Links between two semisymmetric graphs on 112 vertices through the lens of association schemes

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1 Introduction

One of the most striking impacts between geometry, combinatorics and graph theory, on one hand, and algebra and group theory, on the other hand, arise from a concrete necessity to manipulate with the symmetry of the investigated objects. In the case of graphs, we talk about such tasks as identification and compact representation of graphs, recognition of isomorphic graphs and computation of automorphism groups of graphs.

Different classes of interesting graphs are defined in terms of the level of their symmetry, which which is founded on the concept of transitivity. A semisymmetric graph, the central subject in this paper, is a regular graph Γ such that its automorphism group $Aut(\Gamma)$ acts transitively on the edge set $E(\Gamma)$, though intransitively on the vertex set $V(\Gamma)$.

More specifically, we investigate links between two semisymmetric graphs, both on 112 vertices, the graph $\mathcal{N}$ of valency 15 and the graph $\mathcal{L}$ of valency 3. While the graph $\mathcal{N}$ can be easily defined and investigated without serious computations, the manipulation with $\mathcal{L}$ inherently depends on a quite heavy use of a computer.

The graph $\mathcal{L}$, commonly called the Ljubljana graph, see [76], has quite a striking history, due to efforts of a few generations of mathematicians starting from M. C. Gray and I. Z. Bouwer and ending with M. Conder, T. Pisanski and their colleagues; the order of its automorphism group is 168. The graph $\mathcal{L}$ turns out to be a spanning subgraph of the graph $\mathcal{N}$, which has the group S_8 of order 8! as automorphism group.

In our paper we reveal numerous interesting links between $\mathcal{L}$ and $\mathcal{N}$, as well as with diverse combinatorial structures, including association schemes, coherent algebras, symmetric configurations, overlarge sets of Fano planes, partial geometries, etc. For this purpose we exploit tools from algebraic graph theory

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and we also rely on an essential use of computer algebra packages, mainly GAP and COCO.

The main results presented in the paper are various interesting properties of the graphs $\mathcal{L}$ and $\mathcal{N}$, as well as the discovery of a number of new association schemes, namely those on 56 and 112 points, which are non-Schurian.

The style of our exposition, which proceeds from simple cases to more sophisticated and which also tries to reflect the historical development of the topics we discuss will hopefully enable the reader to visit in a sense the "kitchen" of a computer algebra experimentation.

As a rule, most of the collected facts about the investigated structures at the first attempt were obtained with the aid of a computer. Each time, in this paper, special attention is paid to the presentation of a part of the obtained computational results and their transformation to theoretical claims. In a few cases the reader will become also a witness of a successful computer free interpretation and further theoretical generalization of the original computer generated information. Our aim was to combine in one paper the features of at least three different genres: expository text about semisymmetric graphs; tutorial on scientific computation in algebraic graph theory; report about the new results achieved in our research.

Here is a very brief survey of the entire paper. Sections 2-4 provide a brief account of the general preliminaries, introduction to double covers of graphs and a review of semisymmetric graphs. The graphs $\mathcal{N}$ and $\mathcal{L}$ are introduced in Sections 5, 6, the embeddings of $\mathcal{L}$ into $\mathcal{N}$ are examined in Section 8 with the aid of a rank 20 master association scheme $\mathfrak{M}$ on 56 points, which is introduced and investigated in Section 7. The graph $\mathcal{L}$ is the incidence graph of a Ljubljana configuration, the new view of which is presented in Section 9; with the aid of a few auxiliary structures introduced in Section 10 (they are of a definite independent interest), the embeddings of $\mathcal{L}$ into $\mathcal{N}$ are revisited in Section 11, leading to a new example of a so-called Deza family in a Higmanian house (this time on 120 points). In Section 12, paying tribute to the results by I. Dejter et al, we consider one more model for the graph $\mathcal{L}$, this time using the 7-dimensional cube Q_7 and Hamming code $\mathcal{H}_3$ inside of it.

A new family of association schemes on 56 points (which appears as a by-product obtained by exploiting the approach which we follow) is introduced and discussed in Section 13. The last three sections of the text are mainly oriented to the future: promising links with a new concept of two-fold automorphisms of graphs are outlined in Section 14; while Sections 15 and 16 pay attention to many other facets of former and ongoing research, which may be relevant to the continuation of the entire spectrum of ideas and techniques already exploited in the paper. A detailed bibliography aims to help the reader to gain access to all relevant sources of information.

We hope that this paper may assist the creation of new waves of scientific computation to the benefit of both algebra and graph theory.

2 Preliminaries

This project is fulfilled on the edge between a few diverse areas of algebra, combinatorics, graph theory and scientific computation. For this reason it is impossible in principle to provide to the reader a self-contained account of all the necessary background information. Instead, we outline below a very brief guide to the main ingredients of such a background, referring each time to the most appropriate, in our eyes, sources in literature.

The area of finite permutation groups was strongly developed during the last 50 years, in particular through the efforts of H. Wielandt and his followers. The book [29] is commonly regarded nowadays as the standard source in the area. The concept of an invariant relation of a permutation group (G, Ω) was coined and used in [105]. We refer to [67] and [37] for an introduction to this line of Wielandt's methodology.

Invariant binary relations play a central role in our presentation. For a given permutation group (G, Ω) each such relation can be represented as a union of minimal invariant binary relations. The latter ones, following Wielandt, are called *2-orbits of* (G, Ω). The set $2 - orb(G, \Omega)$ of such minimal relations will be the subject of our careful attention for a number of concrete permutation groups.

Association schemes are introduced as a combinatorial axiomatization of significant regularity properties of the 2-orbits of permutation groups. For each transitive permutation group (G, Ω) a pair $(\Omega, 2{-}orb(G, \Omega))$ provides a model of an association scheme; each such model is called a *Schurian* association scheme, cf. [37]. Non-Schurian association schemes are a subject of special interest because they, in a sense, are not predictable on a purely group-theoretical level. The smallest example of a non-Schurian scheme exists on 15 points. The classic book [2] still is an excellent source for this area.

A more general concept of a coherent configuration ([47], [14]) appears as a combinatorial generalization of an arbitrary permutation group, strictly in the same manner as an association scheme relies on a transitive permutation group. Both coherent configurations and association schemes may and should be considered via the impact of the two languages of relations and matrices on each other. The matrix analogue of a coherent configuration, a *coherent algebra*, see [48], is a matrix algebra which is closed with respect to Schur-Hadamard multiplication and transposition and contains the identity matrix I and the all-ones matrix J. Each coherent algebra $\mathfrak{M}$ (regarded as a vector space) has a unique *standard basis* of $(0, 1)$- matrices. These matrices are adjacency matrices of basic graphs of $\mathfrak{M}$.

To each coherent configuration $\mathfrak{M}$, and in particular, to each association scheme, a few natural symmetry groups are defined, namely the automorphism group $Aut(\mathfrak{M})$, the color automorphism group $CAut(\mathfrak{M})$, and the algebraic automorphism group $AAut(\mathfrak{M})$. A few recent papers such as [64], [65] may assist the reader to digest corresponding definitions and to get a helpful context regarding their applications.

We assume from the reader a modest acquaintance with simple concepts from graph theory and the ways how diverse kinds of symmetry of graphs are investigated with the aid of finite permutation groups. The starting context is presented in Chapters 1,2 of [73]; also references in this book may be quite helpful.

In fact, the interplay between graphs and algebra, on which we rely in our paper, during the last few decades was consolidated in what is now called "Algebraic graph theory" (briefly AGT). The classic book [4] and the modern textbook [43] as well as the encyclopedic monograph [10], are all highly recommended guides.

Paying enough attention to the presentation of the subject of AGT, we admit that this paper, according to its genre and audience, is mainly about computer experimentation in AGT. For this purpose we are using a few computer packages, mostly COCO ([36], [37]) and GAP ([42]) with its extension package GRAPE ([98]) which relies on nauty ([89]). Again, we refer to our recent publications, e.g. to [64], [68], [66] for the description of the computer algebra tools used and discussions of the main features of computer algebra experimentation in the areas of coherent configurations and association schemes.

3 Double covers

The operation of a double cover of a graph plays an essential role in our presentation. Below we provide all necessary definitions and illustrating examples as well as discuss properties of this operation. We warn the reader about a possible inconsistency of terminology between the formal definitions which are given below and those sometimes used in the graph theoretical literature. Our main definition strictly follows the one from [55].

Definition 3.1. *Let $\Delta = (V, R)$ be a directed graph. Define a new undirected graph $\Gamma = (V(\Gamma), E(\Gamma))$, such that $V(\Gamma) = V \times \{1, 2\}$, $E(\Gamma) = \{\{(x, 1), (y, 2)\} | (x, y) \in R\}$.*

We will call the graph Γ the *incidence double cover* (briefly IDC) of Δ.

Remark. Note that in the abbreviation IDC the letter I pretty well stands also for both Iofinova and Ivanov. While the graph Δ is considered as a directed graph, the resulting graph Γ is always undirected. In the case of an undirected graph Δ each edge $\{x, y\}$ of Δ should be substituted by the pair of opposite arcs $\{(x, y), (y, x)\}$. For a particular case when the graph Δ is undirected the IDC construction is usually called the standard double cover. Note that the same name was used also in [55]. Many authors however use this constructions exclusively for undirected graphs. Moreover, if the initial graph Δ is directed, it is sometimes automatically substituted by its underlying undirected graph $\overline{\Delta}$ with $E(\overline{\Delta}) = \{\{x, y\} | (x, y \in R\}$. The suggested terminology allows us to avoid any misunderstandings which may occur in the course of consideration of double covers.

A number of simple examples presented below will hopefully provide to the reader a helpful context. Each time we present diagrams of a graph Δ and its corresponding $\Gamma = IDC(\Delta)$.

A few additional remarks about the graphs presented are in order. A directed edge is not a regular graph, and neither is its cover. All other graphs Δ in the examples are regular of valency k, and the corresponding graph Γ is also regular of valency k. Covers of directed and undirected triangles clearly have different valency and are therefore different. The cover of quadrangle is a disconnected graph.

The tournament $P(7)$ has as its vertex set the elements of the finite field $\mathbb{Z}_7$. There is an arc (x, y) in $P(7)$ if and only if $y - x$ is a non-zero square in $\mathbb{Z}_7$, that is $y - x \in \{1, 2, 4\}$. The notation of vertices $0, \dots, 6$ and $7, \dots, 13$ in part (g) for the graph Γ corresponds to the pairs $(0, 1), \dots, (6, 1)$ and $(0, 2), \dots, (6, 2)$.

The automorphism group for all graphs, except those in case (g), are easily comprehended by visual observations. In cases (a) and (b) we get $|Aut(\Gamma)| = 4 \cdot |Aut(\Delta)|$, while $|Aut(\Delta)| = 1$ and 2, respectively. In case (c) $|Aut(\Delta)| = 3$, $|Aut(\Gamma)| = 3! \cdot 2^3 = 48$. In case (d) $|Aut(\Delta)| = 6$ and $|Aut(\Gamma)| = 12$. in case (e) $|Aut(\Delta)| = 8$, $|Aut(\Gamma)| = 2 \cdot 8^2 = 128$.

In case (f) $|Aut(\Delta)| = 10$, $|Aut(\Gamma)| = 20$. Finally, deeper reasonings are required in order to check that in case (g), $Aut(\Delta)$ is the Frobenius group F_{21} of order 21, while $Aut(\Gamma) \cong PSL(3, 2) : \mathbb{Z}_2$ is a group of order 336.

Proposition 3.1. *Let Δ be a graph, $\Gamma = IDC(\Delta)$. Then*

(i) if Δ is a regular graph of valency k, then Γ is also regular of the same valency k;

(ii) if Δ is a bipartite graph then Γ is disconnected;

(iii) if Δ is an undirected graph then the group $Aut(\Gamma)$ contains as a subgroup the direct product $Aut(\Delta) \times \mathbb{Z}_2$.

Proof. Straightforward, see e.g. [111]. □

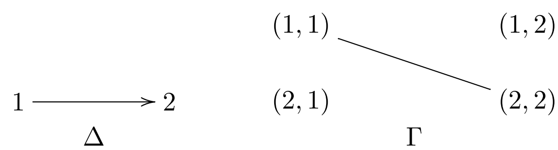
Note that, in our examples for undirected graphs in cases (d) and (f) we get the equality $Aut(\Gamma) = Aut(\Delta) \times \mathbb{Z}_2$, while in cases (b) and (e) the group $Aut(\Delta) \times \mathbb{Z}_2$ appears as a proper subgroup of $Aut(\Gamma)$.

The question of when the equality $Aut(\Gamma) = Aut(\Delta) \times \mathbb{Z}_2$ holds for an undirected graph Δ turns out to be of independent interest. Following [87], we will call such a graph Δ a *stable graph*. Note that while the original definition in [87] was given only for undirected graphs, in this paper it is (in a sense) also extended to directed graphs. According to the above examples, the triangle and the pentagon are stable graphs.

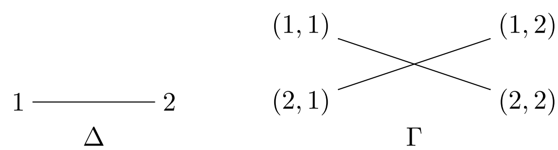
In recent years interest in the investigation of stable graphs has increased; see the short discussion in Section 15.

An alternative way to explain the construction of double cover goes through the use of matrices. For an arbitrary graph (directed arcs and loops are allowed) Δ, denote by $A(\Delta)$ its adjacency matrix. Clearly, any arbitrary square symmetric (0,1)-matrix is the matrix $A(\Delta)$ for a suitable graph Δ. We however

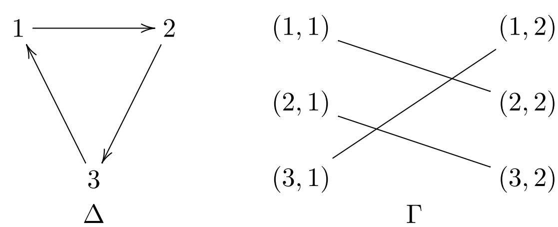
Example 3.1. (a) directed edge



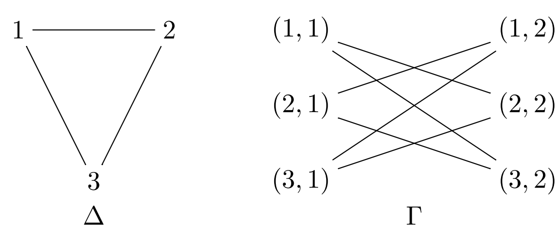
(b) undirected edge



(c) directed triangle



(d) undirected triangle



(e) quadrangle

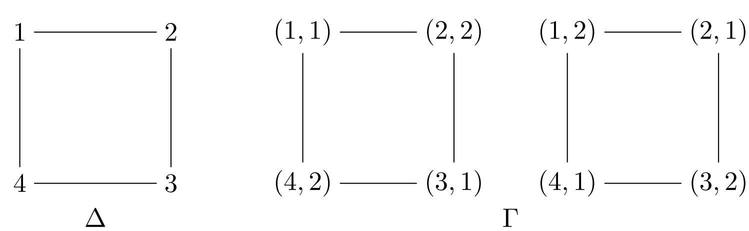
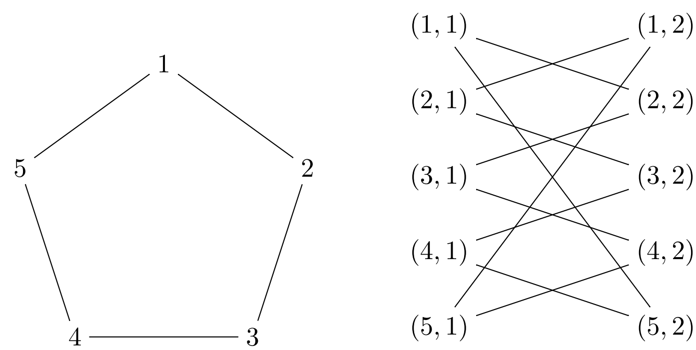


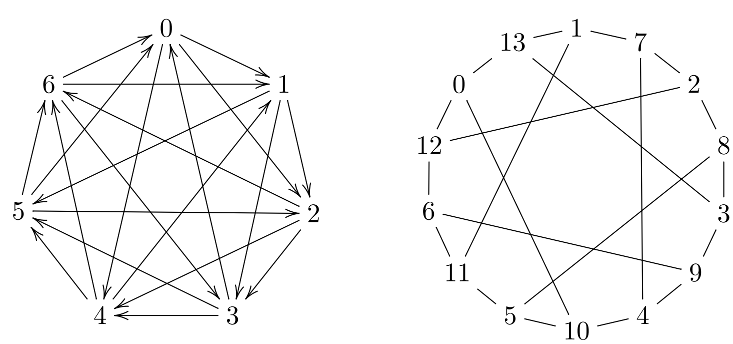
Figure 1: Small IDC covers

(f) pentagon



Δ Γ

(g) tournament $P(7)$



Δ Γ

Figure 2: Small IDC covers (cont.)

may interpret the matrix $A = A(\Delta)$ as the incidence matrix $I(S)$ of a suitable incidence structure S. Here rows of $A = (a_{ij})$ correspond to points of S, while columns correspond to blocks of S. An element a_{ij} is equal to 1 if and only if the point defined by row i is incident to the block defined by row j. The result is that we consider the incidence (Levi) graph of the incidence structure S (cf. [21]). Note that the number of points and blocks in S is equal. Such incidence structures are called *configurations*, if the incidence graph happens to be regular and does not contain quadrangles.

