Links between two semisymmetric graphs on 112 vertices via association schemes

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Abstract

This paper provides a model of the use of computer algebra experimentation in algebraic graph theory. Starting from the semisymmetric cubic graph $\mathcal{L}$ on 112 vertices, we embed it into another semisymmetric graph $\mathcal{N}$ of valency 15 on the same vertex set. In order to consider systematically the links between $\mathcal{L}$ and $\mathcal{N}$ a number of combinatorial structures are involved and related coherent configurations are investigated. In particular, the construction of the incidence double cover of directed graphs is exploited. As a natural by-product of the approach presented here, a number of new interesting (mostly non-Schurian) association schemes on 56, 112 and 120 vertices are introduced and briefly discussed. We use computer algebra system GAP (including GRAPE and nauty), as well as computer package COCO.

Key words: Nikolaev graph, Dejter graph, semisymmetric graph, Ljubljana graph, double cover, association scheme, computer algebra, Deza graph, overlarge set of designs

1. Introduction

Some of the most striking meeting points between geometry, combinatorics and graph theory, on one hand, and algebra and group theory, on the other, appear when investigating the symmetry of a combinatorial object. Semisymmetric graphs are the central topic of this paper. By a semisymmetric graph we mean a graph Γ which is regular (that is all valencies of its vertices are equal), and such that its automorphism group $Aut(\Gamma)$ acts transitively on the edge set $E(\Gamma)$ but not on the vertex set $V(\Gamma)$.

We shall focus our attention on two particular semisymmetic graphs, both on 112 verties: the Nikolaev graph $\mathcal{N}$ having valency 15 and the Ljubljana graph $\mathcal{L}$ with valency 3 (see, e.g., [Conder et al. (2005)], [wikipedia (2010)]). We shall study their properties and links between them. We shall find, for example, that $\mathcal{L}$ is a spanning subgraph of $\mathcal{N}$.

We shall also point out numerous links between $\mathcal{L}$ and $\mathcal{N}$ and a number of combinatorial structures such as assocation schemes and partial geometries. Our aim is to combine in one paper the features of at least three different genres in mathematical literature: an expository text about semisymmetric graphs; a tutorial on scientific computation in algebraic graph theory; and a report about new results achieved in our research.

Writing on the border-line between algebra, combinatorics, graph theory and computer algebra, it is difficult to give a self-contained account of all the necessary background information. Instead we shall outline a brief guide to the main ingredients supported by those references which, in our opinion, best help to fill in our brief sketch.

For permutation groups we recommend the book [Dixon and Mortimer (1996)]. Invariant binary relations play a central role in our presentation. Given a permutation group (G, Ω) we extend, in the natural way, the action (G, Ω) to the action (G, Ω^2). An invariant binary relation is simply a union of the orbits of this action. Following Wielandt, we call the orbits of (G, Ω^2) the *2-orbits* of (G, Ω). The set $2-orb(G, \Omega)$ of all 2-orbits of (G, Ω) will be the subject of our careful attention for a number of concrete permutation groups. This concept of an invariant relation of a permutation group (G, Ω) was coined and used in [Wielandt (1969)]. We refer to [Klin et al. (1988)] and [Faradžev et al. (1994)] for an introduction to this line of Wielandt's methodology.

For each transitive permutation group (G, Ω) the pair $(\Omega, 2 - orb(G, \Omega))$ provides a model of an association scheme; each such a model is called a *Schurian* association scheme, cf. [Faradžev et al. (1994)]. Non-Schurian schemes exist, the smallest one having 15 points.

The classic book [Bannai and Ito (1984)] is still an excellent source for association schemes. For the more general concept of a coherent configuration the reader is refered to ([Higman (1970)], [Cameron (1999)]). For the matrix analogue of a coherent configuration, that is, a *coherent algebra*, see [Higman (1987)].

We assume from the reader a modest acquaintance with simple concepts from graph theory and how a graph's symmetries are related to permutation groups. An introduction to this aspect of graph theory can be found in Chapters 1,2 of [Lauri and Scapellato (2003)], and the references in this book may also be quite helpful.

