permutations (α, β).

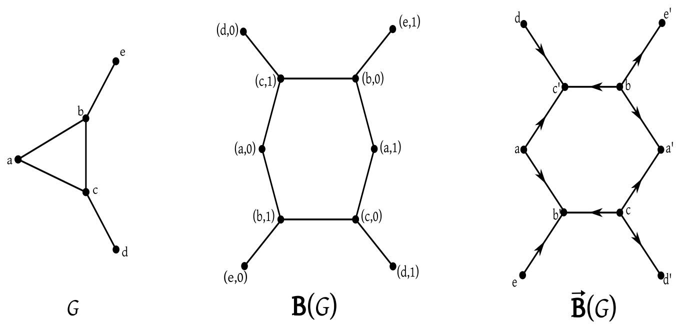

Figure 3: The construction of $\vec{\mathbf{B}}(G)$.

So, let (α, β) be two permutations of $V(G)$. Permute the labels of the left vertices of $\vec{\mathbf{B}}(G)$ according to α and permute the labels of the right vertices of $\vec{\mathbf{B}}(G)$ according to β (we are here tacitly assuming that if $\beta(x) = y$ then $\beta(x') = y'$). Finally construct what we call $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$ as follows. Identify the vertices in $\vec{\mathbf{B}}(G)$ which have the same label up to a prime (that is, identify vertices u and u'), and replace every resulting pair of self-paired arcs by an edge. We can now present this result.

Proposition 1.3. *Let G be a graph. Then if (α, β) is a TF-automorphism of G, then $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$ is isomorphic to G. If (α, β) is a TF-isomorphism from G to the graph H, then $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$ is isomorphic to H.*

Proof. This follows immediately by noting that the arc-set of $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$ is given by all the arcs $(\alpha(u), \beta(v))$ for all arcs (u, v) in G. □

Figure 4 illustrates how, if we start from a graph G, the procedure described above yields $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$ which is isomorphic to the graph H when the pair (α, β) is a TF-isomorphism from G to H. In this example α maps $1, 2, 3, 4, 5, 6, 7$ into $4, 5, 6, 1, 2, 3, 7$ and β maps $1', 2', 3', 4', 5', 6', 7'$ into $1', 2', 3', 4', 5', 6', 7'$.

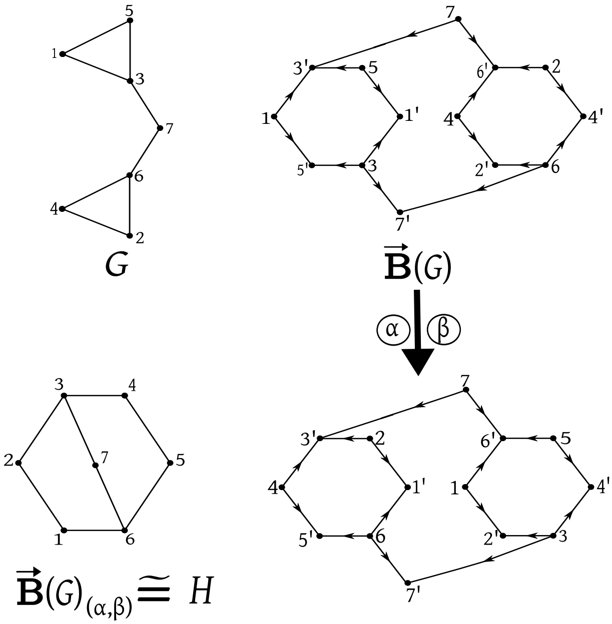

Figure 4: Illustrating a TF-isomorphism via $\vec{\mathbf{B}}(G)$ and $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$.

The next result is therefore quite predictable and follows almost immediately from these considerations.

Lemma 1.4. *Let G be a graph and $\boldsymbol{B}(G)$ its canonical double cover. Then any two-fold automorphism (α, β) induces an automorphism of $\boldsymbol{B}(G)$ that fixes the colour classes.*

Proof. Given any TF-automorphism (α, β) of G we may always define a map $\phi : \mathbf{B}(G) \to \mathbf{B}(G)$ by $\phi(x, 0) = (\alpha(x), 0)$ and $\phi(x, 1) = (\beta(x), 1)$. Clearly ϕ fixes the colour classes. Let us now show that ϕ is an automorphism of G. Let $(u, 0)$ and $(v, 1)$ be adjacent in $\mathbf{B}(G)$. Then u and v are adjacent in G which implies that $\alpha(u)$ and $\beta(v)$ are also adjacent in G since (α, β) is a TF-automorphism. Therefore $(\alpha(u), 0)$ and $(\beta(v), 1)$ are also adjacent in $\mathbf{B}(G)$ by definition so that we can conclude that ϕ is an automorphism of $\mathbf{B}(G)$. If (α, β) is a non-trivial TF-automorphism then ϕ is a non-trivial automorphism of $\mathbf{B}(G)$. □

This result effectively says that the automorphism group of G can be embedded in the colour-class-fixing automorphism group of $\mathbf{B}(G)$, but the latter can be larger, and this happens when G has non-trivial TF-automorphisms. This is the situation which we shall investigate below. The next result is a sort of inverse to the previous one.

