Two-fold Orbital Digraphs and Other Constructions

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Abstract

In this paper we shall present a natural generalisation of orbital graphs. If $\mathbf{\Gamma}$ is a subgroup of $S_n \times S_n$, V and n-set, and $(u, v) \in V \times V$, then the set $\mathbf{\Gamma}(u, v) = \{(\alpha(u), \beta(v)) | (\alpha, \beta) \in \mathbf{\Gamma}\}$ will be the arc-set of a digraph G with vertex-set V. Such a G will be called a two-fold orbital digraph (TOD).

We shall emphasise properties of G which are markedly different from those of orbital graphs, focusing, in particular, on the case when G is disconnected, since this case brings out very sharply differences between orbital graphs and TODs. The close relationship, in this case, between the TOD G and its canonical double covering, is also investigated.

The paper contains several examples intended to make these new concepts and results more clear.

1 Introduction

Development in group theory, such as the classification of finite simple groups, stimulated recent advancements in algebraic graph theory. One of the remarkable tasks accomplished is the determination of all vertex-transitive graphs of order equal to the product of two primes (cf. [6], [12] and [11]).

The construction of orbital digraphs is one of the basic tools in the study of vertex-transitive digraphs. In principle, the group-theoretical method used to construct orbital digraphs may not only be employed to generate all vertex-transitive digraphs (cf. [3] and [7]), but also makes it clear whether these vertex-transitive digraphs are arc-transitive or not.

Orbital digraphs are obtained by a permutation group Γ acting on a set V. Fixing (u, v), $u, v \in V$, $(u, v) \in V \times V$, all pairs $(\alpha(u), \alpha(v))$, with $\alpha \in \Gamma$, form a digraph G, such that $\Gamma \leq Aut(G)$. Some general properties of orbital graphs and their use in algebraic graph theory can be found in [1] and [5].

Here we will present a natural generalization of this construction. Instead of orbitals, we will define two-fold orbitals and instead of orbital digraphs, we will construct two-fold orbital digraphs. A two-fold orbital is obtained by replacing α by a pair of possibly different permutations $(\alpha, \beta) \in \mathbf{\Gamma} \leq \mathcal{S} = S_n \times S_n$. Then, if (u, v) is a fixed element of $V \times V$, the set $\mathbf{\Gamma}(u, v)$ defined by $\mathbf{\Gamma}(u, v) = \{(\alpha(u), \beta(v)) \,|\, (\alpha, \beta) \in \Gamma\}$ is a two-fold orbital. The digraph $G = (V, \mathbf{\Gamma}(u, v))$ having V as vertex set and the two-fold orbital $\mathbf{\Gamma}(u, v)$ as arc-set is then called a two-fold orbital digraph or a $\mathbf{\Gamma}$-orbital digraph.

We will show that although the orbital, that is the arc set of a two-fold orbital digraph, is itself an orbit of a group $\mathbf{\Gamma}$ acting on $V \times V$, two-fold orbital digraphs have properties that are considerably different from those of the usual orbital graphs. For instance, whereas orbital graphs are all vertex-transitive, two-fold orbital graphs are not necessarily so. The action of a pair (α, β) on $V \times V$ has a 'wild' behaviour. We may, for instance, have loops even for $u \neq v$. On the other hand, we will show that there is a close relationship with the automorphism group of the canonical double covering (CDC) of the digraph G, that is, $\mathbf{B}(G)$. In view of the interest in the study of canonical double coverings (cf. [4], [8] and [13]), this close relationship provides further motivation for this research.

The two-fold orbital digraph construction method may therefore be used to build digraphs G, such that their canonical double coverings, that is $\mathbf{B}(G)$, have suitable properties. We may, for example, find pairs of graphs G and $\grave{G}$ with $\mathbf{B}(G) \cong \mathbf{B}(\grave{G})$ (cf. [9] and [10]). Cvetković, Doob and Sachs [2] have shown that such pairs of graphs are cospectral.

Therefore, in view of the above remarks, we hope that investigating the features of two-fold orbital digraphs will be interesting and useful.

The focus of this work is on some properties of two-fold orbital digraphs that differ from those of orbital digraphs. After some preliminary definitions on digraphs and graphs and permutation groups we shall consider, in Sections 2 and 3, two-fold isomorphisms and canonical double coverings. Our main results are then contained in Section 4 where we emphasize disconnected two-fold orbital graphs because whereas in orbital graphs the components are clearly all isomorphic, a very different behaviour is observed for two fold orbital graphs. Then in Section 5 study in more detail a special class of two-fold orbital digraphs, the class of χ-orbital digraphs, where the existence of loops is considered in more carefully. We also characterize bipartite disconnected two-fold orbital graphs and also derive conditions for cycle structures of the isomorphisms α and β which are necessary for (α, β) to be a two-fold isomorphism of a disconnected two-fold orbital graph G. We conclude the paper with some open problems and conjectures.

2 Definitions

2.1 Digraphs and Graphs

A *digraph* $G = (V(G), A(G))$ consists of two sets: a set $V(G)$ whose elements are called vertices and a set $A(G)$ whose elements, called arcs, are ordered pairs of elements of $V(G)$. The sets of vertices will always be finite. An arc (u, v) from u to v, where $u, v \in V(G)$, is denoted by $u \longrightarrow_G v$ (or $u \longrightarrow v$ when G is clear from the context). Arcs $u \longrightarrow u$, consisting of a pair of repeated vertices are sometimes allowed. Such arcs are called *loops.* Multiple arcs, that is, repetition of the same arc $u \longrightarrow v$, are not allowed.

A digraph G is said to be *bipartite* if there exists a partition of $V(G)$ into two classes V_0 and V_1 such that for all arcs $u \longrightarrow v$ either $u \in V_0$ and $v \in V_1$ or $v \in V_0$ and $u \in V_1$. The sets V_0 and V_1 are said to be the *stable parts* (or *sets*) of G.

The number of arcs incident from a vertex v is called its *out-degree*, denoted by $\deg_{out}(v)$ while the number of arcs incident to v is called its *in-degree*, denoted by $deg_{in}(v)$.

An arc $u \longrightarrow_G v$ in $A(G)$, is said to be *self-paired* if the opposite arc $v \longrightarrow_G u$ is also in $A(G)$. If an arc $u \longrightarrow_G v$ is self-paired, then vertices $u, v \in G$ may be 'joined' by a pair of oppositely directed arcs, that is, $u \longrightarrow_G v$ and $v \longrightarrow_G u$. The ordered pairs (u, v) and (v, u) may be substituted by the unordered pair $\{u, v\}$. An unordered pair of vertices $\{u, v\} \in V(G)$ is known as an *edge.* The edge $\{u, v\}$ of digraph G may be denoted by $u \sim_G v$ or $u \sim v$.

A *graph* (or *undirected graph*) $G = (V(G), E(G))$ consists of two disjoint sets : a non-empty set $V(G)$ whose elements are called *vertices* and a set $E(G)$ whose elements, called *edges*, are *unordered* pairs of elements of $V(G)$ in the sense defined above. Therefore, for us *a graph is actually a digraph in which all arcs are self-paired.*

In the case of graphs, the *degree of a vertex* v, denoted by $deg(v)$ is the number of edges in $E(G)$ to which v belongs. A graph is said to be *regular* if all its vertices have the same degree.

2.2 Permutation Groups

A permutation group is a pair (Γ, X) where X is a finite set and Γ is a subgroup of the symmetric group S_X, that is the group of all permutations of X.

Let $v \in X$. The stabilizer of v, under the action of Γ is denoted by Γ_v while the orbit of v is denoted by $\Gamma(v)$.

The *Orbit-Stabilizer Theorem* states that, for any element $v \in X$,

$$|\Gamma| = |\Gamma(v)|.|\Gamma_v|.$$

A permutation group (Γ, X) is said to be a *transitive permutation group* if the elements of X are all in one orbit; Γ is said to act *transitively* on X.

We say that a digraph G is vertex-transitive if $\forall u, v \in V(G)$, there is an automorphism α of G such that $\alpha(u) = v$.

We say that a digraph G is arc-transitive if given any two arcs $a \longrightarrow_G b$ and $c \longrightarrow_G d$, there exists an automorphism α of such that $\alpha(a) \longrightarrow_G \alpha(b) = c \longrightarrow_G d$. This means that a digraph G is arc-transitive if all the arcs of G are in the same orbit under the action of α. In particular, if G is a graph, then it is arc-transitive if, for any two edges $a \sim b$ and $c \sim d$, there exist automorphisms α, β such that $\alpha(a) = c$ and $\alpha(b) = d$, and $\beta(a) = d$ and $\beta(b) = c$.

We say that a graph G is edge-transitive if given any two edges $\{a, b\}$ and $\{c, d\}$, there exists an automorphism α of such that $\{\alpha(a), \alpha(b)\} = \{c, d\}$. This means that a graph G is edge-transitive if all the edges of G are in the same orbit under the action of α.

2.3 Direct Products of Groups

We recall that the direct product $\Gamma_1 \times \Gamma_2$ of two groups Γ_1 and Γ_2 consists of ordered pairs (α, β) where $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$, with multiplication defined by:

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2).$$

Let V be an n-set and S_n be the *symmetric group* on V. We will write $\mathcal{S} = S_n \times S_n$. We shall consider the natural action of $\mathcal{S}$ on $V \times V$ defined as follows:

$$(\alpha, \beta)(u, v) = (\alpha(u), \beta(v)) \textit{ with } (\alpha, \beta) \in \mathcal{S} \textit{ and } (u, v) \in \ V \times V.$$

Suppose $\pi_1, \pi_2 : \mathcal{S} \mapsto S_n$ are defined by $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2(\alpha, \beta) = \beta$. Then, π_1 and π_2 are said to be the *canonical projections* of $\mathcal{S}$ on S_n.

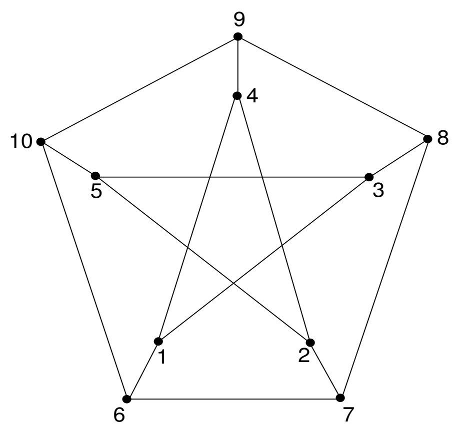

Figure 1: (α, β) *is a TF-automorphism of the Petersen graph where* α, β *are defined by* $\alpha = \beta = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)$

Let $\mathbf{\Gamma} \leq \mathcal{S}$. The *transpose* of $\mathbf{\Gamma}$, written as $\mathbf{\Gamma}^T$ is the set of ordered pairs (β, α) such that $(\alpha, \beta) \in \mathbf{\Gamma}$. If $\mathbf{\Gamma}^T = \mathbf{\Gamma}$, the group is said to be *self-paired*. Clearly, if $\mathbf{\Gamma}$ is *self-paired*, its projections $\pi_i(\mathbf{\Gamma})$ $(i = 1, 2)$ are equal.

If $\Gamma \leq S_n$, then the subgraph $D(\Gamma)$ of $\mathcal{S}$ defined by $D(\Gamma) = \{(\alpha, \alpha | \alpha \in \Gamma\}$ is said to be the *diagonal group* corresponding to Γ.

2.4 TF-isomorphisms

Let G_1 and G_2 be two digraphs. Then (α, β) is a *two-fold isomorphism* from G_1 to G_2, denoted by *TF-isomorphism*, $\iff$ α and β are bijections from $V(G_1)$ to $V(G_2)$ and $u \longrightarrow_{G_1} v \iff \alpha(u) \longrightarrow_{G_2} \beta(v)$ for every pair of vertices $u, v \in V(G_1)$.

If there is a TF-isomorphism between digraphs G_1 and G_2, we say that G_1 and G_2 are *TF-isomorphic* and we write $G_1 \cong^{TF} G_2$. A TF-isomorphism $(\alpha, \beta) : G_1 \rightarrow G_1$, that is, between a digraph and itself, is called a *TF-automorphism*. An example of a TF-automorphism is illustrated in Figure 1. The pair (α, β) is, in fact, a TF-automorphism on the Petersen graph.

The set of TF-*automorphisms* of a digraph G is a group and it is denoted by $Aut^{TF} G$.

Note that although two isomorphic digraphs have isomorphic underlying graphs, two TF-isomorphic digraphs may have non-TF-isomorphic underlying graphs. An example is given in Figure 2.