This paper is mainly about computer experimentation in Algebraic Graph Theory (AGT) (the classic book [Biggs (1993)] as well as [Godsil and Royle (2001)] and [Brouwer et al. (1989)] are highly recommended sources in AGT, while, for example, [Klin et al. (2007)], [Klin et al. (2009)] and [Klin et al. (2009)] are good introductions to the techniques involved in the use of computer algebra). We use a few computer packages, namely COCO ([Faradžev and Klin (1991)], [Faradžev et al. (1994)]) and GAP ([GAP (2008)]) with its extension package GRAPE ([Soicher (1993)]) which relies on *nauty* ([McKay (1990)]).

2. Double covers

The operation of a double cover of a graph plays an essential role in our presentation. The reader must be warned about a possible difference between our definition of double covers and that found in graph theoretical literature. Our main definition follows the one from [Ivanov and Iofinova (1985)]. Here and elsewhere in this paper, if a graph Δ is not a digraph we usually turn it into a digraph by replacing each $\{x, y\}$ with the two arcs (x, y) and (y, x). Such a pair of arcs is said to be self-paired. In general, for a digraph $\Delta = (V, R)$ the digraph Δ^t is defined to be (V, R^t), here $R^t = \{(y, x) | (x, y) \in R\}$.

Definition 1. Let $\Delta = (V, R)$ be a directed graph. Define a new undirected graph $\Gamma = (V(\Gamma), E(\Gamma))$, such that $V(\Gamma) = V \times \{1, 2\}$, $E(\Gamma) = \{\{(x, 1), (y, 2)\} | (x, y) \in R\}$. We will call the graph Γ the *incidence double cover* (briefly IDC) of Δ.

A number of simple cases presented in Example 2.1 will hopefully provide the reader with a helpful context in which to understand better the definition of IDC. In each of the series of figures, we present diagrams of the graph or digraph Δ and the corresponding $\Gamma = IDC(\Delta)$.

The only example needing some comment is (g) which shows the Paley tournament $P(7)$. The vertices of $P(7)$ are the elements of the finite field $\mathbb{Z}_7$. There is an arc (x, y) in $P(7)$ if and only if $y - x$ is a non-zero square in $\mathbb{Z}_7$, that is $y - x \in \{1, 2, 4\}$. The notation of vertices $0, \ldots, 6$ and $7, \ldots, 13$ in part (g) for the graph Γ corresponds to the pairs $(0, 1), \ldots, (6, 1)$ and $(0, 2), \ldots, (6, 2)$. In this case, $Aut(\Delta)$ is the Frobenius group F_{21} of order 21, while $Aut(\Gamma) \cong PSL(3, 2) : \mathbb{Z}_2$ is a group of order 336.

Proposition 2. *Let Δ be a graph, $\Gamma = IDC(\Delta)$. Then*

(i) if Δ is a regular graph of valency k, then Γ is also regular of the same valency k;

(ii) if Δ is a bipartite graph then Γ is disconnected;

(iii) if Δ is undirected graph then the group $Aut(\Gamma)$ contains as a subgroup the direct product $Aut(\Delta) \times \mathbb{Z}_2$.

Proof. Straightforward, see e.g. [Zelinka (1982)]. □

Note that in our example in cases (d) and (f) we get the equality $Aut(\Gamma) = Aut(\Delta) \times \mathbb{Z}_2$, while in all other cases the group $Aut(\Delta) \times \mathbb{Z}_2$ appears as a proper subgroup of $Aut(\Gamma)$. Those undirected graphs Δ for which equality holds are called *stable graphs* following [Marušič et al. (1989)]. The question of when the equality $Aut(\Gamma) = Aut(\Delta) \times \mathbb{Z}_2$ holds for an undirected graph Γ turns out to be of an independent interest.