It seems that the proposed mode of correspondence between graphs and incidence structures was for the first time considered in [30]. A helpful survey of general properties of this correspondence is provided in [13]. Note that this explanation justifies the name "incidence double cover".

Let us now take a new glance at part (g) of Figure 3.1. Here we are coming from the Paley tournament $P(7)$ to its adjacency matrix A, interpreting it as the incidence matrix of the structure with the point set $P = [0, 6]$ and the block set B, the set B consists of the following seven 3-element subsets of P: $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$, $\{0, 4, 5\}$, $\{1, 5, 6\}$, $\{0, 2, 6\}$, $\{0, 1, 3\}$ which are labeled by symbols $7, 8, \ldots, 13$ respectively. The reader, of course, will recognize the diagram of Γ as the Levi graph of the famous Fano plane. This graph Γ is usually called the Heawood graph, see e.g. [46]. Clearly, the group $Aut(\Gamma)$ is much larger then the group $Aut(\Delta)$. Note that both groups will play a significant role further along this paper.

4 Semisymmetric graphs

Recall that an undirected graph $\Gamma = (V, E)$ is called *semisymmetric* if it is regular (of valency k) and $Aut(\Gamma)$ acts transitively on E and intransitively on V.

The proposition below is attributed by F. Harary to Elayne Dauber; its proof appears in [46] and [73].

Proposition 4.1. *A semisymmetric graph Γ is bipartite with the partitions $V = V_1 \cup V_2$, $|V_1| = |V_2|$, and $Aut(\Gamma)$ acts transitively on both V_1 and V_2.*

Interest in semisymmetric graphs goes back to the seminal paper [40], where they were called admissible graphs. The wording "semisymmetric" was suggested in [62].

Example 4.1 (The semisymmetric Folkman graph on 20 vertices)**.** *Let $V_1 = \left\{ {[0,4] \atop 2} \right\}$ be the set of all 2-element subsets of the 5-element set $[0, 4]$. Let $V_2 = [0, 4] \times \{1, 2\}$. Define $V = V_1 \cup V_2$, $E = \{\{\{a, b\}, (a, i)\} | a, b \in [0, 4], i \in \{1, 2\}, a \neq b\}$. It is easy to check that the direct product of the symmetric group S_5 with S_2 acts transitively on the sets V_1,, V_2, E. Moreover, this is the full automorphism group of the resulting graph $\mathcal{F} = (V, E)$. (For the proof it is helpful to notice that $Aut(\mathcal{F})$ acts primitively on V_1 and imprimitively on V_2.)*

At first, interest in semisymmetric graphs was sustained by the representatives of the Soviet school of graph theory. The paper [102] immediately attracted the interest of the researchers from the USSR to the number of open questions about semisymmetric graphs which were posed by Folkman in [40] and repeated in [102]. A general method to construct semisymmetric graphs with the aid of the multi-hypergraphs was suggested by V. K. Titov in [101]. Below we present the semisymmetric graph on 24 vertices, constructed by Titov.

Example 4.2. *Let $V_1 = [0,3] \times \{1,2,3\}$, $V_2 = \left\{ {[0,3] \atop 2} \right\} \times \{4,5\}$, $V = V_1 \cup V_2$, $E = \{\{(x,i),(\{x,y\},j)\} | x,y \in [0,3], x \neq y, i \in \{1,2,3\}, j \in \{4,5\}\}$. We suggest that the reader checks that the resulting graph $\mathcal{T} = (V,E)$ on 24 vertices and valency 6 is a semisymmetric graph with $|Aut(\mathcal{T})| = 2^{13} \cdot 3^5$. Note that the group $Aut(\mathcal{T})$ may be easily described as a generalized wreath product (in the sense of [109]) of the group S_4 acting on orbits of lengths 4 and 6 with groups S_3 and S_2 respectively. Here $|Aut(\mathcal{T})| = 4! \cdot (3!)^4 \cdot (2!)^6$.*

As usual, we refer to Section 15 for more information about the investigation of semisymmetric graphs.

Following [55] let us call a semisymmetric graph $\Gamma = (V,E)$, $V = V_1 \cup V_2$, of *parabolic type* if the stabilizers H_1, H_2 of vertices $x \in V_1$ and $y \in V_2$ respectively are not conjugate in the symmetric group $Sym(V)$. (Note that we slightly modify the original definition in [55].) Otherwise the graph Γ will be called of *non-parabolic type*. In a more naive wording, a semisymmetric graph Γ of parabolic type belongs to an "easy" case of such graphs. This means that one can distinguish vertices x, y with the aid of simple arguments, using suitable numerical or structural invariants of the vertices. The above two examples clearly serve as simple representatives of the easy case.

It turns out that the semisymmetric Ljubljana graph $\mathcal{L}$ on 112 vertices (see Section 6) is one of the smallest examples of the non-parabolic case: here both groups H_1 and H_2 are cyclic groups of order 3, which are conjugate in S_{112}. As a result, the proof of the fact that $\mathcal{L}$ is indeed semisymmetric inherently becomes a more sophisticated task in comparison, e.g. with corresponding proofs for the graphs $\mathcal{F}$ and $\mathcal{T}$ above. For the purpose of investigation of such a case we will use techniques of incidence double covers in conjunction with the ideas which were introduced in [55].

Let Γ be a bipartite graph with partition $V = V_1 \cup V_2$ of its vertices. In what follows we assume that Γ is an edge-transitive regular graph of valency k. Then it follows from Proposition 4.1 that the group $Aut(\Gamma)$ either acts transitively on V or acts intransitively with two orbits V_1 and V_2 of equal length. Let us denote by $Aut^-(\Gamma)$ the subgroup of $Aut(\Gamma)$ which stabilizes each set V_1 and V_2 separately. Then clearly $[Aut(\Gamma) : Aut^-(\Gamma)] = 1$ or 2, depending on whether $Aut(\Gamma)$ acts on V transitively or intransitively, respectively.

Now let G be a finite group and H_1 and H_2 subgroups of G of equal index k. Denote by $\Gamma(G, H_1, H_2)$ the bipartite graph whose vertices are cosets of H_1 and H_2 in G and vertices H_1g_1 ans H_2g_2 are adjacent if and only if $H_1g_1 \cap H_2g_2 \neq \emptyset$. We will call Γ the *coset graph* of G with respect to the pair of subgroups H_1, H_2. The presentation below follows the line which was established in [55].

Lemma 4.2. *1. The edges of the graph $\Gamma(G, H_1, H_2)$ bijectively correspond to the cosets of $H_1 \cap H_2$ in G.*

2. The group G acts transitively on the edge set of the graph $\Gamma(G, H_1, H_2)$.

3. Graph $\Gamma(G, H_1, H_2)$ is connected if and only if $G = \langle H_1, H_2 \rangle$.

4. If $G = \langle H_1, H_2 \rangle$ then each subgroup $N \leq H_1 \cap H_2$, which is normal in both H_1 and H_2 is also normal in G, and acts trivially on the graph $\Gamma(G, H_1, H_2)$.

In what follows it is convenient to assume that such a proper subgroup N does not exist in G, in other words, G acts faithfully on the graph $\Gamma(G, H_1, H_2)$.

Lemma 4.3. *Let $G \leq Aut(\Gamma)$ act edge- but not vertex-transitively on Γ. Let $x \in V_1$, $y \in V_2$, $\{x, y\} \in E(\Gamma)$, $H_1 = G_x$, $H_2 = G_y$. Then Γ is isomorphic to $\Gamma(G, H_1, H_2)$.*

By a triple of groups (G, H_1, H_2) we now mean each triple such that $G = \langle H_1, H_2 \rangle$. Two triples (G, H_1, H_2) and (G', H_1', H_2') are called *equivalent* if the corresponding two coset graphs are isomorphic. Clearly an equivalence class of triples has a unique (up to isomorphism) *maximal* element which is defined by $Aut^-(\Gamma)$ and two stabilizers of the ends of an arbitrary edge in Γ, where Γ is isomorphic to the (isomorphic) coset graphs of the two triples.

Let us call the triple (G, H_1, H_2) *embeddable* into the triple (G', H_1', H_2') if there exists a monomorphism $\phi : G \to G'$ such that $\phi(H_i) = H_i'$, $i = 1, 2$.

Lemma 4.4. *Let the triple (G, H_1, H_2) be embeddable into (G', H_1', H_2') and moreover let $H_i = G \cap H_i'$ for $i = 1, 2$. Then the two triples are equivalent if and only if*

$$[G : G'] = [H_1' \cap H_2' : H_1 \cap H_2].$$

Corollary 4.5. *If triples (G, H_1, H_2) and (G', H_1', H_2') are equivalent then*

$$[G : H_1 \cap H_2] = |E(\Gamma(G, H_1, H_2))| = |E(\Gamma(G', H_1', H_2'))| = [G' : H_1' \cap H_2'].$$

Lemma 4.6. *Let Γ be a bipartite graph as above, $G = Aut^-(\Gamma)$, $\{x, y\} \in E(\Gamma)$, and H_1 and H_2 the stabilizers of x and of y in G, respectively. Then the following two conditions are equivalent:*

(i) the graph $\Gamma = \Gamma(G, H_1, H_2)$ is vertex-transitive;

(ii) there exists an overgroup D of G, containing G as a subgroup of index 2, such that $(H_1^d, H_2^d) = (H_2, H_1)$ for some $d \in D$ (here $H^d = d^{-1}Hd$).

Let us now combine consideration of coset graphs with the construction of IDC.

Lemma 4.7. *Let the triple (G, H_1, H_2) be maximal in its class. Then the following two conditions are equivalent:*

(i) the graph $\Gamma(G, H_1, H_2)$ is the standard double cover of an undirected graph Δ and G acts transitively on both $E(\Delta)$ and $V(\Delta)$;

(ii) $Aut(\Gamma(G, H_1, H_2)) = G \times \langle\tau\rangle$, $\tau^2 = 1$.

Lemma 4.8. *Let (G, Ω) be a transitive permutation group and let R be its connected antisymmetric 2-orbit. Assume that $(x, y) \in R$, $H_1 = G_x$, and $H_2 = G_y$. Assume in addition that the triple (G, H_1, H_2) is maximal in its equivalence class. Then exactly one of the following two possibilities holds:*

(i) $\Gamma(G, H_1, H_2)$ is a semisymmetric graph;

(ii) There exists a permutation group (D, Ω) in which (G, Ω) is a subgroup of index 2 such that $(x^d, y^d) = (y, x)$ for some $(x, y) \in R$ and $d \in D$.

Now we may formulate our main result in this section.

Theorem 4.9 (Criterion of Iofinova-Ivanov)**.** *Let (G, Ω) be a 2-closed permutation group. Assume R is an antisymmetric 2-orbit of (G, Ω), and $\Delta = (\Omega, R)$ is a directed graph. Let $\Gamma = IDC(\Delta)$. Assume that the graph Γ is a connected semisymmetric graph such that $Aut(\Gamma) \cong G$. Then*

(i) Δ is connected;

(ii) Δ is not bipartite;

(iii) Δ and Δ^t are not isomorphic, here $\Delta^t = (\Omega, R^t)$, $R^t = \{(y, x) | (x, y) \in R\}$;

(iv) $Aut(\Delta) = G$.

Proof. The condition (i) is evident. The condition (ii) follows from Proposition 3.1(ii).

Assume that $\Delta^t \cong \Delta$. Then we check that there exists an isomorphism $\phi : \Delta^t \to \Delta$ such that $\phi^2 \in G$. Define $D = \langle G, \phi\rangle$ and check that (D, Ω) is an overgroup of G, and G has index 2 in D. Note also that according to the condition $Aut(\Gamma) = G$ the triple (G, H_1, H_2) is maximal in its class, where H_1 and H_2 are the stabilizers in G of the ends of an arc $(x, y) \in R$. Now to prove (iii) use Lemma 4.7. The proof of (iv) follows from the fact that if G is a proper subgroup of $G' = Aut(\Delta)$ then the triple (G', H'_1, H'_2) for suitably defined H'_1, H'_2 will be equivalent to (G, H_1, H_2), and thus $G' \le Aut(\Gamma)$, providing a contradiction with the assumption of the theorem. □

Remark. The slight difference in arguments which are used in [55], relies on the additional assumption of the primitivity of the group (G, Ω) in [55]. This is why we were not able to provide a stronger result for the necessary and sufficient condition of the incidence double cover to be a semisymmetric graph.

5 Nikolaev graph $\mathcal{N}$

Let us now consider the graph $\mathcal{N}$, which was discovered on October 30, 1977 at Nikolaev (Ukraine). It is the first member of an infinite family of semisymmetric graphs. The result was published in [62], where the term "semisymmetric" was coined. The main motivation of [62] was to provide an affirmative answer to a question of Folkman [40] about the existence of a semisymmetric graph with v vertices and valency k, such that $\gcd(v,k) = 1$. Indeed, for the graph $\mathcal{N}$ we get $\gcd(112, 15) = 1$.

The construction of $\mathcal{N} = (V, E)$ is as follows:

Let the set of vertices $V = V_1 \cup V_2$, $V_1 = \{(a,b)|a,b \in [0,7], a \neq b\}$ and $V_2 = \{X \subseteq [0,7]||X| = 3\}$. The edge set E of $\mathcal{N}$ is $E = \{\{(a,x),\{a,b,c\}\}|x \notin \{a,b,c\}\}$.

Proposition 5.1. *(i) $\mathcal{N}$ is a semisymmetric graph with 112 vertices and valency 15;*

(ii) $Aut(\mathcal{N}) \cong S_8$.

Proof. Clearly, $\mathcal{N}$ is a regular graph of valency 15, and the symmetric group S_8 acts transitively on the sets V_1, V_2 and E. Let $G = Aut(\mathcal{N})$. Regarding $\mathcal{N}$ as the incidence graph of a symmetric incidence structure $\mathfrak{S}$, let us consider the point graph $\mathcal{P}(\mathfrak{S})$ and the block graph $\mathcal{B}(\mathfrak{S})$ defined on the sets V_1 and V_2 respectively. Easy arguments reveal that the automorphism groups of the graphs $\mathcal{P}(\mathfrak{S})$ and $\mathcal{B}(\mathfrak{S})$ are imprimitive and primitive respectively. Therefore graph $\mathcal{N}$ is indeed semisymmetric. Consideration in addition of the 2-closure of the induced symmetric group S_8, acting on the set $\left\{ {[0,7] \atop 3} \right\}$, cf. [60], [61], [35], shows that $Aut(\mathcal{B}(\mathfrak{S})) = S_8$. Therefore, finally we get also that $Aut(\mathcal{N}) \cong S_8$. □

In fact, the proof of Proposition 5.1, outlined above, works for any arbitrary member of the infinite series of semisymmetric graphs introduced in [62]. Note that the idea of the proof, which was actually provided in [62], is of a more elementary nature, basing itself entirely on the counting of simple combinatorial invariants of the vertices in the graph $\mathcal{N}$. We refer to the Section 16 for an additional discussion.

Thus the graph $\mathcal{N}$ serves as a nice example of an "easy" case of semisymmetric graphs: here, as in Example 4.1, the fact that $Aut(\mathcal{N})$ acts intransitively on the set V can be justified by simple arguments of a combinatorial or group-theoretic nature. In what follows, the part V_1 (V_2) will be called left (right) part of the vertex set V of the graph $\mathcal{N}$, respectively.

6 Ljubljana graph $\mathcal{L}$

The number three is the smallest value of the valency of a semisymmetric graph. This implies a natural interest in the investigation of the smallest cubic semisymmetric graphs. Two such small graphs were known for a long while.

The Gray graph has 54 vertices, its automorphism group has order 1296 and is isomorphic to the exponentiation $S_3 \uparrow S_3$ of symmetric group of degree 3 (we use terminology and notation as in [37]). The construction of the Gray graph is quite simple. Start with the Hamming graph $H(3,3)$: the vertices are ternary sequences of length 3, two sequences are adjacent iff they differ in exactly one position. This graph $H(3,3)$ has valency 3 and diameter 3, it is a distance transitive graph. There are 27 natural cliques of size 3, which are defined in a unified manner: fix values of two prescribed coordinates while the remaining one runs through all three possibilities. Points and cliques define a symmetric configuration 27_3, the incidence graph of which is the Gray graph $\mathfrak{G}$. Simple arguments confirm that $Aut(\mathfrak{G}) = S_3 \uparrow S_3$ and $\mathfrak{G}$ is indeed a semisymmetric graph.

The graph $\mathfrak{G}$ was presented for the first time in [7] (see also [8]), where its authorship was attributed to Dr. Marion C. Gray (1932), though Bouwer rediscovered it independently. Since that time the graph $\mathfrak{G}$ was the subject of diverse investigations, see e.g. [83], [84], [20], [95]. A few times it was proved (with and without the use of a computer) that $\mathfrak{G}$ is the smallest cubic semisymmetric graph. The authors' (MK and MZA) own view of the graph $\mathfrak{G}$ and some of its properties is reflected e.g. in [66].