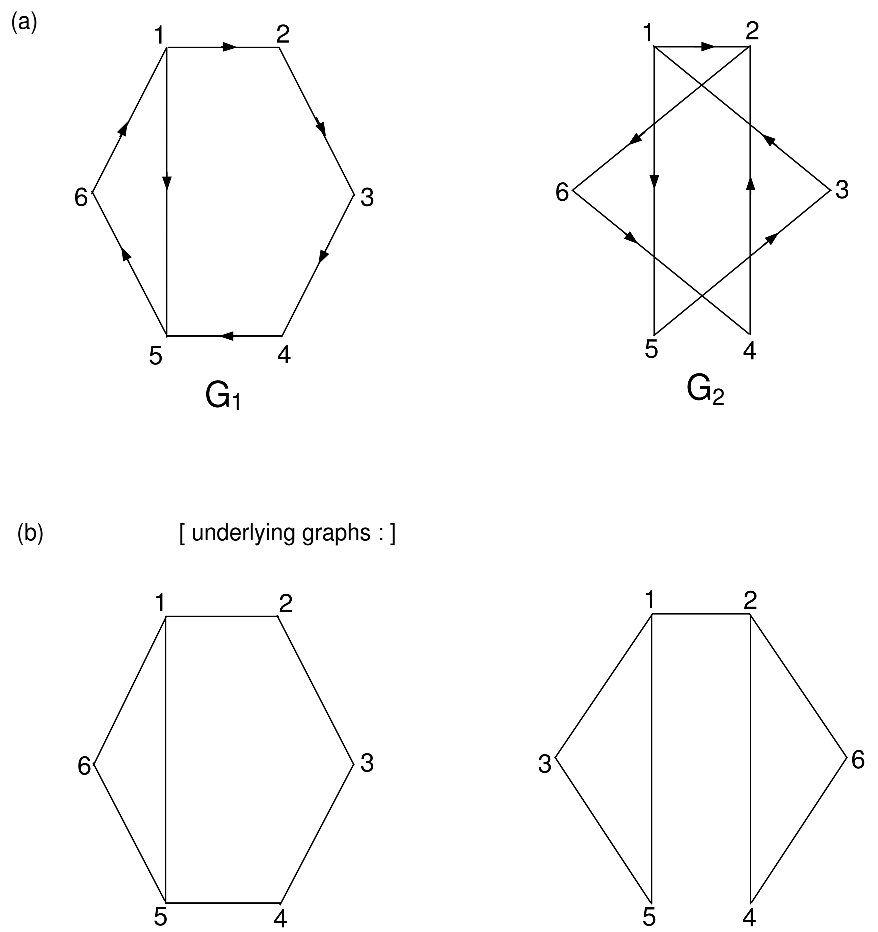

Figure 2: *In this example* $\alpha = (1\ 2\ 3)(4)(5)(6)$, $\beta = (1\ 4)(2\ 5)(3\ 6)$ *and* (α, β) *is a TF-isomorphism from* G_1 *to* G_2. *However,* (α, β) *is not a TF-isomorphism between the underlying graphs. In fact there is no TF-isomorphism between these graphs.*

As we shall see later on, the relationship between TF-isomorphisms and canonical double coverings is more direct than that between TF-isomorphisms and the underlying graphs. Subsequently, we shall use results on canonical double coverings to get results which connect the behaviour of TF-isomorphisms and the properties of the actual digraphs and underlying graphs.

3 Canonical Double Coverings.

3.1 Basic Properties

In the following discussion, the digraphs are finite and without multiple arcs, but loops are allowed.

Let G be a digraph. The *canonical double covering* (CDC) of G is the digraph $\mathbf{B}(G)$, whose vertex set is $V(G) \times \mathbb{Z}_2$ such that there exists an arc

$$(u, \varepsilon) \longrightarrow (v, \varepsilon + 1) \iff u \longrightarrow_G v \text{ exists } (\varepsilon, \in \mathbb{Z}_2).$$

On the same vertex set, we will denote by $\mathbf{B}^+(G)$ the digraph whose arcs are $(u, 0) \longrightarrow (v, 1)$ with $u \longrightarrow v$ in G and we will denote by $\mathbf{B}^-(G)$ the digraph whose arcs are $(u, 1) \longrightarrow (v, 0)$ with $u \longrightarrow v$ in G

Clearly $\mathbf{B}(G) = G \times K_2$, the categorical product of G and K_2, and $\mathbf{B}(G) = \mathbf{B}^+(G) \cup \mathbf{B}^-(G)$. Moreover, $\mathbf{B}^+(G)$, $\mathbf{B}^-(G)$ and $\mathbf{B}(G)$ are bipartite. Note that $\beta : (u, \varepsilon) \mapsto (u, \varepsilon + 1)$ is an *automorphism* of $\mathbf{B}(G)$ and an *isomorphism* from $\mathbf{B}^+(G)$ to $\mathbf{B}^-(G)$.

Figure 3 shows a digraph G, its canonical double covering $\mathbf{B}(G)$ as well as $\mathbf{B}^+(G)$ and $\mathbf{B}^-(G)$. The fact that $\mathbf{B}(G) = \mathbf{B}^+(G) \cup \mathbf{B}^-(G)$ is clearly illustrated. Also note that $\beta : (u, \varepsilon) \mapsto (u, \varepsilon + 1)$ is clearly an automorphism of $\mathbf{B}(G)$ and an isomorphism between $\mathbf{B}^+(G)$ and $\mathbf{B}^-(G)$. In fact, β maps any start-vertex $(x, 0) \in V(\mathbf{B}^+(G))$ to a start-vertex $(x, 1) \in V(\mathbf{B}^-(G))$ and similarly it maps end-vertices of $\mathbf{B}^+(G)$ to end-vertices in $\mathbf{B}^-(G)$.

The proof of the following result is easy.

Proposition 3.1 *The canonical double covering of a bipartite digraph is disconnected.*

Two digraphs G and G' are said to be **B**-*isomorphic* if they have isomorphic canonical double coverings, that is, $\mathbf{B}(G) \cong \mathbf{B}(G')$.

Let $Aut(\mathbf{B}(G))_0$ be the setwise-stabilizer in $Aut(\mathbf{B}(G))$ of the stable parts of $\mathbf{B}(G)$. If G is *connected* and *not bipartite*, then $|Aut(\mathbf{B}(G)) : Aut(\mathbf{B}(G))_0| = 2$. (Note that, under the action of $Aut(\mathbf{B}(G))$ on subsets of vertices of $\mathbf{B}(G)$, the orbit of $\{(u, 0)|u \in G\}$ has only two elements, namely itself and $\{(u, 1)|u \in G\}$.)

Now, let $\tilde{G}$ be a bipartite graph. An involution $\alpha \in Aut(\tilde{G})$ is said to be *switching* if α interchanges the stable parts (or classes) of $\tilde{G}$, and *strongly switching* if furthermore no edge is fixed by α.

Of course, if α is switching, the conjugacy class of α in $Aut(\tilde{G})$ consists only of switching involutions. If α is strongly switching, the conjugacy class of α in $Aut(\tilde{G})$ consists only of strongly switching involutions.

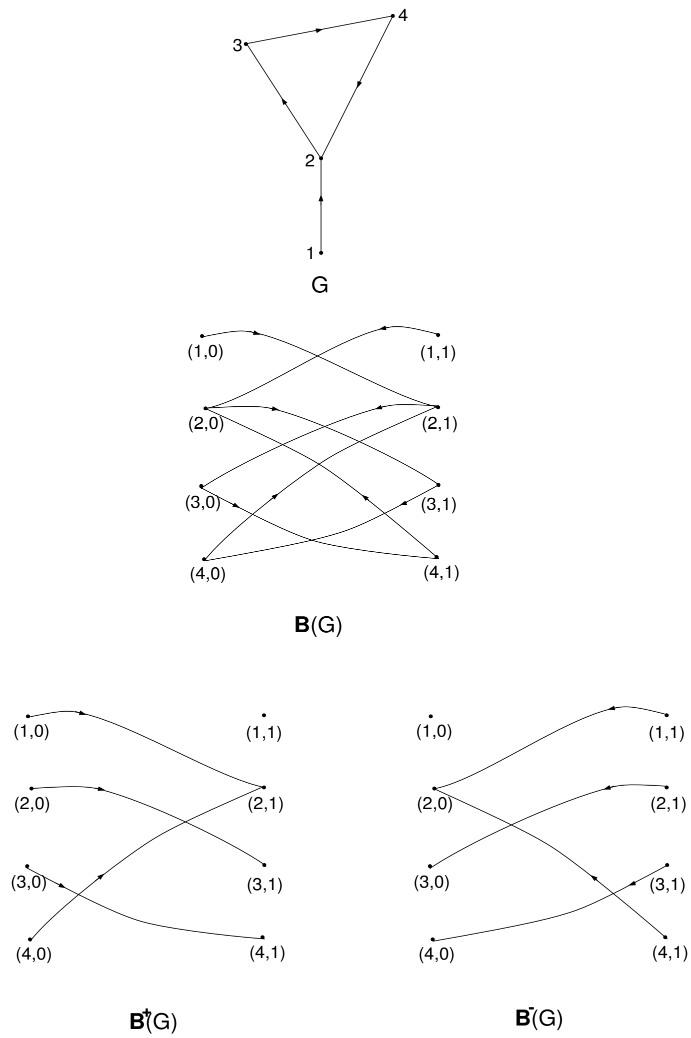

Figure 3: *Digraphs* G, $\mathbf{B}(G)$, $\mathbf{B}^{+}(G)$ *and* $\mathbf{B}^{-}(G)$.

The following result was proved by Porcu. Here, we give a more detailed proof.

Proposition 3.2 ***[Porcu]*** *[10] Let $\tilde{G}$ be a bipartite graph. Then there is a digraph G such that $\tilde{G} \cong \mathbf{B}(G)$ if and only if $\tilde{G}$ has a switching involution α. Loops of G correspond to vertices u such that $u \longrightarrow_{\tilde{G}} \alpha(u)$. In particular, G is loopless if and only if α is strongly switching.*

Proof : If α is a switching involution of $\tilde{G}$ and V_1 and V_2 the classes of $\tilde{G}$, define G by $V(G) = V_0$ and $u \longrightarrow_G v \iff u \longrightarrow_{\tilde{G}} \alpha(v)$. Hence $u \longrightarrow_{\tilde{G}} \alpha(v) \iff u \longrightarrow_G v \iff (u, \epsilon) \longrightarrow_{\mathbf{B}(G)} (v, \epsilon + 1)$. Since all the arcs of $\tilde{G}$ are of the type $u \longrightarrow \alpha(v)$ with $u \in V_0(\tilde{G}) = V(G)$ and $\alpha(v) \in V_1(\tilde{G})$ (because α exchanges the stable parts of $\tilde{G}$), then it follows that $\tilde{G} \cong \mathbf{B}(G)$.

Conversely, suppose that there exists G such that $\mathbf{B}(G) \cong \tilde{G}$. Let us define α by $u \longrightarrow_{\tilde{G}} \alpha(v) \iff u \longrightarrow_G v$. We claim that α is a switching involution. In fact, suppose $u \in V_0(\tilde{G})$. Then $u \longrightarrow_{\tilde{G}} \alpha(v) \Rightarrow \alpha(v) \in V_1$ since $\tilde{G}$ is *bipartite*. Since $\alpha(v) \in V_1$ and $u \in V_0$, then $\alpha(v) \longrightarrow_{\tilde{G}} u \iff \alpha(v) \longrightarrow_{\tilde{G}} \alpha(u) \Rightarrow \alpha(u) \in V_1$ since $\tilde{G}$ is *bipartite*. Hence α exchanges the classes of $\tilde{G}$ and is hence, a switching involution. Furthermore, consider any loop $u \longrightarrow_G u$ in G. Therefore $u \longrightarrow_G u \iff u \longrightarrow_{\tilde{G}} \alpha(u)$. The fact that G is loopless if and only if α ia strongly switching follows easily. □

The digraph G obtained from α and $\tilde{G}$ as in Proposition 3.2 will be denoted by $\tilde{G}_\alpha$. If $\tilde{G}$ has two switching involutions α and β, then we can let G and $\grave{G}$ be two digraphs whose vertex sets are identified with a class V_0 of $\tilde{G}$ and let $G = \tilde{G}_\alpha$ and $\grave{G} = \tilde{G}_\beta$ as constructed above. We will shorten $u \longrightarrow_{\tilde{G}} v$ and $u \longrightarrow_{\tilde{G}_\alpha} v$ into $u \longrightarrow v$ and $u \longrightarrow_\alpha v$ respectively. Similar abbreviations will be used for $\sim$.

Clearly, if $Aut(\tilde{G})$ has several switching involutions, then $\tilde{G} = \mathbf{B}(G)$ for several graphs G. These graphs *may not be* isomorphic. The following result shows the way in which the number of non-isomorphic graphs and the number of conjugacy classes of the switching involutions are related.

Theorem 3.3 ***[Porcu]*** *[10] Let $\tilde{G}$ be a bipartite graph. The number of non-isomorphic digraphs G such that $\tilde{G} \cong \mathbf{B}(G)$ is equal to the number of conjugacy classes of switching involutions in $Aut(\tilde{G})$. The number of non-isomorphic loopless graphs G such that $\tilde{G} \cong \mathbf{B}(G)$ is equal to the number of conjugacy classes of strongly switching involutions in $Aut(\tilde{G})$*

We shall also be needing the next results.

Theorem 3.4 ***[Pacco and Scapellato]*** *[9] Let $G \cong \tilde{G}_\alpha$ and $\grave{G} \cong \tilde{G}_\beta$ be two non-isomorphic digraphs with $\mathbf{B}(G) \cong \mathbf{B}(\grave{G}) \cong \tilde{G}$. Then there exists $\phi \in Aut(\tilde{G}_\alpha) \cap Aut(\tilde{G}_\beta)$ such that $\phi^2 = 1 \neq \phi$ and $\phi = \chi^m|V_0$ for some m where $\chi = \alpha\beta$.*

Furthermore, letting $j = \lfloor \frac{m}{2} \rfloor$, if $u \longrightarrow_\alpha \phi(u)$ then there is a loop in $\chi^j(u)$ in the digraph $\tilde{G}_\alpha$ or $\tilde{G}_\beta$ accordingly to whether m is even or odd respectively.