3. Semisymmetric graphs

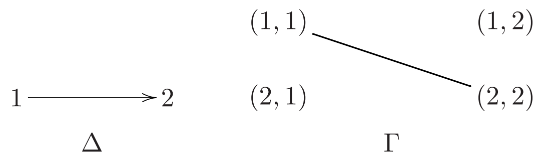
Recall that an undirected graph $\Gamma = (V, E)$ is called *semisymmetric* if it is regular (of valency k) and $Aut(\Gamma)$ acts transitively on E and intransitively on V. The proposition below is attributed by F. Harary to Elayne Dauber, its proof appears in [Harary (1969)] and [Lauri and Scapellato (2003)].

Proposition 3. *A semisymmetric graph Γ is bipartite with the partitions $V = V_1 \cup V_2$, $|V_1| = |V_2|$, and $Aut(\Gamma)$ acts transitively on both V_1 and V_2.*

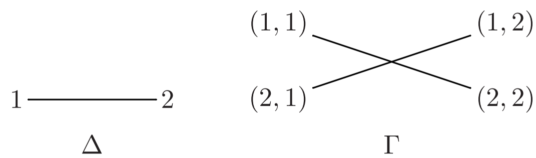
Interest in semisymmetric graphs goes back to the seminal paper [Folkman (1967)], where they were called admissible graphs. The wording "semisymmetric" was suggested in [Klin (1981)].

Following [Ivanov and Iofinova (1985)] let us call a semisymmetric graph $\Gamma = (V, E)$, $V = V_1 \cup V_2$ of *parabolic type* if the stabilizers H_1, H_2 of, respectively, two adjacent vertices $x \in V_1$ and $y \in V_2$, are not conjugate in the symmetric group $Sym(V)$. (Note that we slightly modify the original definition in [Ivanov and Iofinova (1985)].) Otherwise the

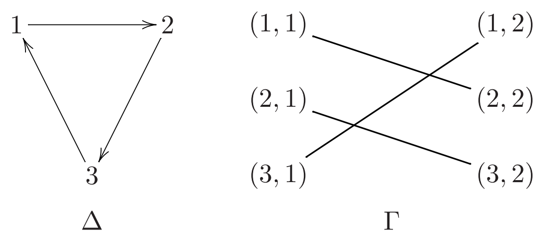
Example 2.1. (a) directed edge



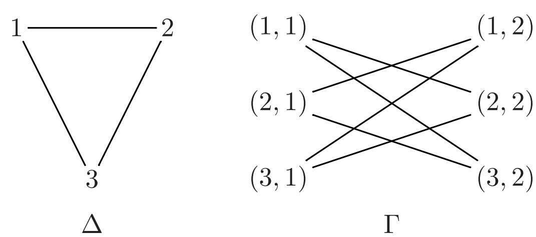
(b) undirected edge



(c) directed triangle



(d) undirected triangle



(e) quadrangle

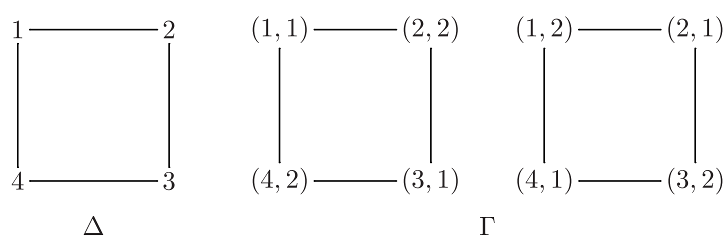
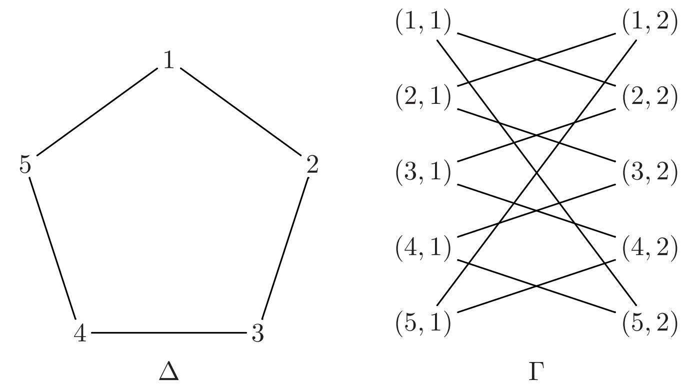