Proposition 1.5. *Let G be a bipartite graph with colour classes X and Y. Let ϕ_1 and ϕ_2 be two automorphisms of G that both fix X and Y set-wise. Define $\alpha(u) = \phi_1(u)$ for any $u \in X$ and $\alpha(w) = \phi_2(w)$ for any $w \in Y$. Define $\beta(u) = \phi_2(u)$ for any $u \in X$ and $\beta(w) = \phi_1(w)$ for any $w \in Y$. Then $(\alpha, \beta) \in Aut^{TF}G$.*

In particular, if the bipartite graph G has a non-trivial automorphism which fixes the colour classes, then G has a non-trivial TF-automorphism.

Proof. Let $\{u, w\} \in \mathrm{E}(G)$. Assume without loss of generality that $u \in X$ and $w \in Y$. Then (α, β) takes arc $u \longrightarrow_G w$ into arc $\phi_1(u) \longrightarrow_G \phi_1(w)$ which exists since $\phi_1 \in \mathrm{Aut}\ G$.

Similarly arc $w \longrightarrow_G u$ is taken into $\phi_2(w) \longrightarrow_G \phi_2(u)$ which exists because $\phi_2 \in \mathrm{Aut}\ G$.

When G has a nontrivial automorphism ϕ which fixes the colour classes define (α, β) as above by taking $\phi_1 = \phi$ and $\phi_2 = \mathrm{id}$. □

Lemma 1.6. *Let G be a disconnected graph with two components isomorphic to H. Then there is a non-trivial TF-isomorphism from G to $\boldsymbol{B}(H)$.*

Proof. Label the vertices in the first copy of H with the suffix 0, and the vertices in the second copy of H with the suffix 1. Then let α map any u_i into $(u, i) \in V(\mathbf{B}(G))$, for any $i = 0$ or 1. Let β map any u_0 into $(u, 1)$ and u_1 into $(u, 0)$. Then (α, β) is a TF-isomorphism from G to $\mathbf{B}(H)$. □

The proof technique presented for the following result will come in very useful in a later construction.

Proposition 1.7. *Let G be a disconnected graph with two components, one of which is H and the other $\boldsymbol{B}(H)$, where $\boldsymbol{B}(H)$ is the canonical double cover of H. Then G is a union of TOGs.*

Proof. Let $V(H) = \{v_1, v_2, \dots, v_n\}$. Add the following vertices to H without any other edges for the moment: $\{w_1, w'_1, w_2, w'_2, \dots, w_n, w'_n\}$. Let γ be the permutation

$$\gamma = (v_1\ w_1\ w'_1)(v_2, w_2, w'_2) \dots (v_n, w_n, w'_n).$$

Let $\tau = (\gamma, \gamma^{-1})$. We shall add edges joining the vertices w_1 and w'_j such that τ is a TF-automorphism of the resulting graph. It will turn out that the graph induced by the w_i and w'_j is the CDC of H.

So, note that any arc (v_i, v_j) under that action of τ has the orbit

$$\{(v_i, v_j), (w_i, w'_j), (w'_i, w_j)\}$$

and the orbit of the arc (v_j, v_i) is

$$\{(v_j, v_i), (w_j, w_i'), (w_j', w_i)\}.$$

The orbit of the edge $\{v_i, v_j)$ is the set of edges

$$\{v_i, v_j\}, \{w_i, w_j'\}, \{w_j, w_i'\}.$$

Therefore, adding these new vertices to complete the orbits for all edges in H, results in the graph $\mathbf{B}(H)$ on the vertices w_i and w_j'. Therefore $H \cup \mathbf{B}(H)$ is a union of TOGs, as required. □

Corollary 1.8. *Let G be a disconnected graph such that it has at least one component being the CDC of some other component. Then G is a TOG or a union of TOGs.*