Corollary 3.5 *In the hypothesis of Theorem 3.4, if both G and $\grave{G}$ are loopless, then there is no vertex u such that $u \longrightarrow_G \phi(u)$ or $u \longrightarrow_{\grave{G}} \phi(u)$.*

Allowing loops, the following proposition would hold.

Proposition 3.6 ***[Pacco and Scapellato]*** *[9] Let G be a digraph with loops such that there exists an involutory automorphism $\phi \in Aut(G)$ with $\phi \neq 1$ and $u \not\twoheadrightarrow_G \phi(u)$ for all $u \in V(G)$. Then there exists a digraph $\grave{G}$ such that $G \ncong \grave{G}$ and $\mathbf{B}(G) \cong \mathbf{B}(\grave{G})$.*

3.2 Automorphisms of a CDC

In this subsection, we shall define some automorphisms of canonical double coverings to which we will be referring in the next subsection. We will also show that there is a link between the canonical double coverings of graphs and the possible TF-isomorphisms between them.

(i) The automorphisms ϕ^+ and ϕ^- of $\mathbf{B}(G)$, induced by any automorphism ϕ of G:

Let ϕ be any automorphism of G. Define ϕ^+ and ϕ^- as follows:

$$\phi^+(u, \varepsilon) = (\phi(u), \varepsilon)$$

$$\phi^-(u, \varepsilon) = (\phi(u), \varepsilon + 1)$$

$\forall u \in V(G)$ and $\varepsilon \in \mathbb{Z}_2$.

Note: ϕ^+ fixes the components and ϕ^- interchanges them. It is easy to check that ϕ^+ and ϕ^- are automorphisms of $\mathbf{B}(G)$.

(ii) The automorphism ξ of $\mathbf{B}(G)$

The automorphism ξ of $\mathbf{B}(G)$ is defined as follows:

$$\xi : (u, \varepsilon) \mapsto (u, 1 + \varepsilon)$$

Note : ξ is an automorphism of $\mathbf{B}(G)$ and an isomorphism between $\mathbf{B}^+(G)$ and $\mathbf{B}^-(G)$.

3.3 TF-isomorphisms and CDC's

Theorem 3.7 *Two digraphs G_1 and G_2 such that $\mathbf{B}(G_1) \cong \mathbf{B}(G_2)$ are TF-isomorphic. The converse holds in the case of graphs.*

Proof: Let G_1 and G_2 be two digraphs such that $\mathbf{B}(G_1) \cong \tilde{G} \cong \mathbf{B}(G_2)$. By definition, there exist maps $\alpha_i, \beta_i : V(G_i) \longrightarrow V(\tilde{G})$ $(i = 1, 2)$ such that $(u, v) \in A(G_i) \iff \alpha_i(u), \beta_i(v) \in A(\tilde{G})$. Letting $\alpha = \alpha_1\alpha_2^{-1}$ and $\beta = \beta_1\beta_2^{-1}$, we have $\alpha, \beta : V(G_1) \longrightarrow V(G_2)$. Moreover, $u \longrightarrow_{G_1} v \iff \alpha_1(u) \longrightarrow_{\tilde{G}} \beta_1(v) \iff \alpha_2^{-1}\alpha_1(u) \longrightarrow_{G_2} \beta_2^{-1}\beta_1(v)$, that is, $u \longrightarrow_{G_1} v \iff \alpha(u) \longrightarrow_{G_2} \beta(v)$. Hence (α, β) is a TF-isomorphism from G_1 to G_2. Therefore, the digraphs $G_i\,(i = 1, 2)$ are TF-isomorphic.

Conversely, assume that $(\alpha, \beta) : (G_1) \longrightarrow (G_2)$ is a TF-isomorphism, with $G_i\,(i = 1, 2)$ being graphs and define $\phi : V(\mathbf{B}(G_1)) \longrightarrow V(\mathbf{B}(G_2))$ by:

$$\phi(w, \varepsilon) = \begin{cases} (\alpha(w), 0) & \varepsilon = 0 \\ (\beta(w), 1) & \varepsilon = 1 \end{cases}$$

Then, for each arc $(u, 0) \longrightarrow (v, 1)$ of $\mathbf{B}(G_1)$ we have $\phi(u, 0) = (\alpha(u), 0)$ and $\phi(v, 1) = (\beta(v), 1)$. Therefore, $((\alpha(u), 0), (\beta(v), 1))$ is an arc of $\mathbf{B}(G_2)$. Clearly, $\phi : \mathbf{B}^+(G_1) \longrightarrow \mathbf{B}^+(G_2)$ is an isomorphism. The underlying graphs $\mathbf{B}(G_1)$ and $\mathbf{B}(G_2)$ are isomorphic, that is, $\mathbf{B}(G_1) \cong \mathbf{B}(G_2)$. □

Note that the above theorem implies that, for graphs, a TF-isomorphism can always induce a ϕ^+ and a ϕ^-.

Example 1:
The graphs G_1 and G_2, in Figure 4 are not TF-isomorphic and, in fact, $\mathbf{B}(G_1) \ncong \mathbf{B}(G_2)$ confirming Theorem 3.7.

Note that the map $\phi : \mathbf{B}(G_1) \rightarrow \mathbf{B}(G_2)$ preserves only the arcs going from $V(G_1) \times \{0\}$ to $V(G_2) \times \{1\}$. Hence it cannot help extend theorem 3.7 to digraphs. In fact, that result does not hold in the more general situation, as shown in Example 2.

Example 2:
The digraphs G_1 and G_2 in Figure 5 have *non-isomorphic canonical double coverings*, that is, $\mathbf{B}(G_1) \ncong \mathbf{B}(G_2)$, but still they are TF-*isomorphic* since there exists a pair (α, β) which is a TF-isomorphism mapping G_1 to G_2.

4 TOD's and CDC's

The main results of this paper are in this section. We first introduce *two-fold orbital digraphs* (TOD's). Then we proceed to show that there is a close re-

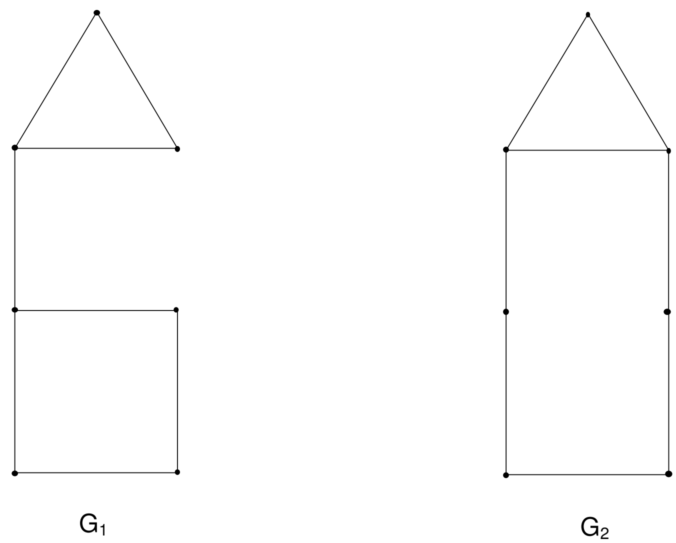

Figure 4: *G_1 and G_2 are not TF-isomorphic. In fact,* $\mathbf{B}(G_1) \ncong \mathbf{B}(G_2)$.

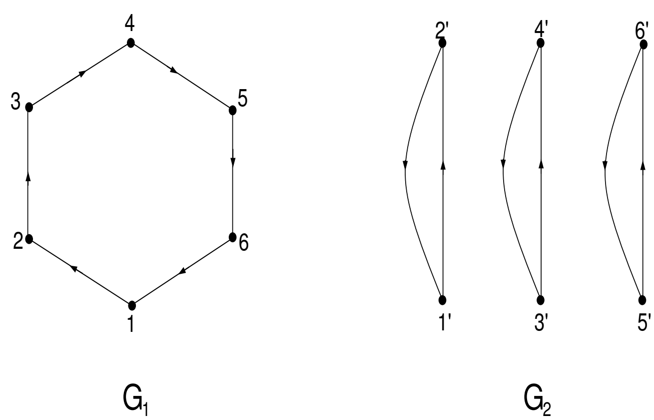

Figure 5: $\mathbf{B}(G_1) \ncong \mathbf{B}(G_2)$. *However, there exists a TF*-isomorphism (α, β) *from G_1 to G_2 defined by $\alpha = id$ and* $\beta = (1\ 5\ 3)(2)(4)(6)$

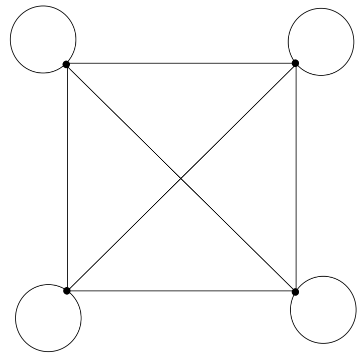

Figure 6: *This graph, has arc set* $\mathbf{\Gamma}(1,2)$ *where* $\mathbf{\Gamma} = D_4 \times S_4 \leq S_4 \times S_4$.

lationship between two-fold orbital digraphs/graphs (TOD's) and canonical double coverings (CDC's) of digraphs/graphs.

Let $\mathbf{\Gamma} \leq \mathcal{S}$ where $\pi_1\mathbf{\Gamma}$ and $\pi_2\mathbf{\Gamma}$ are transitive on the n-set V. For a fixed element (u, v) in $V \times V$ let

$$\mathbf{\Gamma}(u, v) = \{(\alpha(u), \beta(v)) \,|\, (\alpha, \beta) \in \mathbf{\Gamma}\}$$

The set $\mathbf{\Gamma}(u, v)$ is called a *two-fold orbital*.

A two-fold orbital $\mathbf{\Gamma}(u, v)$ is therefore the set of arcs of a digraph having V as vertex set. The digraph $G = (V, \mathbf{\Gamma}(u, v))$ is said to be a *two-fold orbital digraph* (TOD) or a $\mathbf{\Gamma}$-*orbital digraph*.

If G is self-paired then G is a *two-fold orbital graph* (TOG) or a $\mathbf{\Gamma}$-*orbital graph*.

The pair (u, v) is said to be *representative* of G.

It is apparent that orbitals are special cases of two-fold orbitals and that two-fold orbital digraphs correspond to the case where $\mathbf{\Gamma}$ is *diagonal*, that is, $\mathbf{\Gamma} = \{(\alpha, \alpha) \,|\, \alpha \in \Gamma\}$.

Note that even for a self-paired group $\mathbf{\Gamma}$, the graph obtained from $\mathbf{\Gamma}(u, v)$ may not be self-paired. This is known to happen even if $\mathbf{\Gamma}$ is diagonal, that is, in the case of orbital digraphs. On the other hand, $\mathbf{\Gamma}(u, v)$ may be self-paired even when $\mathbf{\Gamma}$ is not self-paired. For example, let $\mathbf{\Gamma} = D_4 \times S_4 \leq S_4 \times S_4$. Although $\mathbf{\Gamma}$ is not self-paired, $A(G) = \mathbf{\Gamma}(1, 2)$ is (cf. Figure 6).

4.1 Disconnected TOG's.

It is known that all connected components of a disconnected orbital digraph are pairwise isomorphic. The properties of a disconnected TOD are not so well-defined. In this section we shall study disconnected $TOD's$, particularly in the case when the components are graphs.

Lemma 4.1 *Let G_i, G_j be components of a disconnected graph G. Let $(\alpha, \beta) \in Aut^{TF}G$ be such that the arc $u \longrightarrow_{G_i} v$ in G_i is mapped to $\alpha(u) \longrightarrow_{G_j} \beta(v)$ in G_j. Then, for any x, neighbour of u in G_i (i.e. $x \sim_{G_i} u$), $\beta(x) \in V(G_j)$.*

Proof: Let $x \sim_{G_i} u$ that is, there exist arcs $u \longrightarrow_{G_i} x$ and $x \longrightarrow_{G_i} u$. Since (α, β) maps $u \longrightarrow_{G_i} v$ to $\alpha(u) \longrightarrow_{G_j} \beta(v)$, we get $\alpha(u) \in V(G_j)$. But $(\alpha, \beta) \in Aut^{TF}(G)$ implies that (α, β) maps any arc of the graph G into some other arc of G. So, let us consider the arc $u \longrightarrow_{G_i} x$, the existence of which is guaranteed by the fact that G_i is connected and each arc is self-paired since G_i is a component of an undirected graph. Therefore, (α, β) maps the arc $u \longrightarrow_{G_i} x$ to some arc $\alpha(u) \longrightarrow_{G} \beta(x)$, whose existence is guaranteed by the fact that (α, β) is a TF-isomorphism. Let $\beta(x) \in V(G_k)$, for some component G_k. But $\alpha(u) \in V(G_j)$ implies that the arc $\alpha(u) \longrightarrow_G \beta(x)$ joins a vertex in G_j to a vertex in G_k; hence $k = j$. That is, $\beta(x) \in V(G_j)$. □

Referring to the proof of the above lemma we should remark that it holds true only for undirected graphs. This is because the arc $x \longrightarrow_{G_i} u$ is mapped to some arc $\alpha(x) \longrightarrow_G \beta(u)$ which might be in a component which is not G_j; as regards the arc opposing $\alpha(u) \longrightarrow_{G_j} \beta(x)$, there must exist some arc $y \longrightarrow_G v$ such that $\alpha(y) = \beta(x)$ and $\beta(v) = \alpha(u)$, but it is possible for $y \longrightarrow_G v$ to be different from the arc $x \longrightarrow_{G_i} u$ (*i.e.* $y \neq x$ and $v \neq u$).