Fig. 1. Small IDC covers

(f) pentagon



(g) tournament $P(7)$

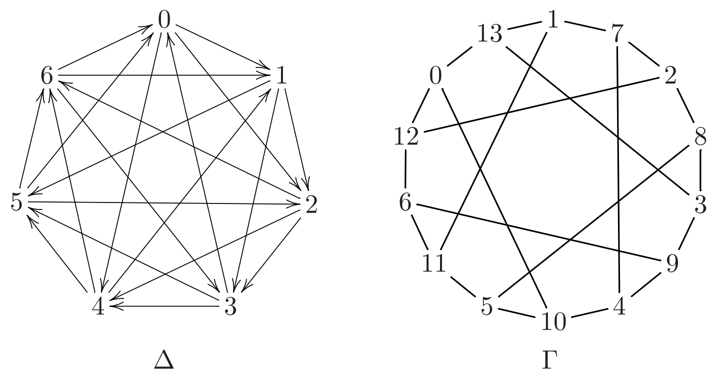


Fig. 2. Small IDC covers (cont.)

graph Γ will be said to be of *non-parabolic type*. In a more naive wording, a semisymmetric graph Γ of a parabolic type belongs to an "easy" case of such graphs. This means that one can distinguish that the vertices x and y are in different orbits with the aid of simple arguments, using suitable numerical or structural invariants of the vertices.

It turns out that the semisymmetric Ljubljana graph $\mathcal{L}$ on 112 vertices (see Section 5) is one of the smallest examples of the non-parabolic case: here both groups H_1 and H_2 are cyclic groups of order 3, and are conjugate in S_{112}. As a result, the proof of the fact that $\mathcal{L}$ is indeed semisymmetric is naturally a more sophisticated task in comparison with that for graphs of parabolic type. For the purpose of investigation of such a case we will use techniques of incidence double covers in conjunction with the ideas which were introduced in [Ivanov and Iofinova (1985)].

Let Γ be a bipartite graph with partition $V = V_1 \cup V_2$ of its vertices. In what follows we assume that Γ is an edge-transitive regular graph of valency k. Then it follows from Proposition 3 that the group $Aut(\Gamma)$ either acts transitively on V or acts intransitively with two orbits V_1 and V_2 of equal length. It might be convenient to denote by $Aut^{-}(\Gamma)$ the subgroup of $Aut(\Gamma)$ which stabilizes each set V_1 and V_2 separately. Then clearly

$[Aut(\Gamma) : Aut^{-}(\Gamma)] = 1$ or 2, depending on whether $Aut(\Gamma)$ acts on V transitively or intransitively respectively.

Now let G be a finite group, and H_1 and H_2 subgroups of G of equal index k. Denote by $\Gamma(G, H_1, H_2)$ the bipartite graph whose vertices are cosets of H_1 and H_2 in G and vertices H_1g_1 and H_2g_2 are adjacent if and only if $H_1g_1 \cap H_2g_2 \neq \emptyset$. We will call Γ the *coset graph* of G with respect to the pair of subgroups H_1, H_2. For our needs, the main result, slightly modified from [Ivanov and Iofinova (1985)], is the following.

Theorem 4 (Criterion of Iofinova-Ivanov)**.** *Let (G, Ω) be a 2-closed permutation group. Assume R is an antisymmetric 2-orbit of (G, Ω), therefore $\Delta = (\Omega, R)$ is a directed graph. Let $\Gamma = IDC(\Delta)$. Assume that the graph Γ is a connected semisymmetric graph such that $Aut(\Gamma) \cong G$. Then*

(i) *Δ is connected;*
(ii) *Δ is not bipartite;*
(iii) *Δ and Δ^t are not isomorphic;*
(iv) *$Aut(\Delta) = G$.*