Proof. Let H and $\mathbf{B}(H)$ be two components of G. Define α and β to be acting on the vertices of H and $\mathbf{B}(H)$ as in the previous proof, and extend them to the other components by the identity action. Then (α, β) is the required TF-automorphism of G, meaning that G is a TOG or union of TOGs. □

2 Connected two-fold isomorphic graphs which are not isomorphic

Let us now discuss a way of constructing connected two-fold isomorphic graphs G_1 and G_2 which need not be isomorphic. We will use the ideas presented in the previous section for this purpose. Consider a graph G_1' consisting of two copies of some non-bipartite graph G and a graph $G_2' = \mathbf{B}(G)$. The graphs G_1' and G_2' are two fold isomorphic by Lemma 1.6. They also have isomorphic CDCs by Theorem 1.1. This is due to the fact that the CDC of G_1' is $\mathbf{B}(G) \cup \mathbf{B}(G)$ and the CDC of G_2' $=\mathbf{B}(G)$ is also equal to two copies of $\mathbf{B}(G)$, since $\mathbf{B}(G)$ is bipartite.

Now we need to add edges to G_1' in order to connect the two components, and corresponding edges to G_2' in order to maintain the TF-isomorphism. Recall the TF-isomorphism (α, β) defined in Lemma 1.6. Label the vertices of the two components of G_1' with suffixes 0 and 1 as in that proof. Let x_0 and y_1 be any two vertices in the two components. Join by an edge these two vertices and also the vertices y_0 and x_1, and call this graph G_1. In G_2', join by an edge the vertices $(x, 0)$ and $(y, 0)$ $(x \neq y)$ and also the two vertices $(x, 1)$ and $(y, 1)$, and call this graph G_2. It is then clear that (α, β) is still a TF-isomorphism from G_1 to G_2. Note that, in general, G_1 and G_2 need not necessarily be non-isomorphic. For example, when $G = C_5$ and x and y are adjacent vertices of G, the graphs produced are isomorphic.

We shall describe another construction because it can help explain Figure 1. Let G_1' and G_2' be as above. Suppose that there is a TF-isomorphism from G_1' to G_2', with

a, b two vertices in different components of G_1' and such that α maps a into x and b into y and β maps a into y and b into x. Now, add a new vertex c to G_1' joined to a and b, respectively, and a new vertex z to G_2' joined to x and y, respectively. Call the two new graphs G_1 and G_2 and extend α and β by letting them both map c to z. It is clear that (α, β) is still a TF-automorphism from G_1 to G_2. Again, G_1 and G_2 need not be non-isomorphic.

This construction is illustrated by the example shown in Figure 5. We start off with one graph being two copies of C_3 and the other a C_6. There is a TF-automorphism (α, β) between the two graphs such that α maps 3 into $6'$ and 6 into $3'$ while β maps 3 into $3'$ and 6 into $6'$. Therefore joining a new vertex 7 to vertices 3 and 6, respectively, and a new vertex $7'$ to $3'$ and $6'$, respectively, we get the resulting graphs in Figure 5 which are TF-isomorphic but not isomorphic.

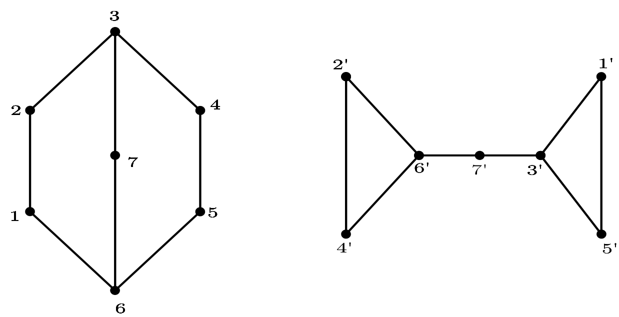

Figure 5: The pair (α, β) where α maps $1, 2 \ldots, 7$, to $4', 5', 6', 1', 2', 3', 7'$, respectively and β maps each $i \in \{1, 2, \ldots, 7\}$ into i' is a TF-isomorphism between these two non-isomorphic graphs.

Repeating this procedure three times we can ultimately obtain a TF-isomorphism between the Petersen graph minus one edge and the second graph in Figure 1 missing, say, edge $\{3, 8\}$. This TF-isomorphism also takes the missing edge from the Petersen graph to the missing edge of the second graph, since both graphs have exactly two vertices of degree 2 and these are the endvertices of the missing edge. Therefore the TF-isomorphism is also a TF-isomorphism between the Petersen graph and the second graph.