Proposition 4.2 *Let G be a graph. Let G_i and G_j, where $|V(G_i)| \geq |V(G_j)|$, be connected components of G such that there exists $(\alpha, \beta) \in Aut^{TF}G$ mapping an arc of G_i to an arc of G_j. Then,*

(a) *Either $G_i \cong^{TF} G_j$ or else $G_i = \mathbf{B}(G_j)$;*

(b) *Or, if G_i and G_j are bipartite, with $V_0(G_k)$ and $V_1(G_k)$ $k = (i, j)$ denoting the stable parts, then $G_i \cong G_j$.*
In particular, $\phi_{\alpha,\beta} : G_i \longrightarrow G_j$ defined by;

$$\phi_{\alpha,\beta}(x) = \begin{cases} \alpha x, & x \in V_0(G_i) \\ \beta x, & x \in V_1(G_i) \end{cases} \quad \text{or} \quad \phi_{\alpha,\beta}(x) = \begin{cases} \alpha x, & x \in V_1(G_i) \\ \beta x, & x \in V_0(G_i) \end{cases}$$

(depending on whether for the arc $u \longrightarrow_{G_i} v$, which is known to be mapped to an arc of G_j, we have $u \in V_0(G_i)$, $v \in V_1(G_i)$ or $u \in V_1(G_i)$, $v \in V_0(G_i)$), is an isomorphism.

Proof: (a) Since G_i and G_j are connected components of an undirected graph G and (α, β) takes an arc of G_i to an arc of G_j, we may define the sets A, $B \subseteq V(G)$ as follows:

$$A = \{\, u \in V(G_i) \mid \alpha(u) \in V(G_j) \,\}$$
$$B = \{\, u \in V(G_i) \mid \beta(u) \in V(G_j) \,\}.$$

If $u \in A$, from $u \longrightarrow_{G_i} v$, it follows that $\alpha(u) \longrightarrow_{G_j} \beta(v)$ since (α, β) is a TF-*isomorphism*. Hence $\beta(v) \in V(G_j)$, that is $v \in B$. Conversely, if $v \in B$ and $u \longrightarrow_{G_i} v$ then $\alpha(u) \longrightarrow_{G_j} \beta(v)$, and $u \in A$. Therefore, in view of the preceding lemma, we may remark that

$$u \in A \iff N_{G_i}(u) \subseteq B \tag{1}$$

$$u \in B \iff N_{G_i}(u) \subseteq A. \tag{2}$$

Also, since G_i and G_j are connected, $A \cup B = V(G_i)$.

We distinguish between two cases:

Case 1 $A \cap B \neq \emptyset$: By the above, if $u \in A \cap B \neq \emptyset$, we have $A = B$. In fact, from (1) and (2), it follows that :

$$u \in A \cap B \Rightarrow N_{G_i}(u) \subseteq A \cap B.$$

Now suppose $w \in N_{G_i}(u)$, then $w \in A \cap B$ and therefore, $N_{G_i}(w) \subseteq A \cap B$. Proceeding like that, we get that all $u \in G_i \in A \cap B$ since G_i is connected. Hence $A = B$. Also, $\forall u \in V(G_i), u \in A \cap B$ so that, from the definition of A and B, it follows that $\alpha(u)$ and $\beta(u)$ are in $V(G_j)$ i.e. α and β take the vertices of G_i to vertices of G_j.

Thus, if $A = B$, we have $|V(G_i)| \leq |V(G_j)|$. Hence, $|A| = |V(G_i)| = |V(G_j)|$. Consequently, $G_i \cong^{TF} G_j$.

Case 2 $A \cap B = \emptyset$: This implies, first of all, that G_i is bipartite, with stable parts A and B.
Next, we have that $\alpha A \cup \beta B = V(G_j)$ for suppose that this is not true, then, there exists a vertex $w \in V(G_j) \backslash (\alpha A \cup \beta B)$ which is adjacent either to a vertex from αA or to a vertex from βB since G_j is connected. Suppose that $w \sim_{G_j} \alpha(u)$ for some $u \in A \subseteq V(G_i)$. Now, $|V(G_i)| \geq |V(G_j)|$ implies that the condition of Lemma 4.1 is satisfied and therefore, for each neighbour x of u, $\beta(x) \in V(G_j)$.

Now, (α, β) is a TF-automorphism over the graph G. This implies that for all u such that $w \sim \alpha(u)$, there exists v such that $\beta(v) = w$ and $u \sim v$. Therefore v must be a neighbour of u. Therefore $v \in V(G_i)$ and, by the previous argument, it is guaranteed that $\beta(v) \in V(G_j)$.

Therefore $\beta^{-1}(w) = \beta^{-1}\beta(v) = v \in N(u) \subseteq V(G_i)$. By definition of (α, β), $\alpha(u) \sim \beta(v)$ if and only if $u \sim v$, that is, $u \longrightarrow v$ exists if $\alpha(u) \longrightarrow \beta(v)$ exists.

Now, we can distinguish two subcases :

Subcase 2.1 Let $\alpha A \cap \beta B \neq \emptyset$. Then we have $\alpha A = \beta B = V(G_j)$. Indeed, suppose that this is not the case. For connectivity reasons, there is a vertex $w \in \alpha A \cap \beta B$ which is adjacent either to a vertex from $\alpha A \setminus \beta B$ or to a vertex from $\beta B \setminus \alpha A$. Let $w \sim_{G_j} y$, where, say, $y \in \alpha A \setminus \beta B$. Then the arc $w \longrightarrow y$ is mapped by $(\alpha^{-1}, \beta^{-1})$ to an arc of G_i. Since G_i is bipartite and $\alpha^{-1}(w) \in A$, we have $\beta^{-1}(y) \in B$, that is $y \in \beta B$, which is a contradiction. Hence $\alpha A = \beta B = V(G_j)$.
We claim that $G_i = \mathbf{B}(G_j)$ i.e. $(u, \varepsilon) \longrightarrow_{G_i} (v, \varepsilon + 1) \iff u \longrightarrow_{G_j} v$. $(\varepsilon \in \mathbb{Z}_2)$.

Consider the arc $u \longrightarrow_{G_i} v$. We can always re-label the start-vertex u as (u, ε) and the end-vertex v as $(v, \varepsilon + 1)$. Without loss of generality, let $u \in A$. Then $v \in B$ since $v \in N_{G_i}(u)$. By definition of A and B, $\alpha(u) \in V(G_j)$ and $\beta(v) \in V(G_j)$. But $(\alpha, \beta) \in Aut^{TF}G$. Hence the pair (α, β) maps arc $u \longrightarrow_G v$ into arc $\alpha(v) \longrightarrow_G \beta(v)$. Since $\alpha(u)$ and $\beta(v)$ are both in $V(G_j)$, then $\alpha(v) \longrightarrow_G \beta(v)$ is an arc in component G_j.

Conversely, suppose $\alpha(u) \longrightarrow_{G_j} \beta(v)$. Since $\alpha(A) = \beta(B) = V(G_j)$, then $\alpha^{-1} : V(G_j) \to A$ and $\beta^{-1} : V(G_j) \to B$. Hence $\alpha^{-1}(\alpha(u)) = u \in A$ and $\beta^{-1}(\beta(v)) = v \in B$.
We claim that the arc $u \longrightarrow_{G_i} v$ exists. Again, this follows from the fact that (α, β) is a TF-*isomorphism* in G which implies that the inverse of (α, β) i.e. $(\alpha^{-1}, \beta^{-1})$ maps the arc $\alpha(u) \longrightarrow_{G_j} \beta(v)$ to the arc $u \longrightarrow_{G_j} v$. We can then re-label any $u \in A$ as (u, ε) and any $v \in B$ as $(v, \varepsilon + 1)$.

Subcase 2.2 Let $\alpha A \cap \beta B = \emptyset$. This implies that G_i and G_j are both bipartite. Define the mapping $\phi : V(G_i) \to V(G_j)$ by the rule $\phi(u) = \alpha(u)$ if $u \in A$ and $\phi(u) = \beta(u)$ if $u \in B$. Then ϕ is a *bijection* and moreover, an *isomorphism*.
Since both components G_i and G_j are *bipartite* and *connected*, $\alpha V_0(G_i) \cap \beta V_1(G_i) = \emptyset$ or $\alpha V_1(G_i) \cap \beta V_0(G_i) = \emptyset$ (depending on the direction of the arc in G_i which is known to be mapped to an arc in G_j). This guarantees that the corresponding $\phi_{\alpha,\beta}$ is a *bijection* and moreover, an *isomorphism*. This gives (**b**).

□

Example 3:

Let X is a digraph. We denote by X^{op} the digraph having the same vertex set as X, that is $V(X^{op}) = V(X)$ and having the arc set defined by ; $u \longrightarrow_{X_{op}} v$ exists $\iff$ $v \longrightarrow_X u$ exists.

Figure 7 illustrates a TF-isomorphism (α, β) on a graph consisting of three vertex-disjoint 4-cycles G_1,G_2 and G_3 on vertices $\{1,2,3,4\} \cup \{5,6,7,8\} \cup \{9,10,11,12\}$, rspectively. $\alpha = (1\ 5\ 9)(2\ 10\ 6)(3\ 7\ 11)\ (4\ 12\ 8)$ and $\beta = (5\ 1\ 9)(10\ 2\ 6)(7\ 3\ 11)\ (12\ 4\ 8)$ *i.e.* $\beta = \alpha^{-1}$. Then (α, β) maps $X \mapsto Y \mapsto Z \mapsto X$ and maps $X^{op} \mapsto Z^{op} \mapsto Y^{op} \mapsto X^{op}$. Note that each of X,Y,Z,X^{op},Y^{op} and Z^{op} is bipartite. This corresponds to Case 2 of Proposition 4.2. If we consider X, for instance, we can define the bipartition sets A and B as in the proof of Proposition 4.2. Therefore, $A = \{1,3\}$ and $B = \{2,4\}$. Note that $\alpha(1) = 5, \alpha(3) = 7$ and that $\beta(2) = 6, \beta(4) = 8$. Therefore, $\alpha(A) = \{5,7\}$ and $\beta(B) = \{6,8\}$ so that $\alpha(A) \cap \beta(B) = \emptyset$. We can repeat the same procedure by considering X^{op} and Z^{op} and then repeat the argument for Y,Z and for Z^{op}, Y^{op} and so on. In each case we may deduce that $\alpha(A) \cap \beta(B) = \emptyset$. In fact, this example corresponds to Subcase 2.2 of the proof of Proposition 4.2 where $A \cap B = \emptyset$ and $\alpha(A) \cap \beta(B) = \emptyset$. Clearly, $G_1 \cong G_2 \cong G_3$ confirming statement **(b)** of Proposition 4.2.

Example 4:

Figure 8 shows a graph G,consisting of two vertex-disjoint 4-cycles G_1 and G_2 with $V(G_1) = \{1,2,3,4\}$ and $V(G_2) = \{5,6,7,8\}$. We recall the definitions of A and B:

$$A = \{\, u \in V(G_1) \mid \alpha(u) \in V(G_2)\,\}$$
$$B = \{\, u \in V(G_1) \mid \beta(u) \in V(G_2)\,\}$$

If $\alpha = (1\ 3)(5\ 7)(2\ 6)(4\ 8)$ and $\beta = (2\ 4)(6\ 8)(1\ 5)(3\ 7)$, then $A = \{2,4\}$ (since $\alpha(2) = 6 \in V(G_2)$ and $\alpha(4) = 8 \in V(G_2)$) and $B = \{1,3\}$ (since $\beta(1) = 5 \in V(G_2)$ and $\beta(3) = 7 \in V(G_2)$). Therefore, $A \cap B = \emptyset$. Also $\alpha(A) \cap \beta(B) = \emptyset$. Therefore the conditions of Case 2, Subcase 2.2 of Proposition 4.2 are satisfied. In fact, we may note again that G_1 and G_2 are bipartite and moreover $\phi_{\alpha,\beta}$ as defined in part **(b)** of the statement of Proposition 4.2 is an isomorphism.