4. Nikolaev graph $\mathcal{N}$

Let us now consider the graph $\mathcal{N}$, which was discovered on October 30, 1977 at Nikolaev (Ukraine). It is the first member of an infinite family of semisymmetric graphs. The result was published in [Klin (1981)], where the term "semisymmetric" was coined. The main motivation of [Klin (1981)] was to provide an affirmative answer to a question of Folkman [Folkman (1967)] about the existence of a semisymmetric graph with n vertices and valency k, such that $\gcd(n, k) = 1$. Indeed, for the graph $\mathcal{N}$ we get $\gcd(112, 15) = 1$. The construction of $\mathcal{N} = (V, E)$ is as follows:

Let the set of vertices $V = V_1 \cup V_2$, $V_1 = \{(a, b) | a, b \in [0, 7], a \neq b\}$ and $V_2 = \{X \subseteq [0, 7] || X| = 3\}$. The edge set E of $\mathcal{N}$ is $E = \{\{(a, x), \{a, b, c\}\} | x \notin \{a, b, c\}\}$.

Proposition 5. *(i) $\mathcal{N}$ is a semisymmetric graph with 112 vertices and valency 15;*
(ii) *$Aut(\mathcal{N}) \cong S_8$.*

Proof. Clearly, $\mathcal{N}$ is a regular graph of valency 15, and the symmetric group S_8 acts transitively on the sets V_1, V_2 and E. Let $G = Aut(\mathcal{N})$. Regarding $\mathcal{N}$ as the incidence graph of a symmetric incidence structure $\mathfrak{S}$, let us consider the point graph $\mathcal{P}(\mathfrak{S})$ and the block graph $\mathcal{B}(\mathfrak{S})$ defined on the sets V_1 and V_2 respectively. Easy arguments reveal that the automorphism groups of the graphs $\mathcal{P}(\mathfrak{S})$ and $\mathcal{B}(\mathfrak{S})$ are imprimitive and primitive respectively. Therefore graph $\mathcal{N}$ is indeed semisymmetric. Consideration in addition of the 2-closure of the induced symmetric group S_8, acting on the set $\left\{ {[0,7] \atop 3} \right\}$, (cf. [Klin (1974)], [Klin (1978)], [Faradžev et al. (1990)]), shows that $Aut(\mathcal{B}(\mathfrak{S})) = S_8$. Therefore, finally we get also that $Aut(\mathcal{N}) \cong S_8$. $\square$

The proof of Proposition 5, outlined above, works for any arbitrary member of the infinite series of semisymmetric graphs introduced in [Klin (1981)]. Note that the actual proof in [Klin (1981)], is of a more elementary nature, basing entirely on the counting of simple combinatorial invariants of the graph $\mathcal{N}$.

Thus the graph $\mathcal{N}$ serves as a nice example of an "easy" case of semisymmetric graph: here, the fact that $Aut(\mathcal{N})$ acts intransitively on the set V can be justified by simple arguments of a combinatorial or group-theoretic nature.

5. Ljubljana graph $\mathcal{L}$

The number three is the smallest possible valency of a semisymmetric graph. This implies a natural interest in the investigation of the smallest cubic semisymmetric graphs. Two such small graphs were known for a long while: the Gray graph $\mathfrak{G}$ on 54 vertices (see, for example, [Bouwer (1968)], [Marušič and Pisanski (2000)]) and a biprimitive graph $\mathfrak{J}$ on 110 vertices which was discovered by A. A. Ivanov [Ivanov (1994)].

A computer based search [Conder et al. (2005)], [Conder et al. (2006)] showed that $\mathcal{L}$ is the unique cubic semisymmetric graph on 112 vertices and the third smallest one after the graphs $\mathfrak{G}$ and $\mathfrak{J}$. In [Conder et al. (2006)], cubic semisymmetric graphs are considered in a much wider context, relying on a beautiful meeting of diverse techniques from group theory, topological graph theory and computer algebra.

The paper [Conder et al. (2005)] indeed provides a lot of important information about the graph $\mathcal{L}$, which is defined quite naturally using voltage assignments.