The second smallest cubic semisymmetric graph is the biprimitive graph on 110 vertices which was discovered by A. A. Ivanov [52]. This graph $\mathfrak{J}$ provides an answer to another question posed by Folkman [40], concerning the existence of a semisymmetric graph of a prime valency which does not divide the number of vertices. There are a few different (though rather close) ways to define the graph $\mathfrak{J}$. First, consider the group $PSL(2,11)$ of order 660. It has subgroups $H_1 \cong A_4$ and $H_2 \cong D_6$, both of index 55, such that $H_1 \cap H_2 \cong E_4$. Therefore $PSL(2,11)$ appears as an amalgam $\langle A_4, D_6, E_4 \rangle$. The vertices of $\mathfrak{J}$ are cosets of A_4 and D_6 in $PSL(2,11)$, while edges are cosets of E_4. Note that $Aut(\mathfrak{J}) \cong PGL(2,11)$ is a group twice larger than $PSL(2,11)$.

Alternative description of the graph $\mathfrak{J}$ (which was also outlined in [52]) relies on the use of an auxiliary structure, namely the Paley design $\mathcal{P}(11)$ with the parameters $(v,k,\lambda) = (11,5,2)$. In this interpretation the vertices of $\mathfrak{J}$ are 55 flags of $\mathcal{P}(11)$ and 55 "double flags", that is 2-element subsets of points belonging to the same $\lambda = 2$ blocks of D. The adjacency is a naturally defined "reverse" inclusion of a flag into a double flag (see [52] for details).

Note that the graph $\mathfrak{J}$ appears also in [55] as the smallest case in the list of five biprimitive cubic semisymmetric graphs. Like all other graphs in the list, $\mathfrak{J}$ is of a parabolic type in the sense of [55].

In 2001, during a brief visit of M. Conder to Ljubljana, he constructed together with Slovenian colleagues a cubic semisymmetric graph on 112 vertices, which was described as a regular $\mathbb{Z}_2^3$-cover of the Heawood graph. According to a suggestion of Conder, the graph was called the Ljubljana graph and denoted by $\mathcal{L}$. A computer based search showed that $\mathcal{L}$ is the unique cubic semisymmetric graph on 112 vertices.

In fact, already in [8] one finds a citation of a private communication of R. M. Foster, who found a cubic semisymmetric graph on 112 vertices with girth

10. However, Foster did not communicate to Bouwer any description of his graph. Thus there was an evident reason to attribute the suggested name, due to a lucky reincarnation of $\mathcal{L}$ which was achieved in the capital of Slovenia. A detailed report about the graph $\mathcal{L}$ was published [19]. Soon the suggested name became well known, see e.g. [76].

A more involved computer search (announced already in [19]) revealed that the graph $\mathcal{L}$ is in fact the third smallest cubic semisymmetric graph (after the graph $\mathfrak{G}$ and $\mathfrak{J}$). Taking into account that the original description of $\mathcal{L}$ in [19] was computer dependent (contrary to the known self-contained descriptions of $\mathfrak{G}$ and $\mathfrak{J}$), the graph $\mathcal{L}$ soon became the subject of further considerations and discussions, see e.g. [79], [9], [77], [20]. Note that in [20] cubic semisymmetric graphs are considered in a much wider context, relying on a beautiful impact of diverse techniques from group theory, topological graph theory and computer algebra.

An interesting paradox was already mentioned in [77]: The graph $\mathcal{L}$ was studied in a series of papers by I. J. Dejter and his coauthors [5], [11], [22], [23], which were not known to the authors of [19] in year 2001. We adopt, in this paper, the existing wording "Ljubljana graph". As sometimes happens in mathematics, some names seem luckier than others, and under this name this graph is now enjoying a new wave of attention.

The paper [19] indeed provides a lot of interesting information about the graph $\mathcal{L}$.

The graph is defined in an evident form with the aid of voltage assignments; cycles of length 10 and 12 are completely classified; $\mathcal{L}$ is proved to be Hamiltonian and thus its LCF code (in the sense of [41]) is provided. The group $Aut(\mathcal{L})$ of order 168 is discussed together with its action on $\mathcal{L}$ and some subgroups. Moreover, it is shown that the edge graph $L(\mathcal{L})$ of $\mathcal{L}$ is a Cayley graph over $Aut(\mathcal{L})$.

We think, however, that there is still an unexploited potential to reconsider the graph $\mathcal{L}$ once more together with the group $Aut(\mathcal{L})$, paying special attention to a few association schemes and coherent configurations naturally related to $\mathcal{L}$, as well as to the embeddings of $\mathcal{L}$ into the graph $\mathcal{N}$.

7 A master association scheme on 56 points

In our attempts to get a new understanding of the graph $\mathcal{L}$ we started from the group $G = A\Gamma L(1,8) := \{x \mapsto ax^\sigma + b | a \in F_8^*, b \in F_8, \sigma \in Aut(F_8)\}$.

Clearly, $|G| = 8\cdot 7\cdot 3 = 168$ and G acts naturally on the set of elements of the Galois field F_8 as a 2-transitive permutation group. Identifying F_8 with the set $[0,7]$, we use the representation $G = \langle g_1, g_2, g_3\rangle$, where $g_1 = (1,2,3,4,5,6,7)$, $g_2 = (0,1)(2,4)(3,7)(5,6)$, $g_3 = (2,3,5)(4,7,6)$, as it appears in [97].

G is a subgroup of S_8, therefore there is a sense to consider again the induced action of G on the same set $V = V_1 \cup V_2$, as it was defined in Section 5.

With the aid of a computer it was discovered that in this way we obtain exactly 8 distinct copies of the same (up to isomorphism) graph $\mathcal{L}$, which are invariant with respect to the induced intransitive action (G, V). Each such copy appears as a spanning subgraph of a suitable copy of $\mathcal{N}$.

The stabilizer of an arbitrary vertex in $\mathcal{L}$ has order 3; thus both stabilizers are isomorphic to the cyclic group $\mathbb{Z}_3$. Therefore, in comparison with the "easy" case of $\mathcal{N}$, this vision of $\mathcal{L}$ stresses that it belongs to a more difficult case. In the following we aim to interpret the graph $\mathcal{L}$ (as well as its embeddings to $\mathcal{N}$), starting from the association scheme formed by the 2-orbits of the transitive permutation group (G, V_1). At the beginning we will essentially rely on the analysis of some computations fulfilled with the aid of computer algebra packages.

Thus let now $\Omega = V_1 = \{(x, y) | x, y \in F_8, x \neq y\}$ and let (G, Ω) be the induced transitive action of $G = A\Gamma L(1, 8)$ on Ω of degree 56.

Proposition 7.1. *The permutation group* (G, Ω) *has rank 20.*

Proof. The rank of a transitive permutation group by definition is equal to the number of orbits of the stabilizer of an arbitrary point. The stabilizer of any point from Ω is similar to the induced cyclic group $(\mathbb{Z}_3, \Omega)$, $Z_3 = \langle \tilde{g}_3 \rangle$, where $\tilde{g}_3$ denotes the action of g_3 on Ω. With the aid of the orbit counting lemma (CFB lemma in [67]), we obtain for the rank r of (G, Ω) that $r = \frac{1}{3}(\binom{8}{2} + 2 \cdot 2) = 20$. $\square$

Using COCO in conjunction with GAP, we construct and investigate our master association scheme $\mathfrak{M} = (\Omega, 2 - orb(G, \Omega))$.

COCO returns a list of representatives of the 20 2-orbits, finds length of the 2-orbits, calculates the intersection numbers of $\mathfrak{M}$, enumerates all mergings of $\mathfrak{M}$ and provides the order of the automorphism group of each merging (together with the rank and subdegrees of each group). GAP allows us to get some extra information, in particular about the basic graphs of $\mathfrak{M}$. The first part of the results obtained is presented below.

Proposition 7.2. *(i) There are 8 pairs of antisymmetric basic relations of valency 3 in* $\mathfrak{M}$.

(ii) All these basic graphs are not bipartite.

(iii) 6 pairs of basic graphs are connected.

(iv) The automorphism group of each of those $6 \cdot 2 = 12$ *connected (di-)graphs is* (G, Ω).

(v) In each pair of connected basic graphs, opposite graphs are not isomorphic.

According to the criterion, presented in Section 4, there is now an evident sense to construct the incidence double cover on 112 vertices, starting from each pair $\{R, R^t\}$ of connected antisymmetric basic graphs of valency 3. Clearly the IDC of R and R^t are isomorphic (undirected) graphs of valency 3. With the aid of GAP we distinguish the 6 pairs into 4 "good" pairs, which all provide isomorphic copies of $\mathcal{L}$ and 2 "bad" pairs, which provide a vertex transitive disconnected graph, isomorphic to 8 copies of the Heawood graph.

i	Rep	Pair	Val	Con	R^t	R^*	$\|Aut\|$	cl	cl v	Aut(v)	rank	#
0	0	(0,1)	1	F	0	0	56!	1	1	$S_{56} \wr S_2$	2	
1	1	(0,2)	3	F	5	7	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	35
2	2	(1,0)	1	F	2	2	$28! \cdot 2^{28}$	3	1	$S_{56} \wr S_2$	3	48
3	4	(1,4)	3	T	12	4	168	4	2	$S_8 \wr F_{21}$	20	
4	5	(2,0)	3	T	8	3	168	4	2	$S_8 \wr F_{21}$	20	
5	6	(0,4)	3	F	1	9	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	35
6	9	(2,5)	3	T	17	14	168	5	3	G	20	
7	11	(4,1)	3	F	9	1	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	39
8	12	(1,2)	3	T	4	12	168	6	2	$S_8 \wr F_{21}$	20	
9	14	(2,1)	3	F	7	5	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	39
10	17	(3,6)	3	T	11	16	168	7	3	G	20	
11	18	(4,6)	3	T	10	13	168	8	3	G	20	
12	20	(4,0)	3	T	3	8	168	6	2	$S_8 \wr F_{21}$	20	
13	23	(5,2)	3	T	16	11	168	8	3	G	20	
14	29	(4,7)	3	T	15	6	168	5	3	G	20	
15	30	(7,5)	3	T	14	17	168	9	3	G	20	
16	32	(5,7)	3	T	13	10	168	7	3	G	20	
17	39	(6,3)	3	T	6	15	168	9	3	G	20	
18	43	(2,4)	3	F	18	18	$14! \cdot (4!)^{14}$	10	4	$S_{14} \wr (S_2 \times S_4)$	3	49
19	44	(4,2)	3	F	19	19	$7! \cdot 48^7$	11	4	$S_{14} \wr (S_2 \times S_4)$	5	24

Table 7.1: 2-orbits of $\mathfrak{M}$ and their covers

To explain better the observed phenomena, we further consider the normalizer $N_{S_{56}}((G,\Omega))$ of G in $S(\Omega)$, which in this case coincides with the group $CAut(\mathfrak{M})$.

The second part of the corresponding computer aided results is presented below.

Proposition 7.3. *(i) $N_{S_{56}}((G,\Omega)) \cong G \times \mathbb{Z}_2$ and has order 336.*

(ii) The quotient group $N_{S_{56}}((G,\Omega))/G$ acts on the 16 antisymmetric 2-orbits as a group of order 2.

(iii) Each "good" 2-orbit R is mapped to a 2-orbit R^* from another "good" pair under this action.

(iv) Each "bad" 2-orbit is mapped to a 2-orbit from another "bad" pair.

Note that the action of the direct factor $\mathbb{Z}_2$ on Ω corresponds to the permutation which transposes each pair (x,y) with (y,x) for distinct elements $x,y \in F_8$.

For the reader's convenience, the main numerical results related to the above propositions are presented in Table 7.1. Here first we list number i of class R_i, a representative $x \in \Omega$ such that $(0,x) \in R_i$, and a description $x = (a,b)$, $a,b \in \mathcal{F}_8$. In the last column of the table we refer to the number of a merging of $\mathfrak{M}$ which is the coherent closure of R_i, according to the list of all mergings, which appears in Supplement A.

Note that we get 11 isomorphism classes of basic graphs of $\mathfrak{M}$, while 4 such classes form the 4 "good" pairs. The IDC covers split into 4 isomorphism classes, described in column "cl v". The class 3 provides the graph $\mathcal{L}$. Again, the information in the last two columns of the table is relevant to coherent closure of basic graphs (which in most cases coincides with $\mathfrak{M}$). F_{21} denotes the Frobenius group of order 21 and degree 7.

8 Embeddings of $\mathcal{L}$ into $\mathcal{N}$

We wish now to understand better all possible embeddings of the graph $\mathcal{L}$ into $\mathcal{N}$. Note that the union of each "good" pair of relations R and R^* is again an antisymmetric relation (of valency 6). Moreover, each such relation is a 2-orbit of the group $\widetilde{\mathfrak{G}} = CAut(\mathfrak{M}) \cong A\Gamma L(2,8) \times \mathbb{Z}_2$. Therefore there is a sense to consider also an association scheme $\widetilde{\mathfrak{M}}$, resulting from the group $\widetilde{\mathfrak{G}}$. In principle, $\widetilde{\mathfrak{M}}$ appears as a merging (#1)of $\mathfrak{M}$. Nevertheless, for us it was more convenient to investigate $\widetilde{\mathfrak{M}}$ independently, using again COCO, and constructing the scheme of 2-orbits of $\widetilde{\mathfrak{G}}$.

We obtain that $(\widetilde{\mathfrak{G}}, \Omega)$ has rank 12 with 4 pairs of antisymmetric 2-orbits of valency 6. For each such 2-orbit we again construct its IDC; for two pairs the resulted cover turns out to be a semisymmetric graph on 112 vertices of valency 6, the automorphism group of which is the group $\widetilde{\mathfrak{G}}$. We prefer to call this graph of valency 6 the *natural* double Ljubljana graph and denote it by $\mathcal{NL}$.

Again GAP is used in conjunction with COCO to obtain our next result.

Proposition 8.1. *(i) The union of edges from IDC $\mathcal{L}$ of a "good" relation R and $\mathcal{L}^*$ of R^* provides a semisymmetric double Ljubljana graph $\mathcal{NL}$ of valency 6 on 112 vertices.*

(ii) $Aut(\mathcal{NL}) = \widetilde{\mathfrak{G}}$.

(iii) $\mathcal{NL}$ appears as an incidence double cover of the antisymmetric 2-orbit $R \cup R^*$ of the group $\widetilde{\mathfrak{G}} = CAut(\mathfrak{M})$.

(iv) Each graph $\mathcal{NL}$ (as well as each graph $\mathcal{L}$) has a unique extension to a copy of graph $\mathcal{N}$ of valency 15 which is invariant with respect to (G, Ω).

Thus we have managed to explain more clearly the essence of the embeddings of a "difficult" case of $\mathcal{L}$ into an "easy" case of $\mathcal{N}$.

Since $Aut(\mathcal{L})$ respects this embedding, we obtain a new proof of the fact that $\mathcal{L}$ is a semisymmetric graph.

It is clear that at this stage all the results which we have presented depend essentially on the use of a computer. In next sections we aim to remove, at least in part, such dependence. For this purpose, additional combinatorial structures will be introduced and investigated.

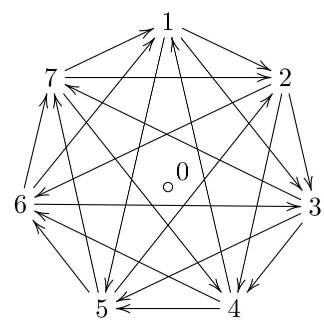


Figure 3: Paley tournament $P(7)$ with isolated vertex

9 The Ljubljana configuration

As was mentioned, each semisymmetric graph may and should be regarded as the Levi graph of a symmetric incidence structure (very frequently it happens to be a configuration), which is not self-dual. Let C and C^T be two such configurations, defined by the graph $\mathcal{L}$. The diagrams of this pair of configurations are depicted in Figure 5 of [19]; they are realized as geometric configurations of points and lines in the Euclidean plane.

Below we develop an alternative, combinatorial approach to the representation and investigation of the two 56_3 Ljubljana configurations and exploit its advantages.

First, let us consider a copy of a Paley tournament $P(7)$ with the vertex set $[1, 7]$ and isolated vertex 0, as it is depicted in Figure 3.

It is easy to check that $Aut(P(7)) = \langle g_1, g_3 \rangle$ (we use notation from Section 7) is a Frobenius group F_{21} of order 21 and degree 7. Recall that this copy of F_{21} is simultaneously the stabilizer of the point 0 in the group $(G, [0, 7]) = (G, F_8)$.

Consider now the orbit $\mathcal{O}$ of this graph $P(7)$ under the action of (G, F_8). This orbit $\mathcal{O}$ contains 8 copies of $P(7)$, where each element from $[0, 7]$ appears exactly once as an isolated vertex. For each copy of $P(7)$ in $\mathcal{O}$ and for each vertex x of $P(7)$ we get the induced subgraph $T(x)$, generated by the out-neighbors of x. Clearly, $T(x)$ is a directed triangle. Denote by $\mathcal{B}$ the collection of all such triangles, $T(x)$. We are ready to present our first construction.