3 Asymmetric graphs

Knowing that non-isomorphic graphs can be TF-isomorphic, it is natural to ask whether a graph G with trivial automorphism group can have a *non-trivial TF-automorphism.* The situation for bipartite graphs is very different from that for non-bipartite graphs. We first start with a few easy but useful results.

Proposition 3.1. *Let G be a graph and let $(\alpha, \beta) \in Aut^{TF}G$. Then (γ, γ^{-1}) is also in $Aut^{TF}G$ where $\gamma = \alpha\beta^{-1}$.*

Proof. Since (α, β) is a TF-automorphism, so is (β, α), because G is a graph, and therefore $(\beta^{-1}, \alpha^{-1})$ is also a TF-automorphism. Therefore

$$(\alpha, \beta)(\beta^{-1}, \alpha^{-1}) = (\alpha\beta^{-1}, \beta\alpha^{-1})$$

is also a TF-automorphism, as required. □

Proposition 3.2. *If (α, β) is a TF-automorphism of a graph G, then either both α and β are automorphisms, or neither α nor β is.*

Proof. Note first that if $\alpha = \text{id}$ then for an arc (u, v) we have $(\alpha, \beta)(u, v) = (u, \beta(v))$, so that β preserves the neighborhood of any vertex u. This implies $\beta \in \text{Aut } G$. For the general case, it remains to show that if α is an automorphism then also β is. Since α is an automorphism, (α, α) and $(\alpha^{-1}, \alpha^{-1})$ are both TF-automorphisms. Therefore, $(\alpha, \beta)(\alpha^{-1}, \alpha^{-1}) = (\text{id}, \beta\alpha^{-1}) \in \text{Aut}^{\text{TF}}G$. Because of the above argument, we get $\beta\alpha^{-1} \in \text{Aut } G$, hence $\beta \in \text{Aut}(G)$. □

Proposition 3.2 suggests looking for α, β which are not automorphisms such that (α, β) is a TF-automorphism. An example has been given in Figure 1. This fact allows us to ask a more ambitious question : Is it possible for a graph G, with $\text{Aut } (G) = \{\text{id}\}$, to have a non-trivial two-fold automorphism, that is, $(\alpha, \beta) \in \text{Aut}^{\text{TF}}G$ such that $\alpha \neq \beta$? In other words, is it possible for an asymmetric graph to be *two-fold symmetric*? The following propositions help us to short-list possibilities.

Proposition 3.3. *If (α, β) is a two-fold automorphism of some graph G and $\gamma = \alpha\beta^{-1}$ has even order, then the graph has a non-trivial automorphism.*

Proof. If $(\alpha, \beta) \in \text{Aut}^{\text{TF}}G$ then by Proposition 3.1, $(\gamma, \gamma^{-1}) \in \text{Aut}^{\text{TF}}G$ and so is (γ^k, γ^{-k}) for any k. Now if γ has even order $2m$, then for $k = m$ we get $\gamma^m = \gamma^{-m}$. Hence γ^m is a non-trivial automorphism. □

Proposition 3.4. *If (α, β) is a non-trivial TF-automorphism of G and S is a subset of $V(G)$ moved by $\gamma = \alpha\beta^{-1}$, then the subgraph H_S induced by S is a union of TOGs.*

Proof. If S is the set moved by γ, it is also moved by γ^{-1}. Since (γ, γ^{-1}) is a TF-automorphism of the graph G then it maps arcs into arcs. If (γ, γ^{-1}) acts on an arc (u, v) of H_S, it maps it to some other arc $(\gamma(u), \gamma^{-1}(v)) = (x, y)$. Since u belongs to a non-trivial orbit of γ, so does $x = \gamma(u)$, hence x belongs to S. Likewise, y belongs to S. □

We shall now see that the problem of finding asymmetric graphs with a non-trivial TF-automorphsim is quite different for the bipartite and non-bipartite cases.

3.1 Bipartite asymmetric graphs

The disconnectedness of $\mathbf{B}(G)$ for bipartite graphs G has a nice consequence.

Theorem 3.5. *If a bipartite graph G has a non-trivial TF-automorphism then it must have a non-trivial automorphism.*

Proof. The idea of the proof is the following. Let (α, β) be a non-trivial TF-automorphism of G. Recall Proposition 1.3. There we saw that G is isomorphic to $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$. But since G is bipartite, $\mathbf{B}(G)$ consists of two copies of G. Therefore $\vec{\mathbf{B}}(G)_{(\alpha,\beta)}$, and hence G, is obtained by identifying pairs of vertices from these two copies of G. This identification gives an automorphism of G. But $\alpha \neq \beta$ and α acts on the labels of the left vertices of $\vec{\mathbf{B}}(G)$ while β acts on the labels of the right vertices. Therefore this automorphism is non-trivial because the final identification cannot match every left vertex labelled u in $\vec{\mathbf{B}}(G)$ (before the action of α) to the corresponding right vertex labelled u' in $\vec{\mathbf{B}}(G)$ (before the action of β).