We however think that there is still an unexploited potential in reconsidering the graph $\mathcal{L}$ once more together with the group $Aut(\mathcal{L})$, paying special attention to a few association schemes and coherent configurations naturally related to $\mathcal{L}$, as well as to the embeddings of $\mathcal{L}$ into the graph $\mathcal{N}$.

6. A master association scheme on 56 points

In our attempts to get a new understanding of the graph $\mathcal{L}$ we started from the group $G = A\Gamma L(1,8) := \{x \mapsto ax^\sigma + b | a \in F_8^*, b \in F_8, \sigma \in Aut(F_8)\}$.

Clearly, $|G| = 8 \cdot 7 \cdot 3 = 168$ and G acts naturally on the set of elements of the Galois field F_8 as a 2-transitive permutation group. Identifying F_8 with the set $[0,7]$, we use the presentation $G = \langle g_1, g_2, g_3 \rangle$, where $g_1 = (1,2,3,4,5,6,7)$, $g_2 = (0,1)(2,4)(3,7)(5,6)$, $g_3 = (2,3,5)(4,7,6)$, as it appears in [Sims (1970)].

G is a subgroup of S_8, therefore there is good reason to consider again the induced action of G on the same set $V = V_1 \cup V_2$, as it was defined in Section 4.

With the aid of a computer it was discovered that, in this way, we obtain exactly 8 distinct copies of the same (up to isomorphism) graph $\mathcal{L}$, which are invariant with respect to the induced intransitive action (G, V). Each such copy appears as a spanning subgraph of a suitable copy of $\mathcal{N}$.

The stabilizer of an arbitrary vertex in $\mathcal{L}$ has order 3; thus both stabilizers of a pair of adjacent vertices are isomorphic to the cyclic group $\mathbb{Z}_3$ and are conjugates in G. Therefore, in comparison with the “easy” case of $\mathcal{N}$, this view of $\mathcal{L}$ stresses that it belongs to a more difficult case. In the following we aim to interpret the graph $\mathcal{L}$ (as well as its embeddings to $\mathcal{N}$), starting from the association scheme formed by the 2-orbits of the transitive permutation group (G, V_1). At the first we will essentially rely on the analysis of some computations carried out with the aid of computer algebra packages.

Thus let now $\Omega = V_1 = \{(x,y) | x, y \in F_8, x \neq y\}$ and let (G, Ω) be the induced transitive action of $G = A\Gamma L(1,8)$ on Ω of degree 56.

Proposition 6. *The permutation group (G, Ω) has rank 20.*

Proof. The rank of a transitive permutation group by definition is equal to the number of orbits of the stabilizer of an arbitrary point. The stabilizer of any point from Ω is similar to the induced cyclic group $(\mathbb{Z}_3, \Omega)$, $Z_3 = \langle \tilde{g}_3 \rangle$, where $\tilde{g}_3$ denotes the action of g_3

on Ω. With the aid of the orbit counting lemma (CFB lemma in [Klin et al. (1988)]), we obtain for the rank r of (G, Ω) that $r = \frac{1}{3}(\binom{8}{2} + 2 \cdot 2) = 20$. $\square$

Using COCO in conjunction with GAP, we could construct and investigate our master association scheme $\mathfrak{M} = (\Omega, 2 - orb(G, \Omega))$.

Proposition 7. *(i) There are 8 pairs of antisymmetric basic relations of valency 3 in* $\mathfrak{M}$.

(ii) All these basic graphs are not bipartite.

(iii) 6 pairs of basic graphs are connected.

(iv) The automorphism group of each of those $6 \cdot 2 = 12$ *connected (di-)graphs is* (G, Ω).

(v) In each pair of connected basic graphs, opposite graphs are not isomorphic.

According to the criterion, presented in Section 3, there is again good reason to construct the incidence double cover on 112 vertices, starting from each pair $\{R, R^t\}$ of connected antisymmetric basic graphs of valency 3. Clearly the IDC of R and R^t are isomorphic (undirected) graphs of valency 3. With the aid of GAP we distinguish the 6 pairs into 4 "good" pairs, which all provide isomorphic copies of $\mathcal{L}$ and 2 "bad" pairs, which provide a vertex transitive disconnected graph, isomorphic to 8 copies of the Heawood graph.