Example 5:

Figure 9 illustrates a still more trivial example. G is a graph on vertices $\{1,2,3,4\}$ with two components G_1 and G_2 consisting of the edges $\{1,2\}$ and

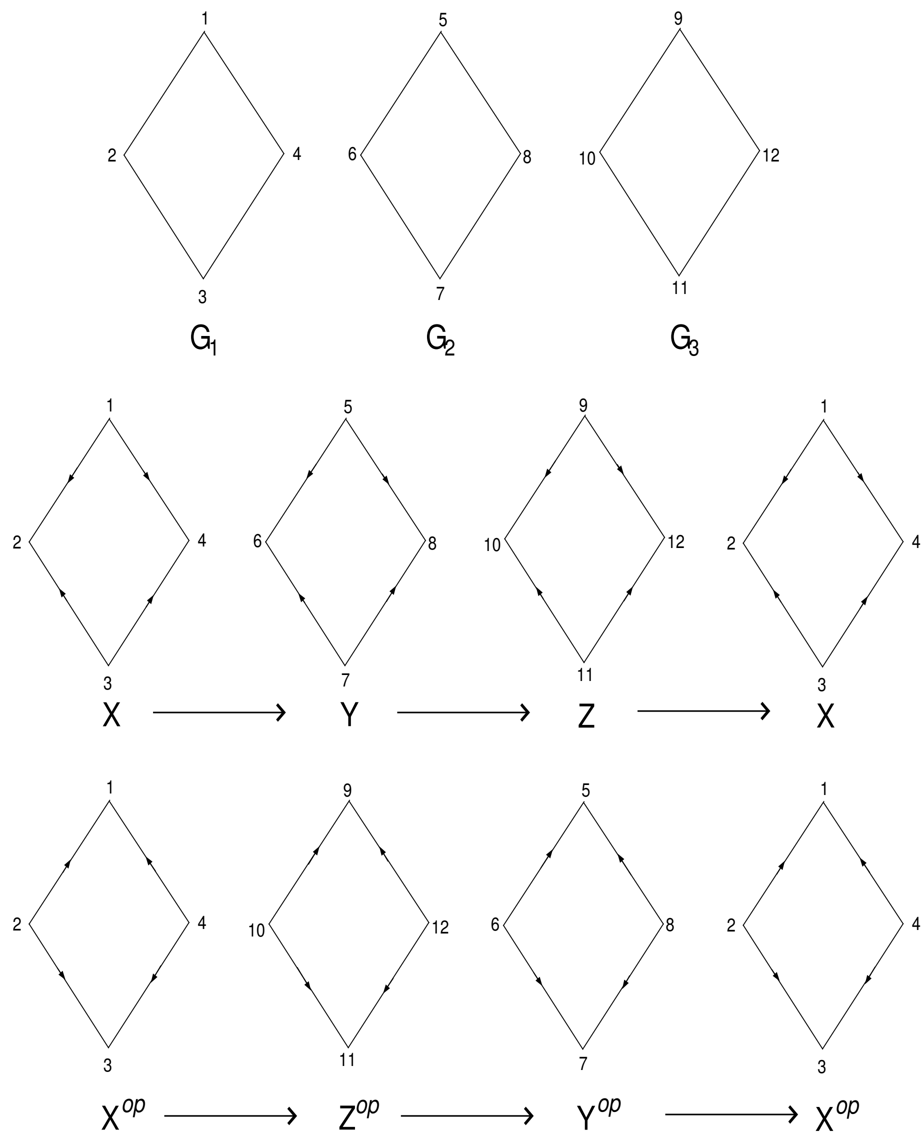

Figure 7: *The case of Example 3.*

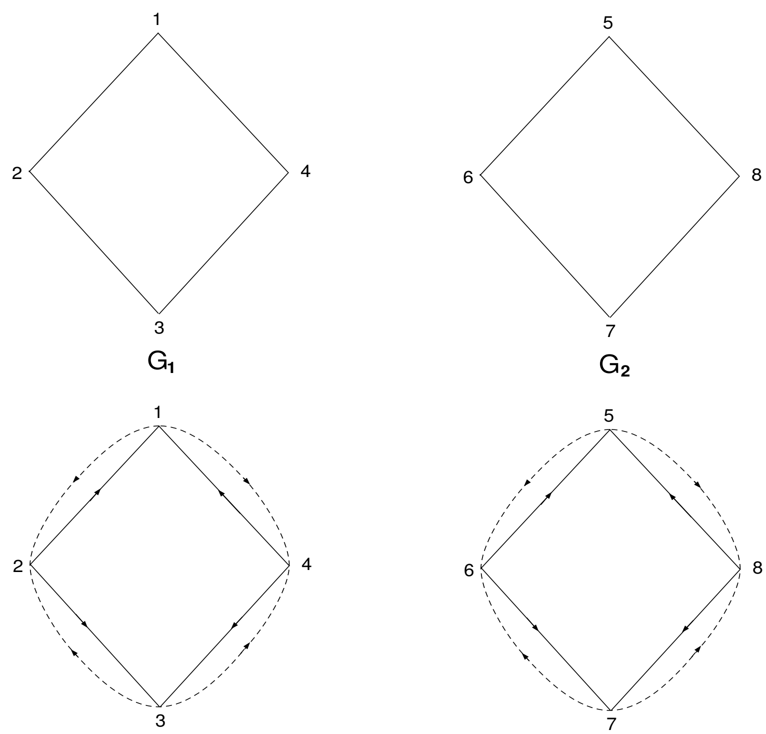

Figure 8: *The case of Example 4*

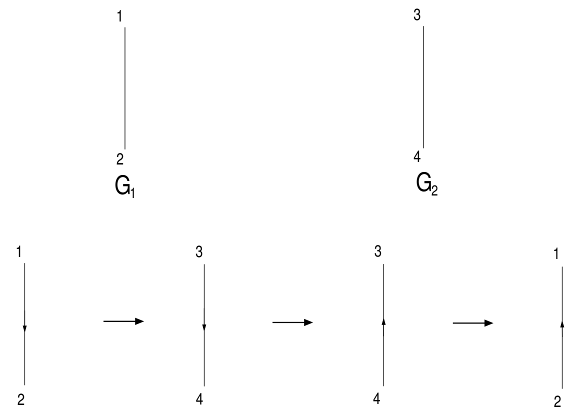

Figure 9: *The case of Example 5.*

$\{3,4\}$ respectively; $\alpha = (1342)$ and $\beta = \alpha^{-1} = (2431)$. One may note that G is a *two-fold orbital* graph with respect to the group $\langle(\alpha, \alpha^{-1})\rangle$. If we maintain the definitions of A and B as in Proposition 4.2, $A = \{1\}$ and $B = \{2\}$. Also $\alpha(A) \cap \alpha(B) = \emptyset$ (since $\alpha(1) = 3$ and $\beta(2) = 4$). The components G_1 and G_2 are in fact, both bipartite and moreover, $\phi_{\alpha,\beta}$ as defined in part **(b)** of the statement of Proposition 4.2 is an isomorphism. Therefore, in this example, (α, β) is a TF-automorphism of G. Its components G_1 and G_2 are isomorphic but (α, β) is not a TF-isomorphism between G_1 and G_2.

Example 6:
Figure 10 shows a disconnected TOG with components G_1, G_2 and G_3 with $|V(G_3)| \leq |V(G_2)| \leq |V(G_1)|$. G is generated by the pair $(\alpha, \beta) \in Aut^{TF}(G)$. The pair (α, β) maps at least one arc of G_3 to an arc of G_2 and at least one arc of G_2 to an arc of G_1. Let us consider the components G_1 and G_2 and maintain the definitions of A and B as proposed in the proof of Proposition 4.2.1. Then $A = \{1,3,5,7,9\}$ and $B = \{2,4,6,8,10\}$. Using the definitions of α and β, we note that $\alpha(A) = \{11,13,15,12,14\}$ and $\beta(B) = \{12,14,11,13,15\}$. Note that $\alpha(A) \cap \beta(B) \neq \emptyset$ and in fact, as in Subcase 2.1 in the proof of Proposition 4.2.1, $\alpha(A) = \beta(B)$. The component G_1 is in fact, the canonical double covering of the component G_2, that is $G_1 = \mathbf{B}(G_2)$ as proposed in the proof (Subcase 2.1) of Proposition 4.2.2. If we then consider the components G_2 and G_3, and again apply the definitions of A and B, it is easy to check that $A = B$ and that $|A| = |B| = |V(G_2)|$. The components G_2 and G_3 are in fact, TF-isomorphic as well as isomorphic. Furthermore, we remark that G_1, G_2 and G_3 taken separately are two fold orbital graphs. Such an observation provides motivation to check whether or

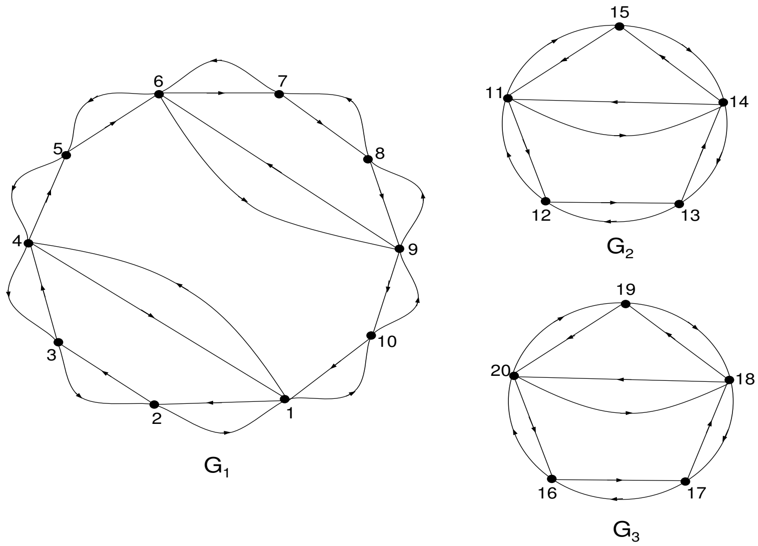

Figure 10: *Graph G has components G_1, G_2 and G_3. G is a TOG where* $\alpha = (1\ 11\ 20\ 6\ 9\ 14\ 18\ 4)(2\ 3\ 13\ 17\ 8\ 7\ 12\ 16)(5\ 15\ 19\ 10)$ *and* $\beta = (1\ 4\ 14\ 18\ 9\ 6\ 11\ 20)(2\ 12\ 16\ 7\ 8\ 13\ 17\ 3)(5\ 10\ 15\ 19)$

not this is always the case and Theorem 4.2.1 will address this question.

Example 7:
In the example illustrated in Figure 11, we consider disconnected digraphs. G_i and G_j ($|V(G_i)| \geq |V(G_j)|$) are the connected components of a *digraph* G. It can be seen that α and β are both fixed-point-free. G is a TOD on $\mathbf{\Gamma}$, $\mathbf{\Gamma} = \langle(\alpha, \beta)\rangle$. Furthermore, we can check that (α, β), for instance, maps arc $1 \longrightarrow_{G_i} 2$ to arc $4 \longrightarrow_{G_j} 5$. However, $G_i \ncong^{TF} G_j$ and $G_i \ncong \mathbf{B}(G_j)$. Therefore, the statement of Proposition 4.2 does not hold in the case of digraphs.

Theorem 4.3 *Let G be a TOG with no isolated vertices and let its connected components be $G_1, ..., G_k$ and:*

$$|V(G_1)| \geq |V(G_2)| \geq, ... \geq |V(G_k)|.$$

Then each $G_i (i = 1, ..., k)$ is still a TOG. Moreover :

(i) *if $|V(G_1) = |V(G_k)|$, then $G_1, G_2, ..., G_k$ are pairwise TF*-isomorphic *:*

(ii) *otherwise, there exists a unique index $r \in \{1, ...k-1\}$ such that $G_1 \cong G_2 \cong ... \cong G_r \ncong^{TF} G_{r+1} \cong^{TF} ... \cong^{TF} G_k$ and $G_1 \cong \mathbf{B}(G_k)$*

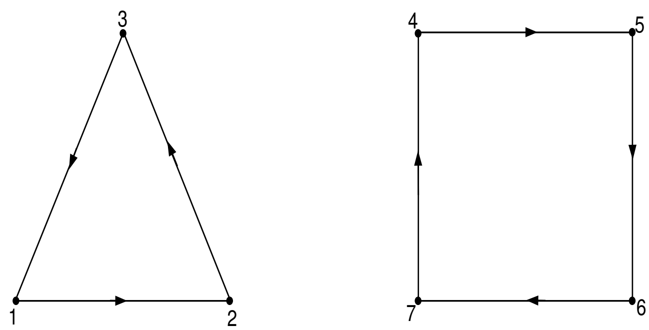

Figure 11: α = *(1456723) and* β = *(1256743). G is a* Γ-TOD , *where* $\Gamma = \langle(\alpha,\beta)\rangle$.

Proof : Consider an arbitrary component G_i. Then, we may distinguish two cases.

Case 1 If G_i is not *bipartite* . In this case, we may assume that any $(\alpha,\beta) \in Aut^{TF}G$ mapping an arc of G_i to an arc of G_i $(i = 1, ..., k)$ restricts to a TF-*automorphism* of G_i.
Let $\mathbf{\Gamma}_i$ denote the group of all such TF-*automorphisms*. Then G_i is a $\mathbf{\Gamma_i} - TOG$. This follows from the proof of Proposition 4.2.