More formally, let G be bipartite with classes X and Y and let (α, β) be a TF-automorphism of G. $\mathbf{B}(G)$ is the union of two disjoint bipartite graphs G' and G'', both of them isomorphic to G. One of them, say G', has classes X_0 and Y_1, whose elements take the forms $(x, 0)$ and $(y, 1)$ respectively, with $x \in X$ and $y \in Y$. Likewise, G'' has classes X_1 and Y_0, with elements of the form $(x, 1)$ and $(y, 0)$. There are three isomorphisms: ϕ' from G to G', γ from G' to G'' and ϕ'' from G'' to G', namely: ϕ' takes v to $(\alpha(v), 0)$ if $v \in X$ and to $(\beta(v), 1)$ if $v \in Y$; γ takes $(v, 0)$ to $(v, 1)$; ϕ'' takes $(v, 1)$ to $\beta^{-1}(v)$ and $(v, 0)$ to $\alpha^{-1}(v)$. Letting ϕ be the composition $\phi'\gamma\phi''$, it is of course an automorphism of G and takes v to $\beta^{-1}\alpha(v)$ if $v \in X$ and to $\alpha^{-1}\beta(v)$ if $v \in Y$. Since α and β are different, ϕ is not the identity. □

Since Theorem 3.5 tells us that no asymmetric bipartite graph can have a non-trivial two-fold automorphism, we may now restrict our attention to asymmetric non-bipartite graphs.

3.2 Non-bipartite asymmetric graphs

Graphs with a trivial automorphism group, that is graphs which are asymmetric, may indeed have a non-trivial TF-automorphism group. In other words, it is possible to find some graph G such that $\mathrm{Aut}(G) = \{\mathrm{id}\}$, but there exists some $(\alpha, \beta) \in \mathrm{Aut}^{\mathbf{TF}} G$ such that $\alpha \neq \beta$. An example is illustrated in Figure 6.

One way of constructing graphs which are asymmetric but which have some non-trivial two-fold automorphism is by starting from an asymmetric non bipartite graph G and its canonical double cover $\mathbf{B}(G)$. The graph $H = G \cup \mathbf{B}(G)$ is a disconnected TOG as described in Theorem 1.2 with $\tau = (\gamma, \gamma^{-1})$ as constructed in the proof. The next step consists of introducing new edges to join G to $\mathbf{B}(G)$ such that τ still remains a two-fold automorphism (that is, we need to include the full two-fold orbitals which contain the new edges) while ensuring that the asymmetry of G stops $\mathbf{B}(G)$ from having any symmetry and no new symmetries are introduced. In the

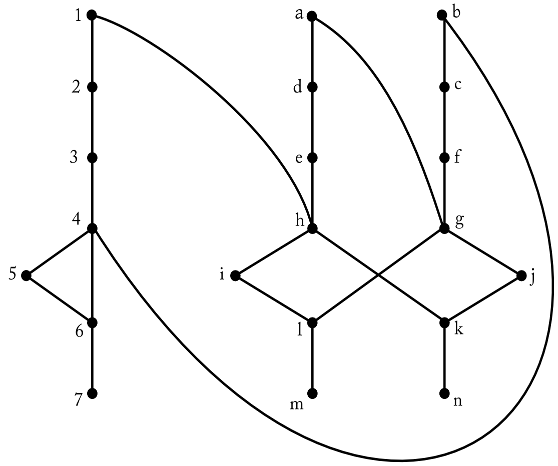

Figure 6: The pair (α, β) where $\alpha = \gamma$ and $\beta = \gamma^{-1}$ and $\gamma = (1\ a\ b)(2\ c\ d)(3\ e\ f)(4\ g\ h)(5\ i\ j)(6\ k\ l)(7\ m\ n)$ is a two-fold automorphism of this asymmetric graph.

example illustrated in Figure 6, the graph G is the smallest asymmetric graph (on seven edges) and the edges $\{1, h\}$, $\{b, 4\}$ and $\{a, g\}$ have been introduced to join G to $\mathbf{B}(G)$ (the vertices of $\mathbf{B}(G)$ are labelled with letters). Another example can be obtained from this by adding the edges $\{1, e\}, \{a, 3\}$ and $\{b, f\}$.

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