To explain better the observed phenomena, we further consider the normalizer $N_{S_{56}}((G, \Omega))$ of G in $S(\Omega)$, which in this case coincides with the group $CAut(\mathfrak{M})$. The second part of the corresponding computer aided results is presented below.

Proposition 8. *(i)* $N_{S_{56}}((G, \Omega)) \cong G \times \mathbb{Z}_2$ *and has order 336.*

(ii) The quotient group $N_{S_{56}}((G, \Omega))/G$ *acts on the 16 antisymmetric 2-orbits as a group of order 2.*

(iii) Each "good" 2-orbit R *is mapped to a 2-orbit* R^* *from another "good" pair under this action.*

(iv) Each "bad" 2-orbit is mapped to 2-orbit from another "bad" pair.

For the reader's convenience, the main numerical results related to the above propositions are presented in Table 1. Here we first list number i of class R_i, representative $x \in \Omega$ such that $(0, x) \in R_i$, and description of $x = (a, b)$, $a, b \in \mathcal{F}_8$. In the last column of the table we refer to the number of a merging of $\mathfrak{M}$ which is the coherent closure of R_i, according to the list of all mergings, which appears in Supplement B.

Note that we get 11 isomorphism classes of basic graphs of $\mathfrak{M}$, while 4 such classes from the 4 "good" pairs. The IDC covers split into 4 isomorphism classes, described in column "cl v". The class 3 provides the graph $\mathcal{L}$.

7. Embeddings of $\mathcal{L}$ into $\mathcal{N}$

We now wish to understand better all possible embeddings of the graph $\mathcal{L}$ into $\mathcal{N}$. Note that the union of each "good" pair of relations R and R^* is again an antisymmetric relation (of valency 6). Moreover, each such relation is a 2-orbit of the group $\widetilde{\mathfrak{G}} = CAut(\mathfrak{M}) \cong A\Gamma L(2, 8) \times \mathbb{Z}_2$. Therefore there is a sense to consider also an association scheme $\widetilde{\mathfrak{M}}$, resulting from the group $\widetilde{\mathfrak{G}}$. In principle, $\widetilde{\mathfrak{M}}$ appears as a merging (#1)

i	Rep	Pair	Val	Con	R^t	R^*	$\|Aut\|$	cl	cl v	$Aut(v)$	rank	#
0	0	(0,1)	1	F	0	0	56!	1	1	$S_{56} \wr S_2$	2	
1	1	(0,2)	3	F	5	7	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	35
2	2	(1,0)	1	F	2	2	$28! \cdot 2^{28}$	3	1	$S_{56} \wr S_2$	3	48
3	4	(1,4)	3	T	12	4	168	4	2	$S_8 \wr F_{21}$	20	
4	5	(2,0)	3	T	8	3	168	4	2	$S_8 \wr F_{21}$	20	
5	6	(0,4)	3	F	1	9	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	35
6	9	(2,5)	3	T	17	14	168	5	3	G	20	
7	11	(4,1)	3	F	9	1	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	39
8	12	(1,2)	3	T	4	12	168	6	2	$S_8 \wr F_{21}$	20	
9	14	(2,1)	3	F	7	5	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4	39
10	17	(3,6)	3	T	11	16	168	7	3	G	20	
11	18	(4,6)	3	T	10	13	168	8	3	G	20	
12	20	(4,0)	3	T	3	8	168	6	2	$S_8 \wr F_{21}$	20	
13	23	(5,2)	3	T	16	11	168	8	3	G	20	
14	29	(4,7)	3	T	15	6	168	5	3	G	20	
15	30	(7,5)	3	T	14	17	168	9	3	G	20	
16	32	(5,7)	3	T	13	10	168	7	3	G	20	
17	39	(6,3)	3	T	6	15	168	9	3	G	20	
18	43	(2,4)	3	F	18	18	$14! \cdot (4!)^{14}$	10	4	$S_{14} \wr (S_2 \times S_4)$	3	49
19	44	(4,2)	3	F	19	19	$7! \cdot 48^7$	11	4	$S_{14} \wr (S_2 \times S_4)$	5	24

Table 1. 2-orbits of $\mathfrak{M}$ and their covers

of $\mathfrak{M}$. Nevertheless, for us it was more convenient to investigate $\widetilde{\mathfrak{M}}$ independently, using again COCO, and constructing the scheme of 2-orbits of $\widetilde{\mathfrak{G}}$.