Let us identify a tuple (x, y) in Ω with vertex y of the copy of $P(7)$ that has x as an isolated point. Here Ω is defined as in Section 7.

Construction 9.1. *Consider the incidence structure $\mathfrak{S} = (\Omega, \mathcal{B})$ with inclusion in the role of incidence relation.*

Proposition 9.1. *(i) $|\mathcal{O}| = 8$, $|\mathcal{B}| = 56$, $\mathfrak{S}$ is a symmetric 56_3 configuration without repeated blocks.*

(ii) $Aut(\mathfrak{S}) = G$.

(iii) The Levi graph of the configuration $\mathfrak{S}$ is isomorphic to $\mathcal{L}$.

Proof. The proof of (i) is a trivial consequence of the 2-transitivity of (G, F_8).

For the proof of the remaining parts, let us consider the point graph $\mathcal{P}$ and the block graph $\mathfrak{B}$ of the configuration $\mathfrak{S}$. Clearly, both graphs have valency 6.

Let us establish that the diameter of the point graph $\mathcal{P}$ is 3; for each of its vertices there are exactly 6, 25 and 24 vertices at distance 1, 2 and 3 respectively. We need also to prove that $Aut(\mathcal{P}) = G$. In principle, it is possible to elaborate a computer free proof, checking that each automorphism of $\mathcal{P}$ which fixes a vertex (say $(0,1)$) and all its neighbors in $\mathcal{P}$ is the identity automorphism. Practically, we advise the use a computer at least for the simple enumeration of the above distance-i subsets ($i \in \{1,2,3\}$) and inspection of the induced subgraphs of $\mathcal{P}$. As a corollary we get that $Aut(\mathcal{P}) \cong (G, \Omega)$ and therefore also $Aut(\mathfrak{S}) \cong (G, \Omega)$.

To prove that $\mathfrak{S} \ncong \mathfrak{S}^T$, we consider the block graph $\mathfrak{B}$, revealing that it has diameter equal to 4. (Moreover we obtain that the distance-4 set for any vertex has cardinality 1.)

Thus, $\mathcal{P} \ncong \mathfrak{B}$ and therefore $\mathfrak{S}$ is not self-dual. This implies that the Levi graph of the configuration $\mathfrak{S}$ is semisymmetric.

To complete the proof, we have to decide which is the Ljubljana graph $\mathcal{L}$. Depending on this, we may then check with the aid of GAP that the Levi graph of $\mathfrak{S}$ is isomorphic to any of the copies of $\mathcal{L}$ which we have constructed above.

Otherwise, we may exploit the fact that $\mathcal{L}$ is unique semisymmetric graph on 112 vertices of valency 3. □

In what follows, the part $\mathcal{P}$ of the vertex set of the current model of $\mathcal{L}$ will be called left, while the part $\mathfrak{B}$ will be called right.

Remark. A quite surprising issue is that the copy $\mathfrak{S}$ of the Ljubljana configuration, defined and considered in this section, does not literally coincide with any of the 8 copies of the configuration which appear from $\mathfrak{M}$ via the IDC procedure. Moreover, our group $(G, \Omega) \neq Aut(\mathfrak{S})$, though of course, the two groups are conjugate in S_{56}.

We will come to the discussion of this phenomenon at the end of our paper.

10 More auxiliary structures

We are now in a position to present a new way of taking a glance at all possible embeddings of the Ljubljana graph $\mathcal{L}$ into the Nikolaev graph $\mathcal{N}$. With this aim in mind, let us consider extra concepts and related combinatorial structures.

An arbitrary permutation group (H, X), according to [3], is called a *geometric* group if it appears as the full automorphism group of a graph or a hypergraph with the vertex set X. Here, a *hypergraph* is a collection of subsets of the set X (*hyperedges*) together with the entire (vertex) set X. In other words, a geometric permutation group can be interpreted as the group of all symmetries of a suitable incidence structure. Let us illustrate this concept with the aid of a simple example.

Example 10.1. *Let* $H = F_{20} = AGL(1,5) = \langle h_1, h_2 \rangle$*, where* $h_1 = (0,1,2,3,4)$*,* $h_2 = (1,2,4,3)$*. It is clear that* H *is a 2-transitive Frobenius group of order 20.*

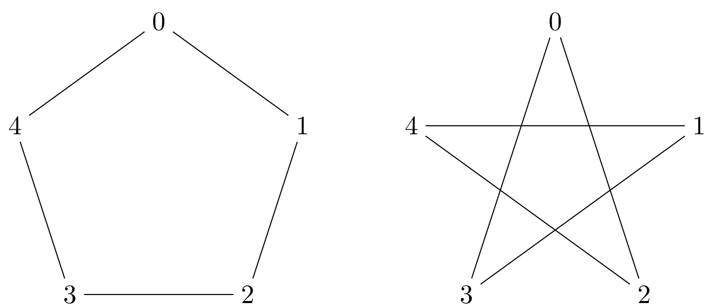


Figure 4: Two complementary pentagons

Therefore H cannot be regarded as $Aut(\Sigma)$ for a suitable graph Σ with vertex set $[0, 4]$. It is easy to observe that the group H cannot also appear as the automorphism group of a suitable hypergraph with the vertex set $[0, 4]$. Nevertheless, H is the full automorphism group of a collection of two complementary pentagons, depicted in Figure 4.

According to [69], let us call a permutation group (H, X) a *geometric group of the second order* if (H, X) is the full automorphism group of a suitable collection of graphs or hypergraphs. As a simple illustration, the above example demonstrates that the group $AGL(1, 5)$ is indeed a geometric group of the second order.

Let us now consider again the group $G = A\Gamma L(1, 8)$. This group, of course, is not the full automorphism group of any graph. Counting orbits of (G, Ω) on the $3-$ and $4-$subsets of F_8, and comparing the obtained numbers with the similar ones for an overgroup $AGL(3, 2)$ of (G, F_8) in $Sym(F_8)$ (cf. e.g. [97]), it is easy to reveal that (G, Ω) is not a geometric group (see [69], for example, for details).

At this point let us consider again one more famous structure, namely the projective plane $\mathcal{F} = PG(2, 2)$, commonly known under the name of Fano plane. The classical picture in Figure 5(a) depicts a difference set model for $\mathcal{F}$.

Indeed, the seven lines of this model are obtained from the line $\{1, 2, 4\}$ via consecutive cyclic shifts with the aid of the permutation g_1 (see Section 7).

In a similar manner one more model (depicted in part (b) of the same Figure) appears from the line $\{3, 5, 6\}$. Let us call these two models of $\mathcal{F}$ the *standard* and the *non-standard* respectively. Clearly, the two models are isomorphic and have disjoint sets of lines. Both models are invariant with respect to the same cyclic group $\langle g_1 \rangle$ of order 7.

According to [18], an *overlarge set* $\mathcal{O}(v, k)$ of Steiner triple systems $S(2, 3, v)$ is a partition of the set of all 3-element subsets of a $(v + 1)$-element set into $v + 1$ disjoint Steiner systems, each of type $S(2, 3, v)$. A similar definition may be formulated for the case of Steiner systems $S(k-1, k, v)$. A pioneering paper, in which such overlarge sets were investigated, is [96] (though the name itself was coined later on).

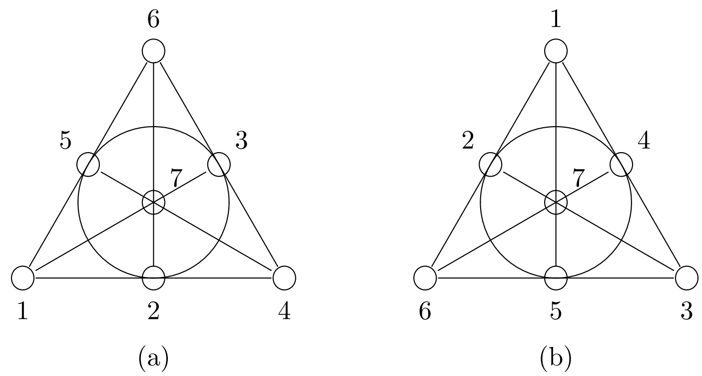


Figure 5: Two models of Fano plane

It was proved in [96] that up to isomorphism, there exist 11 different $\mathcal{O}(7,3)$ on 8 points, having groups of order 1344, 168, 96, 64, 48, 24, 24, 8, 8, 8, 6. Of course, all these groups provide particular examples of geometric groups. Two most symmetric models for $\mathcal{O}(7,3)$ are of a particular interest in this text. Below we briefly repeat the representations of these models, as they are described in [69].

For this purpose, first we introduce a copy E_8 of an elementary Abelian group of order 8 acting regularly on the set $F_8 = [0,7]$. E_8 contains exactly 7 involutions t_i as follows: $t_1 = (0,1)(2,4)(3,7)(5,6)$, $t_2 = (0,2)(1,4)(3,6)(5,7)$, $t_3 = (0,3)(1,7)(2,6)(4,5)$, $t_4 = (0,4)(1,2)(3,5)(6,7)$, $t_5 = (0,5)(1,6)(2,7)(3,4)$, $t_6 = (0,6)(1,5)(2,3)(4,7)$, $t_7 = (0,7)(1,3)(2,5)(4,6)$.

Note that $t_1 = g_2$; it is easy to check that the group $AGL(1,8) = \langle g_1, t_1\rangle$ has order 56, acts sharply 2-transitively on F_8 and is a subgroup of index 3 in our group G.

Construction 10.1. *Let us now regard our standard (non-standard) copy of $\mathcal{F}$ as a copy $\mathcal{F}_0$ carrying an extra isolated point* 0*. Then we define 7 new copies of $\mathcal{F}$ as the respective images $\mathcal{F}_i := \mathcal{F}_0^{t_i}$. Finally , we denote by $\mathcal{O}_S(7,3)$ (or $\mathcal{O}_N(7,3)$) the collection of* 8 *Fano planes $\{\mathcal{F}_0, \ldots, \mathcal{F}_7\}$, depending on which model of $\mathcal{F}$ (standard or non-standard) is used for the initial copy $\mathcal{F}_0$.*

Proposition 10.1. *(i) $\mathcal{O}_S = \mathcal{O}_S(7,3)$ is an overlarge set with the automorphism group $Aut(\mathcal{O}_S)$ isomorphic to $AGL(3,2)$, a group of order 1344.*

(ii) $\mathcal{O}_N = \mathcal{O}_N(7,3)$ is an overlarge set with the automorphism group $Aut(\mathcal{O}_N) = G = A\Gamma L(1,8)$ of order 168.

Proof. A computer free proof is presented in [69], though of course, the reader can easily confirm it with the aid of a computer, or even by routine hand computations. $\square$

Corollary. The group $G = (A\Gamma L(1,8), \Omega)$ is a geometric group of the second order.

Remark. A mysterious (at first sight) distinction between the standard and non-standard models of $\mathcal{F}$ relies on the different roles of the selected copy of E_8 with respect to the two prescribed models of the Fano plane.

Finally, we wish to recall one more concept which is of a definite independent interest.

A *partial geometry* $PG(K,R,T)$ is an incidence structure in which

- each line contains exactly K points;
- each point lies on exactly R lines;
- every pair of distinct points lies on at most one common line;
- every pair of distinct lines contains at most one common point;
- for any line l and point P such that $P \notin l$, there are exactly T lines containing P that intersect the line l.

(We are using definitions and notation exactly as it appeared in the seminal paper by Bose [6], who coined the concept of a partial geometry.)

Bose proved in [6] that the point graph Γ of any partial geometry $PG(K,R,T)$ is an SRG and derived formulas for the parameters of Γ in terms of K, R and T. A nice survey of known partial geometries appears in two papers by F. De Clerck [17], [16].

Partial geometries $PG(8,9,4)$ (their point graph is an SRG with the parameters $(120,63,30,36)$) have a rather striking history. A number of experts discovered a few independent models of such a structure, which later on were proved to be isomorphic. The long standing question about the existence of other non-isomorphic examples was positively resolved by Mathon & Street and independently by MK and S. Reichard (all references are available in [16]). For a few of these new geometries the point graph is isomorphic and coincides with the one described in [12]. Here we are especially interested in that geometry for which the automorphism group is isomorphic to A_8. A model of this A_8-geometry was suggested in [69], it is briefly repeated below.

Construction 10.2. *Start with the non-standard model $\mathcal{F}_N$ of the Fano plane and consider the orbit $\mathcal{P}$ of $\mathcal{F}_N$ under the action of A_8. We get $|\mathcal{P}| = \frac{|A_8|}{|PSL(2,7)|} = \frac{8!}{2\cdot 168} = 120$. Consider also orbits $\mathcal{O}_N^{A_8}$ and $(\mathcal{O}_S^z)^{A_8}$ of non-standard and "skew" standard overlarge sets respectively under the action of A_8. Check that $|\mathcal{O}_N^{A_8}| = 120$, $|(\mathcal{O}_S^z)^{A_8}| = 15$. Define $L = \mathcal{O}_N^{A_8} \cup (\mathcal{O}_S^z)^{A_8}$ and consider incidence structure $(\mathcal{P}, L)$ with incidence defined as inclusion.*

Remark. Here $z = (2,7)(3,6)(4,5)$, therefore $\mathcal{O}_S^z$ is another isomorphic copy of $\mathcal{O}_S(7,3)$. Its advantage over $\mathcal{O}_S$ is that $\mathcal{F}_S^z$ belongs to the orbit of $\mathcal{F}_N$ under the action of A_8, so the Fano planes in the overlarge sets in $(\mathcal{O}_S^z)^{A_8}$ are the same as those in $\mathcal{O}_N^{A_8}$.

Proposition 10.2. *(i) The incidence structure $(\mathcal{P}, L)$ provides a model of $PG(8, 9, 4)$.*

(ii) The automorphism group of this partial geometry is isomorphic to the alternating group A_8.

Proof. A proof is available in [69], it is computer free and relies on a number of combinatorial and group theoretical arguments. □

In the next section we will try to benefit from the consideration of the presented A_8-geometry at least implicitely. For this purpose let us present an alternative vision of both kinds of overlarge sets.

The first structure in the consideration is the affine space $\mathcal{A}(3, 2)$ with the vector space $(\mathbb{Z}_2)^3$ in the role of 8 points, while 14 blocks (aka affine subspaces of dimension 2) are the 7 subgroups of order 4 in $(\mathbb{Z}_2)^3$ and their cosets. The automorphism group of $\mathcal{A}(3, 2)$ is the group $AGL(3, 2)$ of order 1344, it is described as $AGL(3, 2) \cong \mathbb{Z}_2^3 : PSL(3, 2)$. This group acts 3-transitively on the point set $(Z_2)^3$ of cardinality 8.

Recall that $\mathcal{A}(3, 2)$ is the unique (up to isomorphism) Steiner system $S(3, 4, 8)$, that is the set of 14 4-element blocks such that each 3-element subset of points appears as a subset of exactly one block. For the reader's convenience, the list of blocks of a copy of $\mathcal{A}(3, 2)$ (suitable for our purposes) is given below: $\{0, 1, 2, 4\}$, $\{0, 1, 3, 7\}$, $\{0, 1, 5, 6\}$, $\{0, 2, 3, 5\}$, $\{0, 2, 6, 7\}$, $\{0, 3, 4, 6\}$, $\{0, 4, 5, 7\}$, $\{1, 2, 3, 6\}$, $\{1, 2, 5, 7\}$, $\{1, 3, 4, 5\}$, $\{1, 4, 6, 7\}$, $\{2, 3, 4, 7\}$, $\{2, 4, 5, 6\}$, $\{3, 5, 6, 7\}$.

We invite the reader to check that the next observation holds.

Proposition 10.3. *Consider the provided copy of $\mathcal{A}(3, 2)$, let x be an arbitrary point. Consider all 7 blocks containing x and remove x from them. Repeat this procedure for each $x \in [0, 7]$. Then*

(i) For each x we get a copy of the Fano plane with isolated point x.

(ii) The collection of the resulting 8 Fano planes coincides with the standard overlarge set $\mathcal{O}_S(7, 3)$, as it was defined above.

Remark. It is clear that in the reverse procedure one gets from $\mathcal{O}_S(7, 3)$ a copy of $S(3, 4, 8)$. Thus, the automorphism groups of $\mathcal{O}_S(7, 3)$ and of $\mathcal{A}(3, 2)$ indeed coincide. Therefore knowledge of one group automatically implied the knowledge of the other. We mention that though this proposition is not used explicitely below it, in our eyes, sheds extra light on the links between many diverse graphs introduced in our presentation.

Now we wish also to present an alternative vision of the construction of $\mathcal{O}_N(7, 3)$.