Case 2 If G_i is bipartite, then some TF-automorphisms may restrict to TF-automorphisms of G_i.
Let $\mathbf{\Gamma}^{(i)}$ be the set of all $(\alpha,\beta) \in \mathbf{\Gamma}$ mapping an arc of G_i to an arc of G_i and let $\Psi^{(i)}$ be the group generated by the set of induced standard automorphisms $\{\phi_{\alpha,\beta} \mid (\alpha,\beta) \in \Gamma^i \text{ of } G_i\}$ as in Proposition 4.2. Then, G_i is a Ψ-two-fold orbital graph where $\mathbf{\Psi} = \Psi^{(i)} \times \Psi^{(i)}$.

We have thus proved that each G_i $(i = 1, ..., k)$ is still a TOG.

To prove **(i)**:

This follows directly from the proof of Case *1* of the proposition.

To prove **(ii)** :

Assume that there exist r, s such that $1 \leq r < s \leq k-1$ and

$$|V(G_r)| \geq |V(G_{r+1}) = |V(G_s)| > |V(G_{s+1})|.$$

Now, we have that $G_r \cong \mathbf{B}(G_{r+1})$. $G_{r+1} \cong^{TF} G_s$ and $G_s \cong \mathbf{B}(G_{s+1})$.
We had proved (Proposition 3.7) that in the undirected case, if two graphs

G_1 and G_2 are TF-isomorphic, then $\mathbf{B}(G_1) \cong \mathbf{B}(G_2)$. Hence, $G_{r+1} \approx G_s \Rightarrow B(G_{r+1}) \cong B(G_s)$. Therefore $G_r \cong \mathbf{B}(G_s)$. It follows that $G_r \cong \mathbf{B}(\mathbf{B}(G_{s+1}))$. $\mathbf{B}(\mathbf{B}(G_{s+1}))$ is disconnected by virtue of Proposition 3.1. This is a contradiction since G_r is not disconnected. Hence, there exists a unique $r \in \{1, ..., k-1\}$ such that :

$$|V(G_1)| = ... = |V(G_r)| > |V(G_{r+1})| = ... = |V(G_k)|$$

which yields **(ii)**. □

The next result is a partial converse of Theorem 4.3.

Theorem 4.4 *Let $G_1, ..., G_k$ be graphs such that*

$$|V(G_1)| \geq |V(G_2)| \geq, ... \geq |V(G_k)|$$

and $r \in 1, ..., k-1$ such that

(i) *all components G_i $(r+1 \leq i \leq k)$ are $\mathbf{\Gamma}_i$- orbital graphs ;*

(ii) *for all p, q : $1 \leq p \leq q \leq r$, $G_p \cong G_q$;*

(iii) *for all p, q : $r+1 \leq p \leq q \leq k$, $G_p \cong^{TF} G_q$;*

(iv) *for all $i \in \{1, ..., r\}$, there exists $j \in \{r+1, ..., k\}$ such that $G_i \cong \mathbf{B}(G_j)$.*

Then

$$\bar{G} = \bigcup_{i=1}^{k} G_i \qquad \text{is a TOG.}$$

Proof: Denote by $\phi^*_{p,q}$, TF-isomorphisms $G_p \to G_q$, $\ r+1 \leq p \leq q \leq k$. Recall that $G_i \cong^{TF} G_j$ implies $\mathbf{B}(G_i) \cong \mathbf{B}(G_j)$.

Consider the isomorphisms $\phi_{i,j} : G_i \to G_j$ $(1 \leq i < j \leq r)$. Suppose $\phi_{i,j} = \alpha$ is an isomorphism between G_i and G_j, then the pair (α, α) is a TF-isomorphism between G_i and G_j $(1 \leq i < j \leq r)$.

There exists s, t, with $1 \leq s \leq r$ and $r+1 \leq t \leq k$ such that $G_s = \mathbf{B}(G_t)$. Define bijections $\bar{\alpha}, \bar{\beta} : V(G_t) \cup V(G_s) \to V(G_t) \cup V(G_s)$ as follows:

$$\begin{aligned} \bar{\alpha} : \quad & u \leftrightarrow (u,0) \\ & (u,1) \leftrightarrow (u,1) \\ \bar{\beta} : \quad & u \leftrightarrow (u,1) \\ & (u,0) \leftrightarrow (u,0). \end{aligned}$$

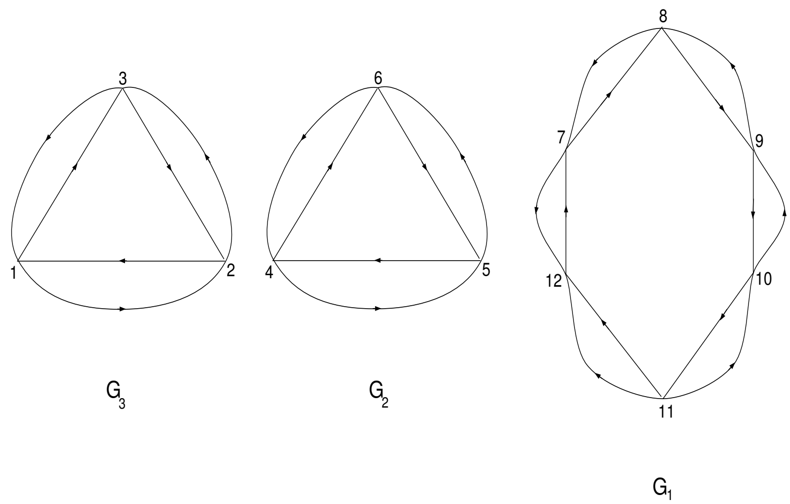

Figure 12: *Graph G has components G_1, G_2 and G_3.* $\alpha =$ (1 4 7 10)(2 5 8 11)(3 6 9 12) *and* $\beta =$ (1 4 10 7)(2 5 8 11)(3 6 9 12)

Then $(\bar{\alpha}, \bar{\beta})$ interchanges the arc $u \longrightarrow v$ with $(u, 0) \longrightarrow (v, 1)$ and fixes the arc $(u, 1) \longrightarrow (v, 0)$. Similarly $(\bar{\beta}, \bar{\alpha})$ interchanges the arc $u \longrightarrow v$ with $(u, 1) \longrightarrow (v, 0)$ and fixes the arc $(u, 0) \longrightarrow (v, 1)$. Therefore, $(\bar{\alpha}, \bar{\beta})$ and $(\bar{\beta}, \bar{\alpha})$ extended to $\bar{G}$ are TF-isomorphisms from $\bar{G}$ onto itself. Hence, let us extend to $\bar{\Gamma}$, the TF-isomorphisms $\phi^*_{p,q}$, the isomorphisms $\phi_{i,j}$ and the elements in $\mathbf{\Gamma}_i$. Then, let $\bar{\mathbf{\Gamma}}$ be the group

$$\langle \mathbf{\Gamma}_i, \phi_{i,j}, \phi^*_{p,q}, (\bar{\alpha}, \bar{\beta}), (\bar{\beta}, \bar{\alpha}) \mid r+1 \leq h \leq k,\ 1 \leq i < j \leq r,\ r+1 \leq p < q \leq k \rangle.$$

It follows that $\bar{G}$ is a $\bar{\mathbf{\Gamma}}$-orbital graph. □

Examples 8 and 9 are included to illustrate Theorem 4.3 and Theorem 4.4, respectively.

Example 8:
Figure 12 shows a disconnected TOG with components G_1, G_2 and G_3 with $|V(G_3)| \leq |V(G_2)| \leq |V(G_1)|$. G is generated by the pair $(\alpha, \beta) \in Aut^{TF}(G)$. We first note that (α, β) maps at least one arc of G_3 to an arc of G_2 and at least one arc of G_2 to an arc of G_1. It is easy to check that $G_1 = \mathbf{B}(G_2)$ and that G_2 and G_3 are isomorphic. In fact being isomorphic, they are also TF-isomorphic. In fact, if we let $\alpha_1 = \beta_1 =$ (1 4)(2 5)(3 6), then $(\alpha_1, \beta_1) = (\alpha_1, \alpha_1)$ is a TF-isomorphism between G_1 and G_3. Finally, we remark that each component of G is itself a TOG.

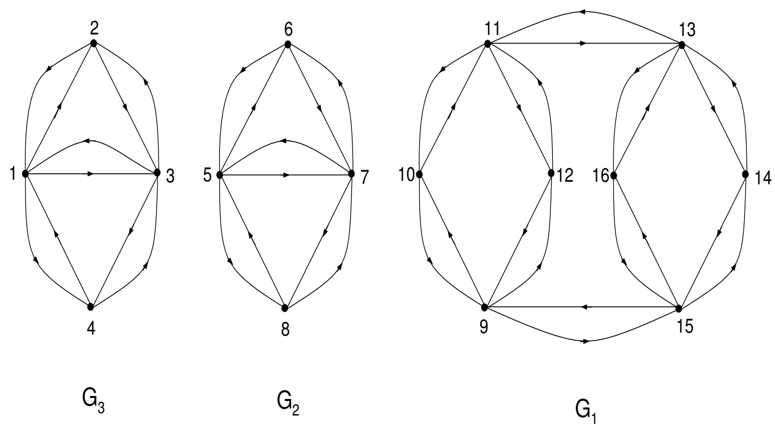

Figure 13: *Graph G has components G_1, G_2 and G_3.* α = (1 5 9 13 3 7 11 15)(2 6 14 10 4 8 16 12) *and* β = (1 5 13 9 3 7 15 11)(2 6 10 14 4 8 12 16)

Example 9:
Figure 13 again shows a disconnected TOG with components G_1, G_2 and G_3 with $|V(G_3)| \leq |V(G_2)| \leq |V(G_1)|$. G is generated by the pair $(\alpha, \beta) \in Aut^{TF}(G)$. The pair (α, β) maps at least one arc of G_3 to an arc of G_2 and at least one arc of G_2 to an arc of G_1. It is easy to check that $G_1 = \mathbf{B}(G_2)$ and that G_2 and G_3 are isomorphic. Again, we a remark that each component of G is itself a TOG.

Theorem 4.5 *Let G be a disconnected TOG, with no isolated vertices and let its connected components be $G_1, ..., G_k$. If one of its components is bipartite then*

either (i) all the components are isomorphic,

or (ii) there exists a unique index $r \in \{1, ...k-1\}$ such that $G_1 \cong G_2 \cong ... \cong G_r \ncong^{TF} G_{r+1} \cong^{TF} ... \cong^{TF} G_k$ and $G_1 \cong \mathbf{B}(G_k)$ where all G_i where $1 \leq i \leq r$ are bipartite and no G_j where $r+1 \leq j \leq k$ is bipartite.

Proof: Let G be a TOG with no isolated vertices and let its connected components be $G_i, ..., G_k$ and

$$|V(G_1)| \geq |V(G_2)| \geq, ... \geq |V(G_k)|.$$

If follows from Theorem 4.3 that

(i) if $|V(G_1) = |V(G_k)|$, then $G_1, G_2, ..., G_k$ are pairwise TF-*isomorphic*;

(ii) otherwise, there exists a unique index $r \in \{1, ...k-1\}$ such that $G_1 \cong G_2 \cong ... \cong G_r \ncong^{TF} G_{r+1} \cong^{TF} ... \cong^{TF} G_k$ and $G_1 \cong \mathbf{B}(G_k)$.

Consider Case**(i)**. Suppose there exists a bipartite component G_r. Consider any G_s, $1 \leq s \leq k$, $s \neq r$. Then $|V(G_r)| = |V(G_s)|$ and $G_r \cong^{TF} G_s$, that is, there exists $(\alpha, \beta) \in Aut^{TF}(G)$ such that

$$\forall (u, v) \in A(G_r),\ (\alpha, \beta)(u, v) = (\alpha(u), \beta(v)) \in A(G_s).$$

As in Proposition 4.2 define $\phi_{\alpha,\beta} : G_i \longrightarrow G_j$ by;

$$\phi_{\alpha,\beta}(x) = \begin{cases} \alpha x, & x \in V_0(G_r) \\ \beta x, & x \in V_1(G_r) \end{cases} \qquad or \qquad \phi_{\alpha,\beta}(x) = \begin{cases} \alpha x, & x \in V_1(G_r) \\ \beta x, & x \in V_0(G_r) \end{cases}$$

depending on whether for the arc $u \longrightarrow_{G_r} v$ is such that $u \in V_0(G_r)$ and $v \in V_1(G_r)$ or $u \in V_1(G_r)$ and $v \in V_0(G_r)$.

Note that (α, β) is a TF-isomorphism between G_r and G_s. Then, $\forall (\grave{u}, \grave{v}) \in A(G_s), (\alpha^{-1}(\grave{u}), \beta^{-1}(\grave{v})) = (u, v) \in A(G_r)$, for some $u, v \in V(G_r)$, such that $\alpha(u) = \grave{u}$ and $\beta(v) = \grave{v}$. Furthermore, recall that $|V(G_r)| = |V(G_s)|$. Therefore $\phi_{\alpha,\beta}$ is clearly well-defined, one-to-one and onto. Hence $\phi_{\alpha,\beta}$ is an isomorphism between G_r and G_s. Consequently, if G_r is bipartite, G_s is also bipartite. It follows that in this case, all the components of G are bipartite and isomorphic. Therefore G itself is bipartite.