Start with a prescribed copy of $S(3, 4, 8)$ and with its group $AGL(3, 2)$ of order 1344. This group has 8 conjugate subgroups of order 168, each isomorphic to our group G. Select a copy of such a group G, and observe that the group G acts transitively on the set X of 56 4-subsets which is complementary to the block set of our Steiner design $S(3, 4, 8)$. Note that, for each point x, the stabilizer G_x of order 21 contains the unique cyclic subgroup $\mathbb{Z}_7$ of order 7. The 28 elements from X which contain x split into two orbits of G_x of the length 7 and 21. Consider the orbit of length 7 and remove x from each 4-subset. Get a copy of the Fano plane with the point set $X \setminus \{x\}$. In such a fashion we obtain eight copies of the Fano plane (with all possible isolated points) which form a copy of $\mathcal{O}_N(7,3)$.

For the reader's convenience in Supplement C we provide a list of all 8 Fano planes in the resulted copy of $O = O_N(7,3)$, when in the role of G our group, as concretely defined in Section 6 is used.

11 Embeddings of $\mathcal{L}$ into $\mathcal{N}$ revisited

Here we are interested in investigating once more all embeddings of $\mathcal{L}$ into a prescribed copy of $\mathcal{N}$, provided certain natural requirements are satisfied. These requirements will be formulated in group theoretical terms.

First, we start from the action of the group S_8 on the set V as it appears in Section 5. It is easy to understand that there are two copies of the graph $\mathcal{N}$ which are invariant with respect to (S_8, V): the one with the edge set E (as in Section 5) and the one with the edge set $E' = \{\{(x,a), \{a,b,c\}\} | x \notin \{a,b,c\}\}$. Both copies $\mathcal{N}$ and $\mathcal{N}'$ have the same group $Aut(\mathcal{N}) = Aut(\mathcal{N}') \cong S_8$. Moreover, these two copies are interchanged by the involution τ, which transposes pairs (a,b) and (b,a) from V_1 and fixes each element of V_2.

Thus, in principle, one may consider the group $S_8 \times S_2$, acting on the set V, and classify all embeddings into either $\mathcal{N}$ or $\mathcal{N}'$. We nevertheless prefer to fix a concrete *master copy* of the Nikolaev graph, say $\mathcal{N}$. In this fashion, the group S_8 still remains our "universal" group.

Let us now consider a concrete copy of the group $G = A\Gamma L(1,8)$ as a subgroup of S_8, and let us investigate all copies of the graph $\mathcal{L}$, which are invariant with respect to this selected group G and which are spanning subgraphs of the master graph $\mathcal{N}$.

As we may easily deduce from the analysis of the master association scheme $\mathfrak{M}$, there exist exactly 8 different copies of $\mathcal{L}$ which are invariant with respect to (G, V). However computer analysis shows that only two of these copies are spanning subgraphs of the same copy of $\mathcal{N}$. Denoting these copies by $\mathcal{L}$ and $\mathcal{L}'$, we observe that $\mathcal{L}$ and $\mathcal{L}'$ belong to different orbits of the group $S_8 = Aut(\mathcal{N})$. Let us summarize this part of our observations.

Proposition 11.1.

1. *For a given copy $\mathcal{N}$ of the Nikolaev graph and the group $Aut(\mathcal{N}) = S_8$ there exist exactly 1920 copies of the graph $\mathcal{L}$, which are invariant with respect to a suitable subgroup of S_8, isomorphic to G.*

2. *480 copies of $\mathcal{L}$ in addition are spanning subgraphs of the master copy $\mathcal{N}$ and form two orbits with respect to S_8.*

3. *For all these 480 copies, the left (right) part of $\mathcal{L}$ goes to the right (left) part of $\mathcal{N}$ respectively.*

4. *Each of the above two orbits of embeddings of $\mathcal{L}$ into $\mathcal{N}$ splits into two orbits of length 120 with respect to the group (A_8, V).*

Now we wish to define a representative $\mathcal{L}$ of one of four orbits of A_8. For this purpose first we need to use the alternative construction of $O_N(7,3)$ as it appears at the end of Section 10.

Let us describe a copy of graph $\mathcal{L}$. It has again vertex set $V = V_1 \cup V_2$, exactly like the graph $\mathcal{N}$ in Section 5. Consider the vertex (a,b) from V_1. Find a copy F_a of the Fano plane in our overlarge set O which does not contain vertex a. Find in this copy of F_a three lines through the point b. Substitute b by a in each of the three lines. Find the three neighbours of (a,b) in our copy $\mathcal{L}$. For example, for the pair $(0,1)$, according to our procedure, we obtain triples $\{0,2,6\}$, $\{0,3,4\}$, $\{0,5,7\}$. It is clear that the resulted graph $\mathcal{L}$ is indeed a spanning subgraph of $\mathcal{N}$ as it appears in Section 5.

In what follows we will call the above copy of $\mathcal{L} = (V,E)$ the *canonical Ljubljana subgraph* of the master copy $\mathcal{N}$ (with respect to a prescribed copy G of the group $A\Gamma L(1,8)$). The orbit of A_8 on the 120 copies of the canonical copy $\mathcal{L}$ should be called the *canonical set* $\mathcal{E}$ of the embeddings of $\mathcal{L}$ into the master copy of $\mathcal{N}$. This canonical set $\mathcal{E}$ is, in fact, the subject of our further investigations.

Remark. Clearly our description of $\mathcal{L}$ depends on the selection of a copy of a group conjugate to G in the congugacy class of A_8. There is a natural bijection between 120 such groups and 120 copies of $O_N(7,3)$, and finally with 120 embeddings of $\mathcal{L}$ into $\mathcal{N}$, forming our canonical set $\mathcal{E}$.

We consider the transitive permutation group $(A_8, \mathcal{E})$ and investigate it with the aid of COCO. Let $\mathcal{X} = (\mathcal{E}, 2{-}orb(A_8, \mathcal{E}))$ be the Schurian association scheme formed by the 2-orbits of $(A_8, \mathcal{E})$.

Proposition 11.2. 1. *The group $(A_8, \mathcal{E})$ has rank 5; valencies of non-reflexive 2-orbits R_i, $1 \le i \le 4$, are 42, 14, 56, 7; all the orbits are symmetric.*

2. *$\mathcal{X}$ has 6 non-trivial mergings $\mathcal{X}_1 = (1,2)$, $\mathcal{X}_2 = (1,3)$, $\mathcal{X}_3 = (3,4)$, $\mathcal{X}_4 = (1,2)(3,4)$, $\mathcal{X}_5 = (1,3,4)$, $X_6 = (1,2,3)$ (we use here brief COCO notation for the mergings).*

3. *The strongly regular graphs corresponding to the mergings X_4, X_5, X_6, have parameters $(120,56,28)$, $(120,14,13)$, $(120,7,6)$ respectively.*

4. $|Aut(\mathcal{X}_4)| = 1290240 = 2^5 \cdot 8!$.

Additional analysis shows that the strongly regular graph generating the rank 3 scheme $\mathcal{X}_4$ is isomorphic to the graph which was described in [12].

This bestows extra interest to the association scheme $\mathcal{X}$. In addition we observe that the basic graph Γ_3 of valency 56 is the Deza graph (in the sense of [27], [34]). We again summarize our computer aided discoveries.

Proposition 11.3. *A rank 5 Schurian association scheme $\mathcal{X}$ is generated by the Deza graph Γ_3.*

The elaboration of a computer free proof of the theorem deserves special attention, though it is out of the scope of the current paper.

According to the terminology introduced in [65], the scheme $\mathcal{X}$ provides an example of a Schurian Deza family in a Higmanian house. To the best of our knowledge, this fact is new. Moreover, it seems that after the Schurian example on 40 points described in [65], the current example on 120 points is the second non-trivial one which appears in the literature.

12 The Dejter approach to the Ljubljana graph

As was already mentioned, the alternative approach to the Ljubljana graph was developed by I. Dejter et al in a sequence of papers [24], [26], [11], [22], [25], [5], [23].

This approach is quite original. It differs very essentially from the other ways already presented.

In a few words,, the idea of Dejter looks as follows. Consider the 7-dimensional cube Q_7 as alternative "universal structure". Consider the subgraph of Q_7 induced by 16 vertices, which correspond to the Hamming code H_3 of dimension 4. Consider subgraph D of Q_7, which is induced by the vertices of $V(Q_7) \setminus H_3$. Check that D is vertex- and edge-transitive graph of valency 6 on 112 vertices. Split the graph D into two isomorphic copies of the graph $\mathcal{L}$. To outline this approach more closely, let us first discuss in more detail some extra necessary classical structures.

The n-dimensional cube Q_n has the vertex set $V(Q_n)$ which coincides with the set of binary sequences of length n, aka vector space of dimension n over the field $\mathbb{Z}_2$. It is convenient also to identify $V(Q_n)$ with the set of characteristic vectors of the power set (that is the set of all subsets) of the n-element set $[0, n-1]$. If necessary, it is convenient to abuse notation identifying the subsets of $[0, n-1]$ with their characteristic vectors. Two distinct elements A, B of $V(Q_n)$ are adjacent if the corresponding symmetric difference $A \triangle B$ has the smallest possible cardinality 1, in other words, if the characteristic vectors of A and B differ in exactly one position.

Denote by $\mathcal{E}_n$ the full automorphism group $Aut(Q_n)$ of the n-cube Q_n. It is well-known that $|\mathcal{E}_n| = 2^n \cdot n!$; the group $\mathcal{E}_n$ (as an abstract group) is isomorphic to the wreath product of symmetric groups of degree n and 2. The action of this wreath product on the set $V(Q_n)$ is usually called by group theorists the product action, see e.g. [29]. Another term, exponentiation, is frequently used in combinatorics (with credits to F. Harary [45]). Following a suggestion by L. A. Kalužnin, we use the notation $\mathcal{E}_n = S_2 \uparrow S_n$, see [59], [67], [37], [58]. Each element of the group $\mathcal{E}_n$ may be represented in the table format $t = (a, b(x))$ where $a \in S_n$, $b(x) : [0, n-1] \to S_2$, a function from $[0, n-1]$ into S_2, and for $x = (x_0, \ldots, x_{n-1}) \in V(Q_n)$, the image x^t is defined as having coordinate with number i equal to $(x_{i^{a^{-1}}})^{b(i)}$. Note that the group $\mathcal{E}_n$ can also be described as a semidirect product $(\mathbb{Z}_2)^n : S_n$ of the elementary Abelian group of order 2^n with S_n.

The next classical structure in our consideration is the Hamming code H_3 (in dimension 7). We regard it as a vector subspace of $V(Q_7)$ of dimension 4. This subset of cardinality 16 of $V(Q_n)$ is the linear span of the rows of the generating matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It can also be defined with the aid of a parity check matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

which in turn may be regarded as a generating matrix of the dual code H_3^D. (We refer to [50] for detailed consideration of these structures and their symmetry properties.)

The 16 vectors from the code H_3 correspond to seven 3-element subsets of $[1, 7]$, their seven complements, the empty subset and the entire set $[1, 7]$. It is easy to check that the seven 3-subsets form the set of lines of a copy $\mathcal{F}_H$ of the same Fano plane. (Note however that, due to the use of the canonical form for the generating matrix A, the copy $\mathcal{F}_H$ does not coincide with any of the copies of $\mathcal{F}$, considered in Section 10.) This easily implies that the automorphism group of H_3 (as it is defined in [50]) coincides with $Aut(\mathcal{F}_H) \cong PSL(3, 2)$ and has order 168. It is also significant to notice that the dual code H_3^D is a subcode of H_3, it corresponds to the line set of $\mathcal{F}_H$ together with the empty subset. Clearly $Aut(H_3^D) = Aut(H_3)$.

Let us now consider H_3 as the induced subgraph of Q_7. It is easy to check that it forms a coclique of Q_7 and thus each vertex from H_3 has all 7 neighbors in the set $V(Q_7) \setminus H_3$, which in this section will be denoted by Ω. The proposition below describes all the information that is required at this stage about the stabilizer F of H_3 in $Aut(Q_7)$. Though this proposition was also obtained with the aid of a computer, it may be justified using computer free arguments (see discussion below).

Proposition 12.1. *(i) $F \cong AGL(3, 2) \times S_2$, has order 2688 and acts transitively and faithfully on the subset H_3 of the set $V(Q_7)$.*

(ii) The group F acts transitively and faithfully on the Ω.

(iii) An equitable partition of Q_7 formed by the two orbits of size 16 and 112 of the group F has matrix $\begin{pmatrix} 0 & 7 \\ 1 & 6 \end{pmatrix}$.

(iv) The subgraph D of Q_7, induced by the vertex subset Ω, has valency 6 and is vertex- and edge-transitive.

(v) $Aut(D) \cong F = AGL(3, 2) \times S_2$.

In what follows we will call the graph D the *Dejter graph*.

We wish to provide for the graph D a self contained computer free description, which allows us also to figure out easily its full automorphism group.

For this purpose we consider ordered flags of the structure $\mathcal{A}(3,2)$ (See Section 10), that is pairs consisting of one of the 8 points and one block incident to it. We consider two kinds of such ordered flags that is (point, block) and (block, point). Clearly there are $8 \cdot 7 = 14 \cdot 4 = 56$ ordered flags of the first kind and the same number of the second kind. We consider the group F acting on the set of 112 flags; here elements of $AGL(3,2)$ act transitively on both sets of ordered flags of the first and second kind respectively, while the group S_2 interchanges simultaneously each flag (x, B) with its counterpart (B, x). Finally we have a transitive action of F on the entire set $\widetilde{\Omega}$ of 112 ordered flags of two kinds. Note also that if the list of 14 vertices of $A(3,2)$ is known then each ordered flag may be identified with a 3-subset of points (we have in addition to carry information about the kind of flag). Thus, for example the flag $(0, \{0,1,2,4\})$ gets the label $(\{1,2,4\}, 1)$, while the opposite flag $(\{0,1,2,4\}, 0)$ is labeled as $(\{1,2,4\}, 2)$. Both ways of coding flags may be useful.

Let us now define adjacency on ordered flags (here distinct letters designate distinct points of the selected copy of $A(3,2)$): flag $(\{a,b,c,d\}, d)$ is adjacent to flag $(a, \{a,d,e,f\})$. It is easy to check that the valency of the resulted graph $\widetilde{D}$ with the vertex set $\widetilde{\Omega}$ is equal to 6: For a given flag $(\{a,b,c,d\}, d)$ we put on the first position in adjacent flag any of the letters a, b, c; after that there are exactly two blocks, different from $\{a,b,c,d\}$ which contain $\{a,d\}$. The defined relation is symmetric.

Proposition 12.2. *(i) $Aut(\widetilde{D}) = F$;*

(ii) the graphs D and $\widetilde{D}$ are isomorphic.

Proof. As usual, both parts of the proposition were confirmed with the aid of a computer. Clearly $Aut(\widetilde{D})$ contains the group F. Suitable ad hoc arguments allow us to prove that $|Aut(\widetilde{D})| = |F|$, this implies (i).

To prove (ii) we have to reconstruct the structure of Q_7 from the graph D. Let us select from the point set of $\mathcal{A}(3,2)$ a distinguished point, say 0. We add to the vertex set of D 14 blocks of $\mathcal{A}(3,2)$ and two special symbols, $\emptyset$ and U.

Let us consider the sets V_1, V_2 as follows:

$V_1 = \{U, \{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{2,3,5\}, \{2,6,7\}, \{3,4,6\}, \{4,5,7\}\}$,

$V_2 = \{\emptyset, \{1,2,3,6\}, \{1,2,5,7\}, \{1,3,4,5\}, \{1,4,6,7\}, \{2,3,4,7\}, \{2,4,5,6\}, \{3,5,6,7\}\}$.

First, we naturally identify these 16 elements of $V_1 \cup V_2$ with the 16 elements of Hamming code $H(7,4)$.

We can identify elements of Q_7 (seen as subsets of $[1,7]$) with subsets of size 3 of V_1 and of V_2, together with the elements of V_1 and V_2. Each subset of $[1,7]$ which is not in $V_1 \cup V_2$ is identified with the elements of V_1 (or V_2) which are of Hamming distance 2 from it. There are always three such elements.

A subset of size 1, $\{a\}$ is identified with three sets from V_1 which include a.

A subset of size 2, is identified with $\emptyset$ and two sets including it in V_2.

A subset of size 3 which is not in V_1 has exactly three sets from V_1 with intersection of size 2.

A subset of size 4 which is not in V_2 has exactly three sets from V_2 with intersection of size 3.

A subset of size 5 is identified with U and two sets from V_1 that it contains.

A subset of size 6 is identified with three sets from V_2 not including its missing element.

This shows us that we can reconstruct Q_7 from $H(7,4)$.

Now we identify the 112 directed flags with subsets of $[1,7]$ which are not in $V_1 \cup V_2$.

We start by identifying arbitrarily each of the seven left flags denoted by the seven 3-subsets that belong to V_1 with one of the singleton subsets of $[1,7]$.

Each two flags from V_1 have a single right flag (from V_2) as common neighbor, since they intersect in one element. This right flag is identified with the union of two singletons. Each of these flags has 0 in the block, but not as the selected element.

Three right flags representing $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, have a single common left flag neighbor (element is 0 and block is 0 and three elements of the right flags). This is identified with $\{a,b,c\}$.