Let us now consider Case(ii) which corresponds to Case**(ii)** of Theorem 4.3. In this case, for all i, j, $r+1 \leq i \leq j \leq k$, $G_i \cong^{TF} G_j \Rightarrow \mathbf{B}(G_i) \cong \mathbf{B}(G_j)$. If there exists $r+1 \leq p \leq k$ such that G_p is bipartite, then its canonical double covering $\mathbf{B}(G_p)$ ($\cong \mathbf{B}(G_i)$, $r+1 \leq i \leq k$) must be disconnected by virtue of Proposition 3.1. Therefore $G_1 \cong \mathbf{B}(G_k)$ is a contradiction since $\mathbf{B}(G_k)$ is disconnected but G_1 is connected. Therefore none of G_p, $r+1 \leq p \leq k$ can be bipartite.

Now suppose that G_q, $1 \leq q \leq r$ is bipartite. Note that this implies that all G_i, $1 \leq i \leq r$ are bipartite since $G_1 \cong G_2 \cong ... \cong G_r$. In particular, G_1 is bipartite and according to Theorem 4.3, $G_1 \cong \mathbf{B}(G_k)$. □

Corollary 4.6 *A disconnected bipartite graph G is a TOG only if its components are all isomorphic.*

Proof: We remark that if the disconnected graph G is bipartite, then all of its components must be bipartite. Therefore, only Case(i) of Theorem 4.5 holds. □

Note that the example shown in Figure 7 illustrates Case(i) of Theorem 4.5. In fact all components are bipartite and isomorphic. Moreover, we remark that in this case, the disconnected graph G is a bipartite TOG. Figure 7 illustrates Case(ii) of Theorem 4.5. In fact, G_1 and G_2 are not bipartite, whereas G_3 is bipartite and $G_1 = \mathbf{B}(G_3)$. In this case, although the disconnected graph G is a TOG, it is not bipartite.

The next theorem shows that for a simple instance of a directed graph, we can conclude that it is necessarily a TOD.

Theorem 4.7 *Disconnected digraph G which is a union of disjoint directed cycles is a TOD.*

Proof: The digraph G is a union of strongly-connected, disjoint directed cycles. Assume that G has only two components G_1 and G_2. Denote the vertices of G_1 with u_i $(i = 1, ..., n)$ and those in G_2 with v_j $(j = 1, ..., m)$. Hence, the arcs in G_i will be u_i, u_{i+1} $(i \in \mathbb{Z}_{n+1})$ and v_i, v_{j+1} $(j \in \mathbb{Z}_{m+1})$ respectively.
Consider the following permutations on $V(G)$:

$$\begin{aligned} \alpha &= (u_0 v_0 v_1 ... v_m u_1 u_2 ... u_n) \\ \beta &= (u_1 v_1 v_2 ... v_0 u_2 u_3 ... u_0) \end{aligned}$$

Each element of $\mathbf{\Gamma} = \langle(\alpha, \beta)\rangle$ takes $u_0 u_1$ to an arc in G, since the image of $u_i u_{i+1}$ (resp. $v_i v_{i+1}$) under the action of $(\rho, \sigma) \in \Gamma$, is either $u_{i+1} u_{i+2}$ or $v_j v_{j+1}$ (resp. $v_{i+1} v_{i+2}$ or u_j, u_{j+1}). Therefore, (ρ, σ) takes arcs of G into arcs of G. On the other hand, each arc of G can be obtained as an image of $u_0 u_1$, under the action of a suitable power of (α, β). In fact, if the arc is $v_i v_{i+1}$ or $u_i u_{i+1}$, we can use respectively, $(\alpha, \beta)^{i+1}$ $(i \in \mathbb{Z}_{m+1})$ or $(\alpha, \beta)^{m+i+1}$ $(i \in \mathbb{Z}_{n+1})$. Hence G is a $\mathbf{\Gamma}$-orbital digraph.

Let us now consider the general case, where the connected components of G are G_i for $i = 1, ..., k$.

Then,

$$G = \bigcup_{i=1}^{k} G_i.$$

For each r, $1 \leq r \leq k-1$, define (α_r, β_r), acting as above on the vertices of the components G_r and G_{r+1} and fixing all the remaining vertices of G. Then G is a $\mathbf{\Gamma}$-TOD with $\Gamma = \langle(\alpha_r, \beta_r) \,|\, 1 \leq r \leq k-1\rangle$. □

4.2 On Disconnected TOG's and the cycle structures of α and β

Theorem 4.8 *Let G be a disconnected graph and let $|V(G)| = n$. Suppose that (α, β) is a TF-automorphism on G. Let $\alpha = \alpha_1\alpha_2\ldots\alpha_r$ where α_i $(1 \leq i \leq r)$ represent disjoint cycles of length $n_{\alpha i}$. Similarly let $\beta = \beta_1\beta_2\ldots\beta_s$ where β_j $(1 \leq j \leq s)$ represent a cycles of length $n_{\beta j}$. Then, if any two numbers from the set $\{n_{\alpha i}, n_{\beta j} \,|\, 1 \leq i \leq r,\ \ 1 \leq j \leq s\}$ are relatively prime, the graph cannot be a TOG.*

Proof: Suppose, for contradiction, that G is a disconnected TOG and that there are at least two from the set $\{n_{\alpha i}, n_{\beta j} \,|\, 1 \leq i \leq r,\ \ 1 \leq j \leq s\}$ which are relatively prime.

We remark that by virtue of Lemma 4.1, all vertices in the cycle α_i must have the same out-degree ρ_i^+. Similarly all vertices in the cycle β_j must have the same in-degree ρ_j^-. Also $\sum_i n_{\alpha_i}\rho_i^+ = \sum_j n_{\beta j}\rho_j^- = |A(G)|$.

Let us consider Theorem 4.3(i). Suppose, without loss of generality, that G has k isomorphic components. Each of these components must have the same number of representatives $p_{\alpha i}$ of any α_i. Therefore $p_{\alpha i} = \frac{n_{\alpha i}}{k}$. Similarly, each of these components must have the same number of representatives $q_{\beta i}$ of any β_i. Therefore $p_{\beta i} = \frac{n_{\beta i}}{k}$. The numbers $p_{\alpha i}$ and $q_{\beta j}$ are integers for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Therefore k must be a common factor of all $n_{\alpha i}$ and $n_{\beta j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Therefore, in Case(i) we have a contradiction.

So, let us consider Case(ii) of Theorem 4.3. Suppose that the TOG consists of k isomorphic components, each having $p_{\alpha i}$ representatives from α_i and l components which are pairwise TF-isomorphic, one of which, namely G_1 is the canonical double covering of the last component G_{k+l}. If G_{k+l} has $p_{\alpha i}$ representatives of α_i, then G_1 would have $2p_{\alpha i}$ representatives of α_i. Since all the other $l-1$ components are TF-isomorphic to G_1, each would have $2p_{\alpha i}$ representatives of α_i. Therefore $2p_{\alpha i}l + p_{\alpha i}k = n_{\alpha i}\ \forall 1 \leq i \leq r$. Therefore $p_{\alpha i} = \frac{n_{\alpha i}}{(2l+k)}$. Since $p_{\alpha i} \in \mathbb{Z}$, then $(2l+k)$ must be a common factor of $n_{\alpha i}$ for all i, $1 \leq i \leq r$. Repeating the argument for the corresponding vertices acted on by β, we may similarly conclude that $(2l+k)$ must be a common factor of $n_{\beta j}$ for all j, $1 \leq j \leq s$. Again, we have a contradiction.

Therefore, we may conclude that if any two from the set $\{n_{\alpha i}, n_{\beta j} \,|\, 1 \leq i \leq r,\ \ 1 \leq j \leq s\}$ are relatively prime, the graph cannot be a TOG. $\square$

The above result may be useful to construct a disconnected TOG with given number of vertices since it eliminates various possibilities for for fixed k, l and n.

It is possible that a similar analysis works for $TOD's$, especially if we do not allow any of the arcs to be self-paired. It is also worth investigating whether the converse of the above theorem is true.

5 A subclass : the χ-orbital digraphs.

A basic difference between orbital graphs and two-fold orbital graphs is that orbital digraphs are arc-transitive whereas, in general, two-fold orbital digraphs are not. In fact, if $u \neq v, (u, v \in G)$, it is possible that $\alpha(u) = \beta(v)$ for some α, β so that $\alpha(u) \longrightarrow \beta(v)$ is a loop.

Here we look for conditions which ensure that two-fold orbital digraphs do not have loops. This could further lead to the possible arc-transitivity of a TOD.

5.1 The χ-orbital digraphs

Consider a group $\Gamma \leq S_n$ and a map $\chi : \Gamma \to \Gamma$. Define $D_\chi(\Gamma) = \{(\alpha, \chi(\alpha)) \,|\, \alpha \in \Gamma\} \subseteq \mathcal{S}$.

Proposition 5.1 *The $D_\chi(\Gamma)$ is a subgroup of $\mathcal{S}$ if and only if χ is an homomorphism on Γ. Moreover $D_\chi(\Gamma)$ is self-paired if and only if $\chi^2 = 1$.*

Proof: $D_\chi(\Gamma)$ is a subgroup of $\mathcal{S}$ if and only if

$$(\alpha, \chi(\alpha))(\beta, \chi(\beta)) = (\alpha\beta, \chi(\alpha)\chi(\beta)) = (\alpha\beta, \chi(\alpha\beta)),$$

that is, $\chi(\alpha)\chi(\beta) = \chi(\alpha\beta)$ for any $\alpha, \beta \in \Gamma$, that is, if and only if χ is a homomorphism on Γ.

Now assume that $D_\chi(\Gamma)$ is *self-paired*. Then $(\chi(\alpha), \alpha) \in D_\chi(\Gamma)$ for all $\alpha \in \Gamma$. This means that $(\chi(\alpha), \alpha) = (\beta, \chi(\beta))$ for a suitable β, so that χ interchanges α and β. This holds for all α so that $\chi^2 = 1$.

Conversely, if $\chi^2 = 1$, consider $(\alpha, \chi(\alpha)) \in D_\chi(\Gamma)$. Now, if we let $\beta = \chi(\alpha)$, then $(\chi(\alpha), \chi(\chi(\alpha))) = (\chi(\alpha), \chi^2(\alpha)) = (\chi(\alpha), \alpha) = (\beta, \chi(\beta)) \in D_\chi(\Gamma)$. Hence $D_\chi(\Gamma)$ is self-paired. □

Note that, if $\chi = 1$, then $D_\chi(\Gamma)$ is diagonal and therefore $D_\chi(\Gamma)$ would give rise to orbital digraphs.

A $D_\chi(\Gamma)-$*two-fold orbital digraph* will be referred to as a χ-*orbital digraph* or a χ-TOD for short.

Example 10:
Figure 14 shows an example in which the choice of (u, v) determines whether

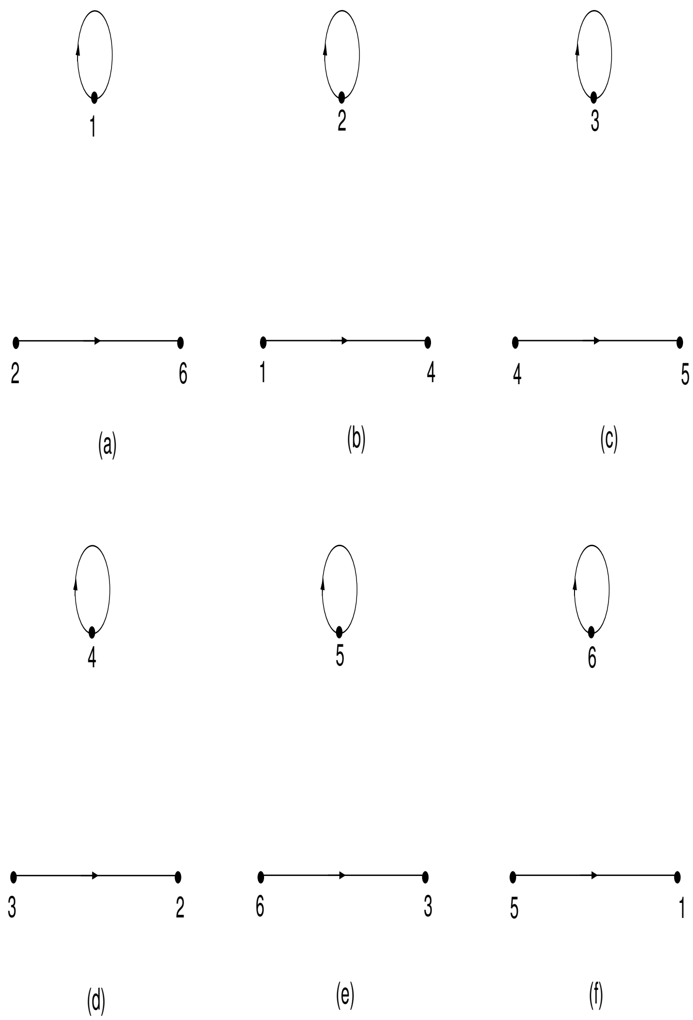

Figure 14: *The figure shows digraphs constructed on the vertex set* $V(G) = \{1, 2, 3, 4, 5, 6\}$. *The digraphs shown are formed by the arc sets (a)* $D_\Gamma(\lambda)(2,6)$, $(b)D_\Gamma(\lambda)(1,4)$, $(c)D_\Gamma(\lambda)(4,5)$, $(d)D_\Gamma(\lambda)(3,2)$, $(e)D_\Gamma(\lambda)(6,3)$ *and* $(f)D_\Gamma(\lambda)(5,1)$ *respectively. The pair* $(\alpha, \beta) \in \mathbf{\Gamma} = \Gamma \times \Gamma \subset S_6 \times S_6$ *is defined by* $\alpha = (12)(34)(56)$ *and* $\beta = (13)(25)(46)$ *and* $\chi = \alpha\beta$.