The automorphism of Q_7 defined by completion corresponds to exchanging the kinds automorphism, so identification with subsets of sizes 4,5,6 is defined. All that is left to check is that adjacency of sets of sizes 3 and 4 is the same as directed flags adjacency, which is immediate. □

The main reason for our interest in the Dejter graph in the context of this paper was the fact, originally observed by Dejter et al, that D splits into two copies of the graph $\mathcal{L}$. The proof of the existence of such a split depends on the selected model of the Dejter graph. An outline of such a proof for the model D, that is an induced subgraph of Q_7, was provided in [11].

We start with a given vertex, $(0,\{0,1,2,4\})$. Its 6 neighbours are $(\{0,1,3,7\},1)$, $(\{0,1,5,6\},1)$, $(\{0,2,3,5\},2)$, $(\{0,2,6,7\},2)$, $(\{0,3,4,6\},4)$, $(\{0,4,5,7\},4)$.

There are eight ways to select 3 out of its 6 neighbours in a way that is in a sense compatible with $Aut(\widetilde{D})$: Four by selecting an element x not in $\{0,1,2,4\}$ and taking the three neighbours with set containing x, and the other four by taking the neighbours with sets not containing x. Each of those sets of 3 neighbours extends in one of two ways to a graph isomorphic to $\mathcal{L}$ which is compatible with a subgroup $AGL(3,2)$ of $Aut(\widetilde{D})$. For each such subgraph, its complement graph in $\widetilde{D}$ is also isomorphic to $\mathcal{L}$, so we get a partition of the edge set of $\widetilde{D}$ into two sets of edges of a subgraph isomorphic to $\mathcal{L}$.

Proposition 12.3. *Let $\widetilde{D}$ be a Dejter graph with the automorphism group $AGL(3,2)$. There exists exactly 8 splits of the edge set of $\widetilde{D}$ into two complementary sets of edges of $\widetilde{\mathcal{L}}$ such that the group $Aut(\widetilde{\mathcal{L}})$ appears as a subgroup of index 8 in $AGL(3,2)$. Each such subgroup provides exactly one split.*

Remark. The presented model $\widetilde{D}$ of the Dejter graph and description of its split into two copies of $\widetilde{\mathcal{L}}$ strongly relies on the presented construction of $\mathcal{O}_N(7,3)$, which (in a sense) is an alternative to the one in [69]. In a wider context such kind of dependence is discussed in Section 15.

Finally we mention a few new interesting non-Schurian association schemes which were discovered in the course of the investigation of the centralizer algebra of the group $F = Aut(D)$.

The centralizer algebra W of the group F has rank 16. In fact, it is a direct product of two centralizer algebras of orders 56 and 2 and ranks 8 and 2 respectively. With the aid of COCO we investigated all coherent subalgebras of W. There are 81 such (non-trivial) subalgebras, the ranks of which vary between 11 and 3. It turns out that a few of those algebras are non-Schurian: a quite rare occurrence for the direct product of two Schurian algebras, provided that each of them contains only Schurian subalgebras. Of a special interest is a commutative subalgebra of rank 7 and valencies $1, 6, 7, 7, 21, 28, 42$, having the same group F of order 2688. There is also a pair of isomorphic rank 5 non-Schurian algebras with valencies $1, 7, 28, 28, 48$, which belongs to the intersection of classes I and II in the sense of [49]. (We refer to the paper [65] where the problem of investigation of association schemes of rank 5, as it was formulated in [49] is considered in the flavor of computer algebra experimentation.) The automorphism group in this case is isomorphic to $S_2 \times (E_{64} : PSL(3,2))$, has order 21504 and rank 10. In our eyes, these non-Schurian coherent algebras deserve a special attention in the future.

13 Some association schemes on 56 points

We now come back to the consideration of all merging schemes of our master association scheme $\mathfrak{M}$ on 56 points as it was presented in Section 7. At this stage our interest does not stem exclusively from the consideration of the graphs $\mathcal{L}$, $\mathcal{NL}$ and $\mathcal{N}$.

Recall that there are altogether exactly 50 merging schemes, which split into 43 isomorphism classes under the action of the group $CAut(\mathfrak{M})$. The isomorphism classes are as follows (only those consisting of two schemes are listed; the remaining ones form classes consisting of a single scheme): $\{2, 3\}$, $\{18, 19\}$, $\{22, 33\}$, $\{35, 39\}$, $\{37, 41\}$, $\{38, 40\}$, $\{46, 47\}$ (we refer here to the mergings in Supplement A).

Among the 50 merging schemes not all are of an equal interest. Here we pay most attention to non-Schurian schemes. Altogether there are 8 such schemes, two of rank 6 and six of rank 5.

Both mentioned schemes of rank 6 (#12 and #15) have the same automorphism group $CAut(\mathfrak{M})$ of order 336 and the same set of valencies $1, 1, 6, 12, 12, 24$, are non-symmetric, though commutative. They are not algebraically isomorphic. The symmetrization of a non-symmetric commutative scheme forms also an association scheme (see e.g. [2]); for both schemes #12 and #15 this is the same scheme #27. The latter scheme is nothing else but the centralizer algebra of the wreath product $\widetilde{AGL(3,2)} \wr S_2$ of order $2^{28} \cdot 1344$. Here the group $\widetilde{AGL(3,2)}$ denotes the induced action of the group $AGL(3,2)$ (considered in its natural action of degree 8) on the set $\left\{ {[0,7] \atop 2} \right\}$ of all 2-subsets of the 8-element set $[0,7]$.

The centralizer algebra $\mathfrak{n} = V(\widetilde{AGL(3,2)}, \left\{ {[0,7] \atop 2} \right\})$ of order 28 has rank 4 and valencies of basic graphs are $\underline{1,3,}12,12$. To explain it, consider the triangular graph $T(8)$, its complement $\overline{T(8)}$ and define two elements from $\left\{ {[0,7] \atop 2} \right\}$ to be adjacent in R if their union is a block of $S(2,3,8)$. Then the relations $T(8)$, $\overline{T(8)} \setminus R$, R are all 2-orbits of the group $AGL(3,2)$ acting on the set $\left\{ {[0,7] \atop 2} \right\}$. It is easy to check that this group is 2-closed.

The relations of the merging Schurian scheme #27 may be defined combinatorially as "blowup" of the relations of $\mathfrak{n}$. Thus we get rank 5 scheme with the valencies of basic graphs $1,1,6,24,24$.

As was mentioned, the two non-Schurian schemes #12 and #15 may be explained in a unified manner as non-symmetric fissions of the Schurian scheme #27. For this purpose we have to consider a concrete copy of overlarge set $\mathcal{O}_N(7,3)$. Its selection depends on the selection of a concrete copy of G as a subgroup of the group $AGL(3,2)$ (cf. Section 10). Provided that a concrete copy of $\mathcal{O}_N(7,3)$ is considered, we define "orientation" of one of the two relations $T(8)$ or $\overline{T(8)} \setminus R$. Any of two such orientations define a pair of directed graphs of valency 12 on the set V_1.

Such a definition may be provided in a routine manner, depending on selected copy of $\mathcal{O}_N(7,3)$.

In each of these two cases, the fact that the resulted color graph of rank 6 defines an association scheme follows from computer inspection. A computer free proof may be provided with the aid of ad hoc counting arguments.

Remark. In both cases it follows from the description that if the pair $((a,b),(a,c))$ $(((a,b),(c,d)))$ belongs to a defined directed graph, then the pair $((b,a),(c,a))$ (respectively $((b,a),(d,c))$) belongs to the same directed graph. This implies immediately that the automorphism group of the defined relation is invariant with respect to the group $\mathbb{Z}_2$ which reverses all pairs of the form $\{(x,y),(y,x)\}$. In other words, both association schemes have automorphism group of order 336. In fact, this is the order of the entire automorphism group.

Let us now consider the six non-Schurian association schemes of rank 5. Just one of them, namely #29 is symmetric. It has valencies $1,3,4,24,24$. Its automorphism group has order 672 and is twice larger than $CAut(\mathfrak{M})$. The Schurian association scheme #5 defined by the same group of order 672 has rank 8 and is non-symmetric and non-commutative. The scheme #29 appears as its symmetrization. This is a rather unusual situation which requires special attention (cf. end of Section 15), which is, however, out of the scope of the presentation in the framework of this paper.

Now we pay attention to the remaining five association schemes of rank 5, which are not symmetric.

One of these schemes, namely #25 with valencies $1,1,12,12,30$ stands alone. Its automorphism group coincides with $CAut(\mathfrak{M})$. In a sense, it is a by-product of the scheme #15. Indeed, the symmetric relation of valency 30 is the union of relations of valency 6 and 24 and is the 2-fold blow-up of the graph $\overline{T(8)}$.

The non-Schurian schemes $\#21, 22, 33, 34$ together with the Schurian scheme #23 form a family of five algebraically isomorphic schemes with the valencies $1, 3, 24, 24, 4$. The schemes $\#22, 33$ are combinatorially isomorphic (with the aid of any permutation from $CAut(\mathfrak{M}) \setminus Aut(\mathfrak{M})$) and both have the automorphism group coinciding with G. The symmetrization of all these schemes is scheme #44, which corresponds to the 2-orbits of the iterated wreath product $S_7 \wr S_2 \wr S_4$. The scheme #21 has group of order 1344 and rank 8 (corresponding to the scheme #6). The scheme #34 has group of order 672 and rank 8 (scheme #5 respectively). The Schurian scheme #23 has group of order 10752 which is isomorphic to $E_8 \times AGL(3, 2)$.

We believe that this family of algebraically isomorphic association schemes is new and deserves special attention. The results of the investigation are in progress and will be published elsewhere together with a more wide panorama of the entire collection of all mergings of $\mathfrak{M}$, both Schurian and non-Schurian.

For the reader's convenience in Supplement B we provide the Hasse diagram for all (up to isomorphism) 43 merging association schemes of the scheme $\mathfrak{M}$. The labeling coincides with the one in Supplement A. Note that each double circle in the diagram substitutes a pair of isomorphic schemes by one of its representatives.

14 Two-fold isomorphisms and related concepts

Let (G, Ω) be a transitive permutation group, R its antisymmetric 2-orbit, $\Delta = (\Omega, R)$, such that $Aut(\Delta) = G$. Let us consider the graph $\Gamma = IDC(\Delta)$. Assume that Γ is a semisymmetric graph and $Aut(\Gamma) = G$.

Then there is a definite sense to call the graph Δ a *stable* directed arc-transitive and vertex-transitive graph.

The Iofinova-Ivanov criterion, presented in Section 4, presents a necessary condition for Δ to be a stable graph. We know that if, in addition, the permutation group (G, Ω) is primitive, then this necessary condition also becomes sufficient.

As soon as (G, Ω) is imprimitive this condition is not sufficient. Indeed, relations $3, 4, 8, 12$ of the master association scheme (see Table 7.1) provide counterexamples: all properties in the criterion are satisfied, however, the corresponding directed graphs are not stable.

In our eyes, the observed phenomenon is one of the most significant by-product results in this paper. Further clarification of this phenomenon is necessary. One of the helpful concepts to be considered for such purpose is two-fold automorphisms and two-fold 2-orbits, as they were recently introduced by Scapellato, JL et al, see [70], [72].

Let Γ be an arbitrary directed graph (loops and mixed directed and undirected edges are allowed, though the case of purely directed graphs, that is those defined by antisymmetric relations will be mostly significant in this section). In other words, the adjacency matrix $A(\Gamma)$ is an arbitrary square $(0,1)$-matrix. Motivated by a possibility to consider simultaneously the same matrix $A(\Gamma)$ as the incidence matrix $I(S)$ of an incidence structure S, we wish to go further and introduce a suitable language for the systematic consideration of the symmetries of matrices in such a way that some impact between combinatorial and formal algebraic understanding of the symmetries will be achieved.

For this purpose let us consider two copies of the vertex set V of graphs in the consideration, say $V_1 = V \times \{1\}$ and $V_2 = V \times \{2\}$ and let $\widetilde{V}^2 := V_1 \times V_2$. Clearly there is a natural bijection between the elements of V^2 and $\widetilde{V}^2$, when the image of $(x, y) \in V^2$ is $((x, 1), (y, 2)) \in \widetilde{V}^2$, or briefly $(x_1, y_2) \in \widetilde{V}^2$. Here elements of V_1 may be called *origins*, while elements of V_2 *ends* of the arcs from $\widetilde{V}^2$.

Let $\widetilde{Sym(V)}^2 := Sym(V_1) \times Sym(V_2)$. An arbitrary element $(g_1, g_2) \in \widetilde{Sym(V)}^2$ will be called a *two-fold permutation* of elements from V (which by definition acts on $\widetilde{V}^2$), or briefly TF permutation. Any subgroup $\widetilde{G}$ of $\widetilde{Sym(V)}^2$ will be called a *TF permutation group* of the set V (again, by definition it acts on the set $\widetilde{V}^2$).

Of course, we will consider orbits of the action of $(\widetilde{G}, \widetilde{V}^2)$, which will be called *TF orbits.* Here, by definition, for $x_1 \in V_1$, $y_2 \in V_2$, we denote

$$Orb_{\widetilde{G}}((x_1, y_2)) := \{(x_1^{g_1}, y_2^{g_2}) | (g_1, g_2) \in \widetilde{G}\}.$$

Typically, the sign $\widetilde{G}$ may be omitted if it is clear from the context. We may also consider some stabilizers in a given group $\widetilde{G}$. For us a few kinds of stabilizer will play a significant role, namely for $x_1 \in V_1$, $y_1 \in V_1$, $y_2 \in V_2$:

$\widetilde{G}_{x_1} = \{(g_1, g_2) \in \widetilde{G} | x_1^{g_1} = x_1\}$,
$\widetilde{G}_{x_1,y_1} = \{(g_1, g_2) \in \widetilde{G} | x_1^{g_1} = x_1 \wedge y_1^{g_1} = y_1\}$,
$\widetilde{G}_{x_1,y_2} = \{(g_1, g_2) \in \widetilde{G} | x_1^{g_1} = x_1 \wedge y_2^{g_2} = y_2\}$.

In principle, an arbitrary subgroup of $\widetilde{Sym(V)}^2$ may become a subject of consideration. Nevertheless, for the purposes of the current paper we will be mainly interested in a few kinds of TF groups, which may be attributed in terms of symmetries of graphs.

To an arbitrary graph $\Gamma = (V, R)$, $R \subseteq V^2$, we attribute its *TF copy* $\widetilde{\Gamma} = (V_1 \cup V_2, \widetilde{R})$, where $\widetilde{R} = \{(x_1, y_2) \in \widetilde{V}^2 | (x, y) \in R\}$.

Two graphs $\Gamma_1 = (V, R_1)$ and $\Gamma_2 = (V, R_2)$ are called *TF isomorphic* if there exists a TF isomorphism $(g_1, g_2) \in \widetilde{Sym(V)}^2$ between the TF copies $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$. In other words, Γ_1 and Γ_2 are TF isomorphic if

$\widetilde{\Gamma}_1^{(g_1,g_2)} = \widetilde{\Gamma}_2$ for a suitable (g_1, g_2), where $\widetilde{\Gamma}_1^{(g_1,g_2)} := \{(x_1^{g_1}, y_2^{g_2}) | (x, y) \in \Gamma_1\}$. Of course, we can also consider *TF automorphisms* of a given graph Γ and the concept of the *TF automorphism group* of the given graph Γ turns out to be correctly defined for a graph $\Gamma = (V, R)$:

$$TFAut(\Gamma) := \{(g_1, g_2) \in \widetilde{Sym(V)}^2 | (x_1^{g_1}, y_2^{g_2}) \in \widetilde{R} \text{ for all } (x_1, y_1) \in \widetilde{R}\}.$$

Following [71], we distinguish a few significant subgroups of the group $TFAut(\Gamma)$, namely:

usual automorphism group $\widetilde{Aut}(\Gamma) := \{(g,g)|g \in Aut(\Gamma)\}$;

symmetric part of the TF group, namely $\Sigma TFAut(\Gamma) := TFAut(\Gamma) \cap DTFAut(\Gamma)$, where the dual TF group $DTFAut(\Gamma)$ of Γ is the automorphism group of the *dual* TF copy $D\widetilde{\Gamma}$ of Γ, that is of $D\widetilde{\Gamma} = \{(x_1,y_2)|(y_1,x_2) \in \widetilde{\Gamma}\}$.

Clearly, the dual TF copy of Γ is the TF copy of the *transposed* graph $\Gamma^T = (V, R^T)$, where $R^T := \{(y,x)|(x,y) \in R\}$. The word dual stems from the fact that the dual copy $D\widetilde{\Gamma}$ corresponds to the dual incidence structure with respect to the incidence structure S with $I(S) = A(\Gamma)$.

We also consider other subgroups of $TFAut(\Gamma)$:

$LTFAut(\Gamma) := \{(g,e)|(g,e) \in TFAut(\Gamma)\}$;

$RTFAut(\Gamma) := \{(e,g)|(e,g) \in TFAut(\Gamma)\}$;

$NTFAut(\Gamma) := \{(g_1,g_2)|(g_1,g_2) \in TFAut(\Gamma) \wedge g_1, g_2 \in Aut(\Gamma)\}$,

each ingredient of TF automorphism from $NTFAut(\Gamma)$ *naturally* restricts to an automorphism of Γ. Here e is the identity permutation. Clearly, the *left* and *right* groups $LTFAut(\Gamma)$ and $RTFAut(\Gamma)$ are normal subgroups of $TFAut(\Gamma)$.

Proposition 14.1. *Any TF orbit of the group $TFAut(\Gamma)$ is a union of TF copies of usual 2-orbits of the group $Aut(\Gamma)$.*

Proof. The proof is evident because the isomorphic image $\widetilde{Aut}(\Gamma)$ of the group $Aut(\Gamma)$ is a subgroup of $TFAut(\Gamma)$. □

Let us call an arc-transitive (directed) graph $\Gamma = (V, R)$ a *TF stable graph* if $\widetilde{R}$ is still the TF orbit of the group $TFAut(\Gamma)$. We hope that the concept of TF stable graph may play a significant role in the investigation of diverse kinds of the stability, which were formulated with the aid of the double covers of graphs.

We conclude with a couple of reasonably friendly examples with the aim of creating for the reader a helpful context in the newly defined TF world.

First, let us stress that, up to notation, for an arbitrary graph Γ with adjacency matrix $A(\Gamma)$ and corresponding incidence structure S with the incidence matrix $I(S) = A(\Gamma)$, the groups $TFAut(\Gamma)$ and $Aut(S)$ are clearly isomorphic. This helps to manipulate successfully with these groups, switching back and forth between the two natural interpretations of the same square $(0,1)$-matrix A.

Example 14.1 (A circulant graph on 6 vertices.)**.** *Let us consider the circulant graph $\Gamma = Cay(\mathbb{Z}_6, \{1,4\})$ (we refer to [92] for more information about circulant graphs, their automorphism groups, and corresponding Schur rings, a particular case of association schemes). It is more convenient for the current purposes to label the vertices of Γ by elements from $[1,6]$ as in Figure 6(a) instead of those in $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$. We can easily depict the same graph in a different way, presented in Figure 6(b).*

Here a double arrow from "metavertex" $\{x,y\}$ to the "metavertex" $\{u,v\}$ substitutes 4 arrows $(x,u),(x,v),(y,u),(y,v)$. In other words, our graph Γ is a

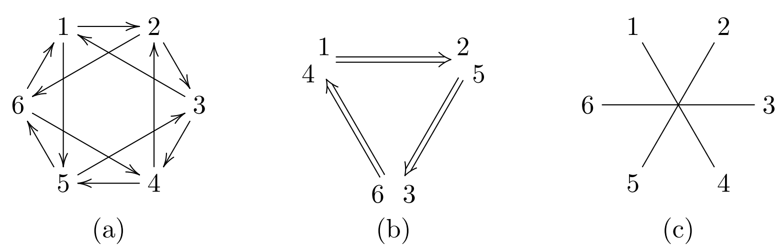


Figure 6: Circulant graph Γ and related graphs

"blow-up" of a directed triangle, where each vertex of the triangle is substituted by a coclique of size 2.

This new interpretation of the graph Γ *immediately implies that* $H = Aut(\Gamma) = \mathbb{Z}_3 \wr \mathbb{Z}_2$ *is the wreath product of cyclic groups* $\mathbb{Z}_3$ *and* $\mathbb{Z}_2$*, and thus* $|Aut(\Gamma)| = 3 \cdot 2^3 = 24$.

The 2-orbits of the group H *are arc sets of the Cayley graphs* $Cay(\mathbb{Z}_6, \{0\})$, $Cay(\mathbb{Z}_6, \{1,4\})$, $Cay(\mathbb{Z}_6, \{2,5\})$ *and* $Cay(\mathbb{Z}_6, \{3\})$*, denoted by* R_0, R_1, R_2, R_3. *Here* R_0 *consists of loops,* $R_2 = R_1^T$*, while the graph* $\Gamma_3 = (\mathbb{Z}_6, R_3)$ *appears in Figure 6(c).*

Clearly, the adjacency matrix $A(\Gamma)$ *looks as follows:*

$$A = A(\Gamma) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Considering the corresponding incidence structure $S = (\mathcal{P}, \mathcal{B})$ *with the point set* $\mathcal{P} = [1,6]$*, we can easily detect that the set* $\mathcal{B}$ *of blocks consists of six elements, namely, each of the subsets* $\{1,4\}$, $\{2,5\}$, $\{3,6\}$ *appears as a multiple block with multiplicity 2. The automorphism group* $Aut(S)$ *has therefore structure* $S_3 \wr E_4$ *of the wreath product of* S_3 *with* E_4*. Here* E_4*, the elementary Abelian group of order* 4*, is generated by permutations switching the end of an edge and two copies of the edge with the same ends, while* S_3 *is responsible for the permutation of the three edges. The order of the group* $Aut(S)$ *is* $3! \cdot 4^3 = 2^7 \cdot 3 = 384$*. Note that the provided combinatorial vision of the group* $Aut(S)$ *is indeed quite evident.*

Let us now switch back to the strict consideration of the matrix A *and its TF symmetries. To simplify notation, the elements of* V_1*, that is rows of* A*, were denoted by their numbers, while the elements of* V_2*, that is columns of* A*, by the symbols* $\tilde{i}$*. Let us list a few permutations of* $\widetilde{G} = TFAut(\Gamma)$*, which can be clearly seen from the picture. In some of these ordered pairs of permutations, one of the components is the identity permutation which is suppressed for ease of notation. It is clear whether the shown permutation is the first or the secend entry in the pair because the second entry is the one acting on columns which are denoted*

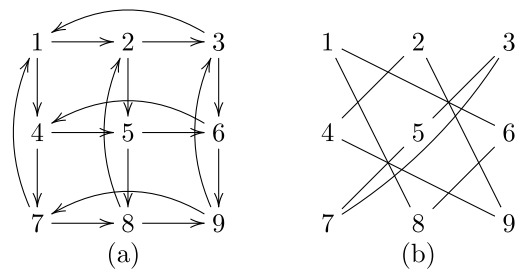


Figure 7: Oriented graph $P(9)$ and auxiliary graph Δ

by the ˜ sign. Here $a_1 = (1,4)$, $b_1 = (2,5)$, $c_1 = (3,6)$, $a_2 = (\tilde{1},\tilde{4})$, $b_2 = (\tilde{2},\tilde{5})$, $c_2 = (\tilde{3},\tilde{6})$, $h = (1,2,3)(\tilde{1},\tilde{2},\tilde{3})(4,5,6)(\tilde{4},\tilde{5},\tilde{6})$, $i = (1,2)(4,5)(\tilde{2},\tilde{3})(\tilde{5},\tilde{6})$.

The group $\langle a_1, b_1, c_1, a_2, b_2, c_2, h, i\rangle$ *generated by the above 8 permutations, clearly has structure* $E_{64} : S_3$ *and thus is of order 384. It is a subgroup of* $TFAut(\Gamma)$*. At this stage we claim that this is the entire group* $\widetilde{G} = TFAut(\Gamma)$*, just comparing its order with the order of the isomorphic group* $Aut(S)$*.*

The reader is now welcome to check that
$\widetilde{G}_1 = \langle b_1, c_1, a_2, b_2, c_2, hi\rangle$ *has order* 64*,*
$\widetilde{G}_{1,\tilde{2}} = \langle b_1, c_1, a_2, b_2, hi\rangle$ *has order 32,*
$\widetilde{G}_{1,\tilde{1}} = \langle b_1, c_1, b_2, c_2\rangle$ *has order 16.*

Therefore, using the Lagrange theorem for permutation groups, we obtain that $|TF((1,\tilde{2}))| = [\widetilde{G} : \widetilde{G}_{1,\tilde{2}}] = 12$, $|TF((1,\tilde{1}))| = [\widetilde{G} : \widetilde{G}_{1,\tilde{1}}] = 24$*. (Here,* $TF((a,\tilde{b}))$ *represents the orbit of the pair, consisting of row* a *and column* $\tilde{b}$*.) Because* $12 + 24 = 36 = 6^2$*, we already are getting the full list of the TF orbits of the group* $\widetilde{G}$*. Clearly this list consists of* R_1 *and* $R_0 \cup R_2 \cup R_3$*. In particular, we obtain that the graph* Γ *is TF stable.*

As an extra simple exercise, we list below a few other subgroups of $\widetilde{G}$*, using the notation introduced above. Here*
$\widetilde{Aut}(\Gamma) = \langle a_1a_2, b_1b_2, c_1c_2, h\rangle \cong Aut(\Gamma)$
$\Sigma TFAut(\Gamma) = TFAut(\Gamma)$,
$LTFAut(\Gamma) = \langle a_1, b_1, c_1\rangle \cong E_8$,
$RTFAut(\Gamma) = \langle a_2, b_2, c_2\rangle \cong E_8$,
$NTFAut(\Gamma) = \langle a_1, b_1, c_1, a_2, b_2, c_2, h\rangle \cong \mathbb{Z}_3 \wr E_4$.

Example 14.2. *The diagram of graph* Γ*, depicted in Figure 7, is borrowed from [39], where it was used for the discovery of two new strongly regular graphs on 512 vertices. This graph is an "orientation" of the graph* $L_2(3)$*; note that* $L_2(3)$ *is isomorphic to the Paley graph* $P(9)$*.*

It is easy to understand that the automorphism group $G = Aut(\Gamma)$ *this time is isomorphic to the wreath product* $\mathbb{Z}_2 \wr \mathbb{Z}_3$ *and has order* $2 \cdot 3^2 = 18$*. Let* $V = [1,9]$ *be the vertex set of* Γ*. The set* $2-orb(G,V)$ *consists of five relations* R_0*,* R_1*,* R_2*,* R_3*,* R_4*, where* R_1 *is the arc set of* Γ*,* $R_2 = R_1^T$*, while* R_3 *and* R_4 *are two orientations of the complement of* $L_2(3)$*; the complement is isomorphic* *to the underlying graph of Γ. It is important to mention that the four graphs $\Gamma = \Gamma_1$, Γ_2, Γ_3, Γ_4 are all isomorphic.*

First, we again consider the incidence structure S such that $I(S) = A(\Gamma)$. In this case each block of S is an edge of the auxiliary graph Δ, depicted in Figure 7(b).

This graph Δ is $3 \circ K_3$, that is a disjoint union of three copies of K_3. Thus the group $\widetilde{G} = Aut(S)$ is again easily understood with the aid of purely combinatorial arguments: $\widetilde{G} = S_3 \wr S_3$ has order $3!(3!)^3 = 1296$.

The reader is invited to check that this time $\widetilde{G}_{1,\tilde{2}} \cong S_3 \times S_2 \times S_2$ and has order 24; $\widetilde{G}_{1,\tilde{1}} \cong (S_2 \wr S_3) \times S_2$ and has order 144; $\widetilde{G}_{1,\tilde{3}} \cong S_2 \wr S_3$ and has order 72. Therefore the TF orbits of the pairs $(1,\tilde{2})$, $(1,\tilde{1})$ and $(1,\tilde{3})$ have the lengths 54, 9, 18 respectively. Thus, because $54 + 9 + 18 = 9^2$, we get a complete list of the TF orbits as follows: R_0, $R_1 \cup R_2 \cup R_4$, R_3. Note that this time the graph Γ is not TF stable, because its arc set merges with two other 2-orbits into a TF orbit of the group $\widetilde{G}$.

Comparing the two examples, one can say that while all TF automorphisms of the graph Γ in Example 14.1 are, in a sense, predictable, in Example 14.2 we get extra surprises, due to the fact that the underlying graph of Γ and its complement are isomorphic.

We conclude this section with an extra notice. The new concept of a stable graph, suggested here, expresses its flexibility. On one hand, a TF stable graph does not have to be a stable directed graph. On the other hand, the fact that graph Γ is not TF stable reflects some extra features of the symmetry of the initial graph Γ, which may remain hidden from the observer on first sight. This confirms the hopeful efficiency of TF stability for further more systematic investigations.

15 A wider scope

Computer algebra experimentation in AGT is the main subject of this paper. Our goals were to demonstrate to the reader:

- which kinds of computations appear in this area;
- which computer packages may be helpful and how they can be used;
- how routine data, produced with the aid of a computer, should be presented and analyzed;
- why is it that in many cases the computer should be considered as the most reliable tool for producing proofs of claims about diverse properties of the structures being investigated;
- what is a successful computer free interpretation of the results of computations;

- how the impact of ideas and techniques, mobilized from diverse areas of algebra and combinatorics may help to reach a theoretical generalization of the computer results achieved.

To approach these goals, we started from the consideration of the two concrete graphs $\mathcal{L}$ and $\mathcal{N}$ on 112 vertices and were simply following all the ongoing logics of a computer aided investigation of association schemes and diverse structures which were naturally linked to the starting graphs. Discovery of a number of new, quite interesting, association schemes on 56,112 and 120 points serves as additional reward and confirms that our methodology is natural: it allows us to detect deep links of the investigated graphs with many other structures considered in AGT, which were hidden at first sight.

The computer packages used are the subject of ongoing extension, improvement and advancement: for example, new releases of GAP are announced, on average, once every two years. Similarly, during recent years, certain efforts were made in order to start a transformation of COCO into a new package COCO-II for GAP cf. [66].

While from the side of computer algebra AGT appears just like one of many possible subject areas, for the authors this is the central stage of our research interests. This is why in the current section we take liberty to discuss in more detail a number of topics, related to the structures, axiomatical systems and lines of investigation that were considered in this paper.

The paper [101] was very influential, promoting further research of semisymmetric graphs: Besides [62], [52] and [55] we wish also to mention a contribution of A. V. Ivanov, see [56], [57], who classified with the aid of a computer all semisymmetric graphs with at most 30 vertices. Up to isomorphism there are just 5 such graphs on 20, 24 and 28 vertices, those which were already described in [40], [8] and [101]. A significant corollary of the attained result is the negative answer to question 4.2 from [40] about the existence of a semisymmetric graph with 30 vertices. In fact, the techniques of constructive enumeration of the incidence structures, developed by Andrei Ivanov, still have potential to be used for advanced enumeration and deeper investigation of semisymmetric graphs with a relatively small number of vertices, cf. [106].

During the last decade a new wave of interest in semisymmetric graphs was initiated, in particular due to the efforts of D. Marušič and his colleagues, see [31], [82], [85].

Cubic semisymmetric graphs are considered on both theoretical and computer aided approaches [80], [81], [103], [20].

There is a number of interesting lines of investigation of semisymmetric graphs. Below we briefly discuss a few of these lines.

Recall that a semisymmetric graph Γ is called *biprimitive* ([55]) if $Aut(\Gamma)$ acts primitively on both parts of the bipartite vertex partition of Γ. Biprimitive (semisymmetric) graphs of a small order are, of course, of a definite interest; typically they are considered up to a bipartite complement. The examples on 80, 126 and 990 vertices were already mentioned in [54]. In fact the first two are the first members of two infinite series, see also [63]. It was proved in [31] that the two examples on 80 vertices of valency 4 and 36 are the smallest biprimitive ones.

It is well-known from the time of Folkman [40] that a regular edge-transitive graph of order $2p$ or $2p^2$ (p a prime) is necessarily vertex-transitive. In this context, among many papers related to the Gray graph (see e.g. references in [66]) the one of the central interest is [81], where the Gray graph is characterized as the unique cubic semisymmetric graph of order $2p^3$, p a prime.

Serious tools from group theory are mobilized in many investigations of the semisymmetric graphs, see e.g. [33], [93].

A geometric approach in conjunction with the use of diverse algebraic techniques was successfully used, see e.g. [74], [75], [32], [91].

It is significant to mention that, using covering techniques, see e.g. [4], [55], one may get from one semisymmetric graph Γ an infinite series of such graphs of the same valency; thus in general, the problem of the classification of all semisymmetric graphs is in a sense intractable. This is why special attention should be paid to irreducible examples (with respect to covering). In particular, such examples of a non-parabolic type should be investigated systematically.

Though below we are coming back to a review of semisymmetric graphs (in conjunction with a few other topics), unfortunately, a complete and comprehensive survey of all facets of this line in AGT goes beyond the scope of the current paper.

The concept of stability coined in [87] in the case of undirected graphs has attracted attention of very serious investigations, such as [99], [100], [108].

On the other hand, in a number of papers very skillful diverse techniques are demonstrated in order to consider links between vertex-transitive antisymmetric 2-orbits and their IDCs, see [86], [106], [107].

We hope that the concept of two-fold orbits, which was outlined in Section 14, will help in future to consolidate diverse efforts of combinatorial, geometric and algebraic nature in order to consider double covers of both directed and undirected graphs, basing on a more uniform algebraic background. It is worth also to mention, that double covers are considered in graph theory in wider contexts, see e.g. [51], [88].

Methods of graph coverings, as they were discussed, say in [4] at the initial stage were of a more abstract nature. For example, Biggs admits (at the bottom of p. 152) that the three first graphs in his infinite series have 234 and about 2^{21} and 2^{100000} vertices respectively. It took a while in order to develop systematic methods of topological graph theory (in a sense of [44]). Nowadays voltage assignments appear as an efficient tool to construct and investigate semisymmetric graphs, see [80], [78], [103], [38].