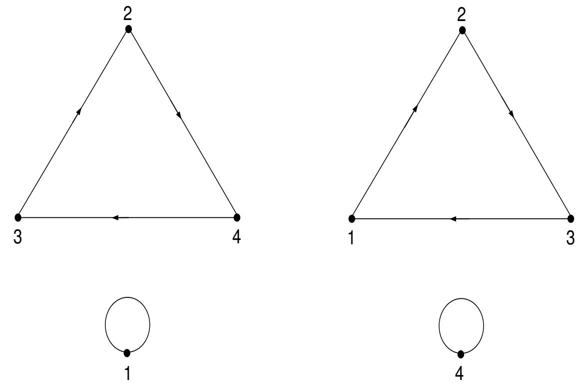

Figure 15: *The case of Example 11.*

or not the resulting two fold orbital $\mathbf{\Gamma}(u,v)$ has loops . Only the cases when the choice $(u,v), u \neq v$ that yield loops are given.

Example 11:
Refer to Figure 15. Consider the group $\Gamma = \{id, (12)(34), (13)(24), (14)(23)\}$. Set $\rho = (12)(34)$, $\sigma = (14)(23)$, $\tau = (13)(24)$ and let $\chi : \Gamma \to \Gamma$ be the *automorphism* $\chi = (\rho\sigma\tau)$. Figure 15 represents $D_\chi(\Gamma)(1,1)$ and $D_\chi(\Gamma)(1,2)$, respectively.

We shall now deal with the problem of detecting *loopless χ-orbital digraphs* $D_\chi(u,v)$ since the condition that $u \neq v$ is not sufficient to avoid loops, as it has already been observed.

In fact, for a χ- orbital graph to be loopless, it is necessary that, for all $\alpha \in \Gamma$:

$$\alpha(u) \neq \chi(\alpha(v)) \tag{3}$$

This condition does not hold in the example illustrated in Figure 15, whatever choice of (u,v) is made. As regards the example illustrated in Figure 14, whether or not this condition holds, depends on the choice of (u,v).

On the other hand, if Γ is *not transitive* and if u and v belong to distinct orbits, then it is easily seen that condition (3) is fulfilled.

We shall now define a subset of Γ, based on the automorphism χ on Γ, whose algebraic properties will help us determine when χ-orbital digraphs have loops.

Let $\Gamma \leq S_n$ be transitive and $\chi \in Aut(\Gamma)$. We define the set $M_\chi(\Gamma)$ as follows:

$$M_\chi(\Gamma) = \{(\chi(\alpha))^{-1}\alpha \mid \alpha \in \Gamma\}$$

Note that, if Γ is abelian, then $M_\chi(\Gamma)$ is a subgroup of Γ.

Proposition 5.2 *Let $\Gamma \leq S_n$ and $\chi \in Aut\Gamma$.*

(i) if $M_\chi(\Gamma) = \Gamma$, then the graph $\mathcal{D}_\chi(\Gamma)(u,v)$ always has loops ;

(ii) if $M_\chi(\Gamma) \neq \Gamma$ and Γ is regular, then there exists (u,v) such that $D_\chi(\Gamma)(u,v)$ has no loops ;

(iii) if there exists $v \in X \subseteq S_n$ such that $\Gamma_v \subseteq M_\chi(\Gamma)$ and if $M_\chi(\Gamma) < \Gamma$, then there exists (u,v) such that $D_\chi(\Gamma)(u,v)$ has no loops.

Proof: Equation (3) is equivalent to:

$$(\chi(\alpha))^{-1}\alpha(u) \neq v \tag{4}$$

If $M_\chi(\Gamma) = \Gamma$, the element in Γ, taking u to v, has the form $(\chi(\alpha))^{-1}\alpha$, contradicting (4). Therefore part(i) of the proposition follows.

From now on, assume that $M_\chi(\Gamma) \neq \Gamma$. Let us fix $\beta \in \Gamma \backslash M_\chi(\Gamma)$ and $u \in X$, and consider $v = \beta(u)$. If there exists $\alpha \in \Gamma$ such that $(\chi(\alpha))^{-1}\alpha(u) = v$, then the element $\grave{\beta} = (\chi(\alpha))^{-1}\alpha$ fulfills the equation $v = \grave{\beta}(u)$. It follows that $\beta \notin M_\chi(\Gamma)$ and $\grave{\beta} \in M_\chi(\Gamma)$. Therefore, $\beta \neq \grave{\beta}$. This is impossible since Γ is regular. This proves part (ii) of the proposition.

If $M_\chi(\Gamma) < \Gamma$ and $M_\chi(\Gamma)$ is a subgroup, the same argument as above yields $\beta^{-1}\grave{\beta} \in \Gamma_u$ and $\grave{\beta} \in M_\chi(\Gamma)$. This implies that $\beta \in M_\chi(\Gamma)$ which is a contradiction. This proves part (iii) of the proposition. □

From the previous proposition, it is clearly useful to know whether for some automorphism χ of Γ, $M_\chi(\Gamma)$ is or is not equal to Γ. We present below two results which settle this question in particular circumstances.

Proposition 5.3 *Let Γ be an abelian group and m be an integer such that $(m, |\Gamma|) = 1$. Define $\chi \in Aut\Gamma$ by $\chi : \alpha \mapsto \alpha^m$. If $(m-1, |\Gamma|) = 1$, that is, if $|\Gamma|$ is odd, then $M_\chi(\Gamma) = \Gamma$, otherwise $M_\chi(\Gamma) < \Gamma$.*

Proof: Recall that $M_\chi(\Gamma) = \{(\chi(\alpha))^{-1}\alpha \,|\, \alpha \in \Gamma\}$. But $\chi(\alpha) = \alpha^m$ with $m \in \mathbb{Z}$ and $(m, |\Gamma|) = 1$. Also $(\chi(\alpha))^{-1}\alpha = \alpha^{-m}.\alpha = \alpha^{1-m}$. It follows that $M_\chi(\Gamma) = \{\alpha^{1-m} \,|\, \alpha \in \Gamma\}$. Since, from Lagrange's Theorem, the order of a subgroup of Γ is a divisor of $|\Gamma|$, then there exists $p \in \mathbb{Z}$ such that

$|(1-m)|p = |\Gamma|$. Therefore $p = |\Gamma|/|1-m|$. If $(m-1, |\Gamma|) = 1$, then $p = |\Gamma|$ and $M_\chi(\Gamma) = \Gamma$. Otherwise, $p < |\Gamma|$ and $M_\chi(\Gamma) < \Gamma$.
□

For a group Γ, $[\Gamma, \Gamma]$ is defined to be the group generated by all elements $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$, for all $\alpha, \beta \in \Gamma$.

Recall that a group Γ is nilpotent of class 2 when $[\Gamma, \Gamma] \leq Z(\Gamma)$, the centre of Γ.

Proposition 5.4 *Let Γ be a nilpotent group of class 2 and let χ be an inner automorphism of Γ induced by $\tau \in \Gamma \setminus Z(\Gamma)$, that is, $\chi : \alpha \mapsto \tau^{-1}\alpha\tau$. Then $M_\chi(\Gamma) < \Gamma$.*

Proof: Note that $M_\chi(\Gamma) = \{(\chi(\alpha))^{-1}\alpha \mid \alpha \in \Gamma\} = \{\tau^{-1}\alpha^{-1}\tau\alpha \mid \alpha \in \Gamma\} = \{[\tau, \alpha] \mid \alpha \in \Gamma\}$, since

$$\begin{aligned} \chi(\alpha)^{-1} &= \chi(\alpha^{-1}) \quad (\because \chi \text{ is an automorphism}) \\ &= \tau^{-1}\alpha^{-1}\tau \quad (\text{by definition}). \end{aligned}$$

We have $[\mu, \omega] \in Z(\Gamma)$ (that is, $\mu^{-1}\omega^{-1}\mu\omega \in Z(\Gamma)$) for all $\mu, \omega \in \Gamma$ since Γ is nilpotent of class 2. Therefore $[\tau, \mu][\tau, \omega] = [\tau, \mu\omega]$, since,

$$\begin{aligned} [\tau, \mu\omega] &= \tau^{-1}(\mu\omega)^{-1}\tau\mu\omega \\ &= \tau^{-1}\omega^{-1}\mu^{-1}\tau\mu\omega \\ &= \tau^{-1}\omega^{-1}\mu^{-1}\mu\tau[\tau, \mu]\omega \\ &= [\tau, \mu]\tau^{-1}\omega^{-1}\mu^{-1}\mu\tau\omega \quad (\because [\tau, \mu] \textit{ commutes}) \\ &= [\tau, \mu]\tau^{-1}\omega^{-1}\tau\omega \\ &= [\tau, \mu][\tau, \omega]. \end{aligned}$$

Hence $M_\chi(\Gamma) \leq \Gamma$. The map $\mu \mapsto [\tau, \mu]$ is not injective since $[\tau, \tau] = [\tau, 1] = 1$. Therefore $M_\chi(\Gamma)$ is a proper subgroup of Γ.
□

Example 12 :
Let Γ be a nilpotent group of class 2, and $\mathcal{H} \leq M_\chi(\Gamma)$. Consider the action of Γ on the set of right cosets of $\mathcal{H}$, defined by :

$$\mathcal{H}\omega \longmapsto \mathcal{H}\omega\alpha.$$

Then, $\mathcal{H} \leq M_\chi(\Gamma)$. In view of Proposition 5.4, this is Case(iii) of Proposition 5.2. Hence, the graph $D_\chi(\Gamma)(u, v)$ has no loops.

6 Conclusion

We have established a close link between two-fold orbital digraphs and canonical double coverings. Now, it seems quite useful to build up a general theory of two-fold orbital digraphs. Of course, much more work needs to be done to achieve this. The following two problems seem to be important:

- **Problem 1**. Characterize those digraphs that are $\mathbf{\Gamma}$-orbital for a suitable group $\mathbf{\Gamma}$.
- **Problem 2**. Study the behaviour of the construction of two-fold orbital digraphs with respect to various graph-theoretical properties.

It might also be worth investigating whether the converse of Theorem 4.5 is true.

As regards disconnected TOG's, we present the following conjectures.

- **Conjecture 1**. A disconnected TOG with no isolated vertices is a disconnected line graph of some graph G.
- **Conjecture 2**. Any disconnected TOG whose components are all isomorphic is an iterated line graph $L^n(G)$ of some graph G, where G is a disconnected bipartite TOG.
- **Conjecture 3**. The subgraph of a disconnected TOG made up of all isomorphic components of the disconnected TOG is an iterated line graph $L^n(G)$ of some graph G, where G is a disconnected bipartite TOG.

We believe that the study of two-fold orbital digraphs has a potentially high impact. We hope that this work provides an adequate basis for further research.

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References

[1] N.L. Biggs, A.T. White, Permutation groups and combinatorial structures, *LMS Lecture Note Series*, No.33, Cambridge University Press, London, 1979.

[2] D.M. Cvetković, M. Doob, H. Sachs, Spectra of graphs, Cambridge University Press, *Johann Ambrosius Barth*, Heidelberg, 1995.

[3] M. Conder, P. Lorimer, C.E. Praeger, Constructions for arc-transitive digraphs, *J. Austral. Math. Soc* A-**59** (1995), 61-80.

[4] M. Hofmeister, Counting double covers of graphs, J. Graph Theory **12** (1988), 437-444.

[5] J. Lauri, R. Scapellato, Topics in Graph Automorphisms and Reconstruction, *Cambridge University Press*, 2003.

[6] D. Marušič, R. Scapellato, Classifying vertex-transitive graphs whose order is a product of two primes, *Combinatorica*, **14**(2)(1994), 187-201.

[7] D. Marušič, R. Scapellato, Classification of vertex-transitive pq-digraphs, *Atti. Ist. Lombardo (Rend.Sci.)*, A-**128**, no.1 (1994), 31-36.

[8] R. Nedela, M. Skoviera. *Regular maps of canonical double coverings of graphs*, *J. Combin. Theory Ser. B*, **67** (1996), 249-227.

[9] W. Pacco, R. Scapellato, Digraphs having the same canonical double covering, *Discrete Math.*, **173** (1997), 291-296.

[10] L. Porcu, Sul raddoppio di un grafo, *Atti. Ist. Lombardo (Rend.Sci.)*, A-**110** (1976), 453-480.

[11] C.E. Praeger, M-Y. Xu, Vertex-primitive graphs of order a product of two distinct primes, *J. Comb. Theory Ser. B* **59** (1993), no.2, 245-266.

[12] C.E. Praeger, R-J. Wang, M-Y. Xu, Symmetric graphs of order a product of two distinct primes, *J. Comb. Theory Ser. B* **58** (1993), no.2, 299-318.

[13] B. Zelinka, On double covers of graphs, *Math. Slovaca* **32** (1982), 49-54.