A promising potential lies, in our eyes, in the conjunction of the techniques of topological graph theory and those of coherent configurations, in particular of half-homogeneous configurations (in the sense of [64]).

A general class of half-homogeneous configurations, namely Wallis-Fon-Der-Flass (briefly WFDF) configurations was recently introduced and investigated, see [64], [65]. It is worth mentioning that the WFDF configuration on 28 points in [64] which appears via the induced action of the group E_8 on the set $\left\{ {[0,7] \atop 2} \right\}$, has a natural covering by the WFDF configuration on the set Ω of cardinality 56, as it appears in Section 7. (Just consider E_8 as a subgroup of the group G.) We are sure that the consideration of the latter may explain, in a more unified manner, the origin of a number of new association schemes on 56 points which were introduced in this paper.

The idea of a TF automorphism, TF groups, TF orbits may be traced to a few mathematicians (already mentioned in Section 14), in particular to Bohdan Zelinka (1940-2005) whose scientific heritage in AGT still remains underrated. Some of his papers, such as [110], [112] (as well as [111]) may provide helpful insights to an interested reader.

During last decade the authors Scapellato, JL and their collaborators (in particular we wish to mention graduate thesis [90]), made some systematic efforts to exploit these TF concepts for the purposes of traditional graph theory. Section 14 in the current paper aims to outline possible attractive new applications of the developing techniques of TF strictly inside of AGT.

As one interesting further example on the way of the use of the TF approach we recommend the consideration of the Doyle-Holt graph on 27 vertices and its automorphism group of order 54 (see [104], [66] and references in it). The double cover of this graph (as well as of a corresponding antisymmetric 2-orbit) may provide a worthy training ground for better understanding of similarities and distinctions in the behavior of IDC for directed and undirected graphs.

Another challenge for researchers goes through the consideration of amalgams of groups (as they already appear in [55]) and further to the theory of diagram geometries (see e.g. [94], [53]). A great advantage of this language is that it provides a natural, adequate, in a sense coordinate free vision of all links between the considered structures and their symmetries. We have brought to the attention of the reader a few attempts to create models for the graph $\mathcal{L}$ and to digest the embeddings of $\mathcal{L}$ into $\mathcal{N}$. Each way considered by us was reasonably simple and provided the reader with immediate access to the investigated objects. Nevertheless, depending on the selected "universal" group, to which we were referring (say $AGL(1,8)$, $AGL(3,2)$, A_8 or $S_2 \uparrow S_7$) we were forced each time to start consideration from scratch and to observe already investigated interactions from a new angle. Involvement of suitable group amalgams, even for the case of the current project, may allow the presentation of all the links we have studied in a much more standard manner. Of course, the price for such an advantage would be the substitution of the exploited impact of computational and combinatorial techniques by much more complicated group-theoretical technology. Also our favorite non-Schurian association schemes may remain hidden in the relative shadow under the strong radiation of this kind of algebraic sunlight.

Finally, it remains to mention an interesting speculation (due to Misha Muzychuk) about a possible origin of some of our non-Schurian association schemes. In general, the center of a non-commutative coherent algebra is not necessarily also a coherent algebra. Nevertheless, this may be true, for example, for some non-commutative association schemes (a very simple example is provided by the symmetrization of the thin scheme of quaternions on 8 points). We refer to [15] for the discussion of symmetrization of association schemes in a close context. In this framework the investigation of the center of the above mentioned WFDF coherent configuration on 56 points, as well as of its mergings may be of a definite interest.

16 Concluding comments

We use this opportunity to reveal some details about the history of a note [62]. As was mentioned above, the results were obtained in 1977. Next year the author MK got an invitation to participate in the conference at Szeged and tried to get a permission from the authorities to attend this event. At the time of "iron curtain" such a permission was not given. Fortunately, Laci Babai was visiting USSR during fall 1978. He kindly picked up the manuscript prepared by MK, and published its English version, elaborated by LB, in the proceedings of the Szeged conference. The English wording "semisymmetric", used in [62] was suggested by Laci. The entire picture of the circumstances of this travel of L. Babai is presented in [1].

In fact, during the preparation of the note [62], LB substituted part of the original group-theoretical reasonings with a more elaborate use of counting of suitable simple numerical invariants of the vertices in the two parts of the considered bipartite graph. The original arguments of MK, though also quite simple, relied on the investigation of the "induced symmetric group" see [60], [37]. By chance, a self-contained treatment containing the description of the automorphism group of the Johnson association scheme $J(n, 3)$ was recently presented in [28].

This project was originated during a short visit of the author MK to Ljubljana in the summer of 2003. MK learned about the Ljubljana graph from Tomo Pisanski and quickly realized its relevance to the graph $\mathcal{N}$. However, it took a while to elaborate information about all links between the graphs $\mathcal{L}$ and $\mathcal{N}$. The preliminary version of this paper was presented by MK as an invited lecture at the mini-conference "Tomo is sixty", Ljubljana, June 2009. Later on, the two authors from Beer Sheva learned about the TF approach, elaborated by Scapellato, JL et al, and suggested to JL to merge efforts. The lecture, presented by MZA at Maribor, emerged from this extended partnership.

It is impossible to overevaluate the significance and relevance of the use of a computer algebra technology to all the stages of the fulfillment of this project. This is why we wish to conclude the text with a clear formulation of a few tasks for further research which are, in our eyes, strictly linked to this area of scientific computation.

Task 1. To fulfill constructive enumeration of all (up to isomorphism) small semisymmetric graphs (thus extending the results in [57]), paying a special attention to the graphs of a non-parabolic type. It seems that relying in particular on currently available GAP catalogs of transitive permutation groups, this task may be fulfilled at least to $n = 86$ vertices.

Task 2. To consider the WFDF configuration on 56 points discussed above; in particular, to enumerate and to investigate all its merging association schemes. Special attention should be paid to the resulting algebraic mergings and algebraic twins in a way similar to the one exploited in [64].

Task 3. To arrange an extensive computer aided experimentation in order to measure the efficiency of the Iofinova-Ivanov criterion, as it is modified in Theorem 4.9. The goals are to find new nice examples of the cases where it does not work, and/or to reach further strengthening of the criterion, relying on the properties of the revealed examples.

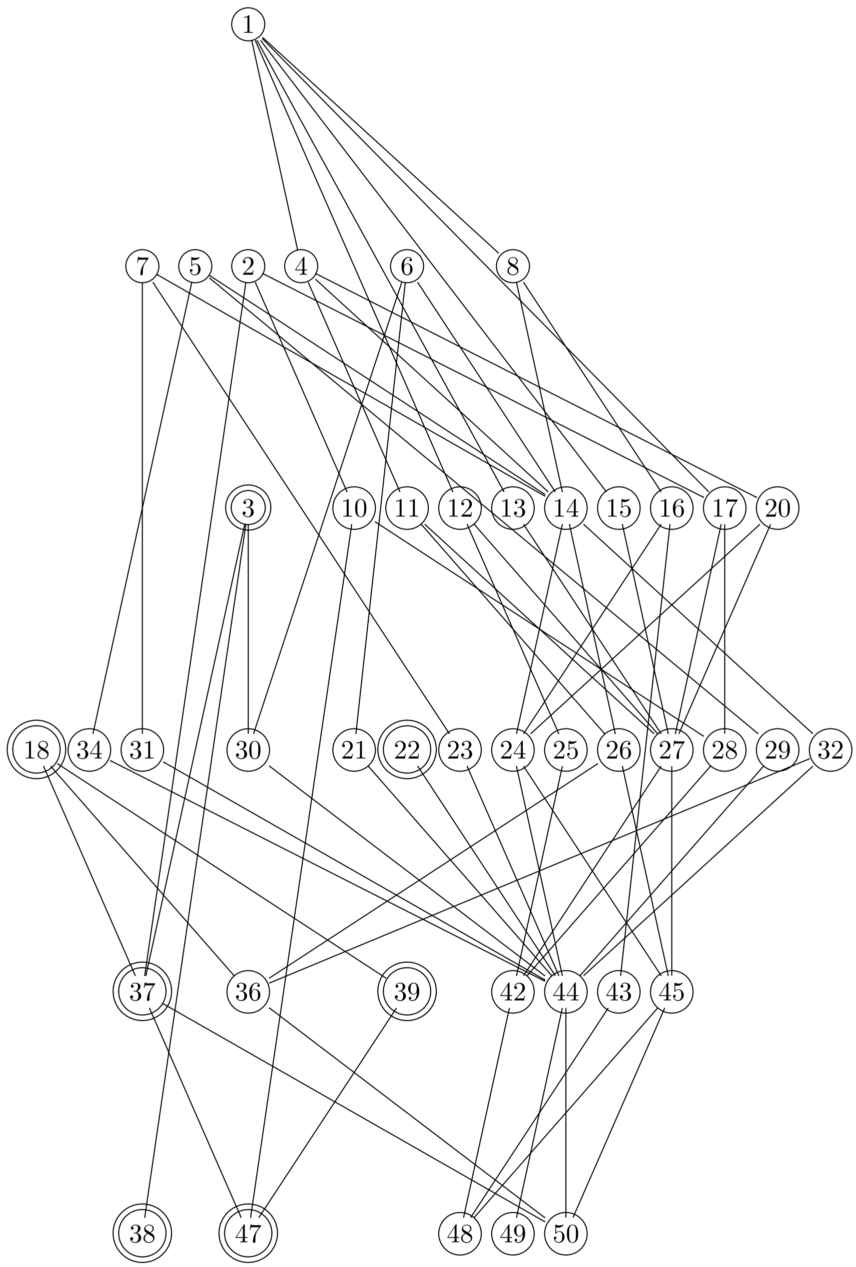
Acknowledgments

It is a great pleasure to thank officially (after more than 30 years!) Laci Babai for his assistance in the publication of [62]. Conversations with Tomo Pisanski in 2003 about the graph $\mathcal{L}$ were very helpful. We thank Italo Dejter, Marston Conder and Misha Muzychuk for helpful material, discussions and communication. Collaboration of the authors on diverse projects, jointly with Sven Reichard, Raffaelle Scapellato, Russell Mizzi and Andy Woldar was very fortunate and influential. Finally, we wish to thank Viktor Levandovskyy for his kind and constructive attention to this project.

Supplement A. List of mergings of association scheme $\mathfrak{M}$

No.	rank	merging	unmerged	$\lvert Aut \rvert$
1	12	(1,7)(5,9)(3,4)(12,8)(6,14)(17,15)(10,16)(11,13)	2,18,19	336
2	8	(1,5)(6,17,10,11,13,16,14,15)(7,9)(18,19)(3,8)(12,4)	2	1344
3	8	(2,19)(4,8,6,17)(7,9)(10,11)(1,12,13,15)(5,3,16,14)	1	1344
4	8	(1,3,4,7)(5,12,8,9)(6,11,13,14)(17,10,16,15)	2,18,19	21504
5	8	(1,3,6,11)(5,12,17,10)(4,7,13,14)(8,9,16,15)	2,18,19	672
6	8	(1,3,13,14)(5,12,16,15)(4,6,7,11)(8,17,9,10)	2,18,19	1344
7	8	(1,4,11,14)(5,8,10,15)(3,6,7,13)(12,17,9,16)	2,18,19	1344
8	8	(1,6,7,14)(5,17,9,15)(3,4,11,13)(12,8,10,16)	2,18,19	2688
9	8	(1,5)(2,19)(3,12,14,15)(13,16)(4,6,9,10)(8,17,7,11)	2	1344
10	7	(1,5)(6,17,10,11,13,16,14,15,18,19)(7,9)(3,8)(12,4)	2	40320
11	7	(18,19)(1,3,4,7)(5,12,8,9)(6,11,13,14)(17,10,16,15)	2	$2^{28} \cdot 168$
12	6	(6,17,10,11,13,16,14,15)(18,19)(1,12,8,7)(5,3,4,9)	2	336
13	6	(1,5,3,12,4,8,7,9)(6,17,14,15)(10,11,13,16)(18,19)	2	2688
14	6	(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	2,18,19	$2^{28} \cdot 3^8 \cdot 7$
15	6	(1,5,3,12,4,8,7,9)(18,19)(6,10,16,14)(17,11,13,15)	2	336
16	6	(1,5,6,17,7,9,14,15)(3,12,4,8,10,11,13,16)	2,18,19	21504
17	6	(1,5,7,9)(3,12,4,8)(6,17,10,11,13,16,14,15)(18,19)	2	2688
18	6	(2,18,19)(1,3,4,6,11,13,14)(5,12,8,17,10,16,15)	7,9	846720
19	6	(2,18,19)(3,4,6,7,11,13,14)(12,8,17,9,10,16,15)	1,5	846720
20	6	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15)	2,18,19	172032
21	5	(2,19)(1,3,8,17,9,10,13,14)(5,12,4,6,7,11,16,15)	18	1344
22	5	(2,19)(1,12,4,17,7,10,16,14)(5,3,8,6,9,11,13,15)	18	168
23	5	(2,19)(1,12,4,17,9,11,16,14)(5,3,8,6,7,10,13,15)	18	10752
24	5	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)	2,18,19	$2^{32} \cdot 3^9 \cdot 5 \cdot 7$
25	5	(6,17,10,11,13,16,14,15,18,19)(1,12,8,7)(5,3,4,9)	2	336
26	5	(18,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	2	$2^{49} \cdot 3^8 \cdot 7$
27	5	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15)(18,19)	2	$2^{31} \cdot 168$
28	5	(1,5,7,9)(3,12,4,8)(6,17,10,11,13,16,14,15,18,19)	2	80640
29	5	(1,5,3,12,6,17,10,11)(2,19)(4,8,7,9,13,16,14,15)	18	672
30	5	(1,5,3,12,13,16,14,15)(2,19)(4,8,6,17,7,9,10,11)	18	10752
31	5	(1,5,4,8,10,11,14,15)(2,19)(3,12,6,17,7,9,13,16)	18	10752
32	5	(2,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	18	$2^{49} \cdot 3^{15} \cdot 7$
33	5	(2,19)(1,3,8,6,7,10,16,15)(5,12,4,17,9,11,13,14)	18	168
34	5	(2,19)(1,3,8,6,9,11,16,15)(5,12,4,17,7,10,13,14)	18	672
35	4	(2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)	1,5	$2^7 \cdot 3^{10} \cdot 5 \cdot 7^9$
36	4	(2,18,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)		$21 \cdot (8!)^7$
37	4	(1,5,3,12,4,8,6,17,10,11,13,16,14,15)(2,18,19)(7,9)		$8 \cdot (7!)^2$
38	4	(1,5,3,12,7,9,13,16,14,15)(2,10,11,19)(4,8,6,17,18)		40320
39	4	(1,5,2,3,12,4,8,6,17,10,11,13,16,14,15,18,19)	7,9	$2^7 \cdot 3^{10} \cdot 5 \cdot 7^9$
40	4	(1,5,4,8,6,17,7,9,10,11)(2,13,16,19)(3,12,14,15,18)		40320
41	4	(1,5)(2,18,19)(3,12,4,8,6,17,7,9,10,11,13,16,14,15)		$8! \cdot (7!)^2$
42	4	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15,18,19)	2	$2^{28} \cdot 8!$
43	4	(1,5,6,17,7,9,14,15,19)(3,12,4,8,10,11,13,16,18)	2	$8 \cdot 9!$
44	4	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(2,19)	18	$7! \cdot (2 \cdot 24^2)^7$
45	4	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(18,19)	2	$7! \cdot (2^4 \cdot 4!)^7$
46	3	(1,5)(2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)		$8! \cdot (7!)^8$
47	3	(1,5,2,3,12,4,8,6,17,10,11,13,16,14,15,18,19)(7,9)		$8! \cdot (7!)^8$
48	3	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)	2	$2^{28} \cdot 28!$
49	3	(1,5,2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,19)	18	$(4!)^{14} \cdot 14!$
50	3	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(2,18,19)		$7! \cdot (8!)^7$

Supplement B. Hasse diagram of mergings of $\mathfrak{M}$



Supplement C. List of $\mathcal{O}_N(7,3)$

0	$\{1,2,6\},\{3,5,6\},\{2,3,7\},\{4,6,7\},\{1,3,4\},\{1,5,7\},\{2,4,5\}$
1	$\{0,2,7\},\{0,3,6\},\{2,4,6\},\{0,4,5\},\{5,6,7\},\{3,4,7\},\{2,3,5\}$
2	$\{0,1,3\},\{0,4,7\},\{3,5,7\},\{0,5,6\},\{1,6,7\},\{1,4,5\},\{3,4,6\}$
3	$\{0,1,5\},\{1,4,6\},\{0,6,7\},\{1,2,7\},\{2,5,6\},\{4,5,7\},\{0,2,4\}$
4	$\{0,1,7\},\{1,2,3\},\{3,6,7\},\{1,5,6\},\{0,3,5\},\{0,2,6\},\{2,5,7\}$
5	$\{0,1,2\},\{2,3,4\},\{1,4,7\},\{2,6,7\},\{0,4,6\},\{0,3,7\},\{1,3,6\}$
6	$\{0,1,4\},\{2,4,7\},\{0,2,3\},\{3,4,5\},\{1,2,5\},\{1,3,7\},\{0,5,7\}$

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