This way we obtained that $(\widetilde{\mathfrak{G}}, \Omega)$ has rank 12 with 4 pairs of antisymmetric 2-orbits of valency 6. For each such 2-orbit we again construct its IDC; for two pairs the resulting cover turns out to be a semisymmetric graph on 112 vertices of valency 6, the automorphism group of which is the group $\widetilde{\mathfrak{G}}$. We prefer to call this graph of valency 6 the *natural* double Ljubljana graph and denote it by $\mathcal{NL}$.

Again GAP was used in conjunction with COCO to obtain our next result.

Proposition 9. *(i) The union of edges from IDC $\mathcal{L}$ of a "good" relation R and $\mathcal{L}^*$ of R^* provides a semisymmetric double Ljubljana graph $\mathcal{NL}$ of valency 6 on 112 vertices.*

(ii) $Aut(\mathcal{NL}) = \widetilde{\mathfrak{G}}$.

(iii) $\mathcal{NL}$ appears as an incidence double cover of the antisymmetric 2-orbit $R \cup R^*$ of the group $\widetilde{\mathfrak{G}} = CAut(\mathfrak{M})$.

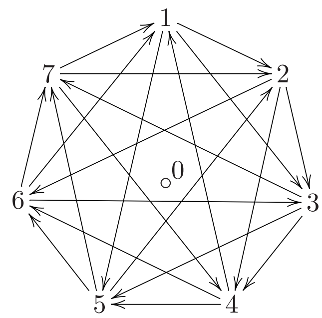


Fig. 3. Paley tournament $P(7)$ with isolated vertex

(iv) Each graph $\mathcal{NL}$ (as well as each graph $\mathcal{L}$) has a unique extension to a copy of the graph $\mathcal{N}$ of valency 15 which is invariant with respect to (G, Ω).

Thus we have managed to explain more clearly the essence of the embeddings of a "difficult" case of $\mathcal{L}$ into an "easy" case of $\mathcal{N}$. Since $Aut(\mathcal{L})$ respects this embedding, we obtain a new proof of the fact that $\mathcal{L}$ is a semisymmetric graph.

It is clear that at this stage all the results presented depend essentially on the use of a computer. In next sections we aim to remove, at least in part, such dependence. For this purpose, additional combinatorial structures will be introduced and investigated.

8. The Ljubljana configuration

We maintain that each semisymmetric graph may and should be regarded as the Levi graph (in the sense of [Coxeter (1950)]) of a symmetric incidence structure (very frequently it happens to be a configuration), which is not self-dual. Let C and C^T be two such configurations, defined by the graph $\mathcal{L}$. The diagrams of this pair of configurations are depicted in Figure 5 of [Conder et al. (2005)]; they are realized as geometric configurations of points and lines in the Euclidean plane.

Below we develop an alternative, combinatorial approach to the representation and investigation of the two 56_3 Ljubljana configurations and exploit its advantages.

First, let us consider a copy of a Paley tournament $P(7)$ with the vertex set $[1, 7]$ and isolated vertex 0, as it is depicted in Figure 3.

It is easy to check that $Aut(P(7)) = \langle g_1, g_3 \rangle$ is a Frobenius group F_{21} of order 21 and degree 7. Recall that this copy of F_{21} is simultaneously the stabilizer of the point 0 in the group $(G, [0, 7]) = (G, F_8)$.

Consider now the orbit $\mathcal{O}$ of this graph $P(7)$ under the action of (G, F_8). This orbit $\mathcal{O}$ contains 8 copies of $P(7)$, where each element from $[0, 7]$ appears exactly once as an isolated vertex. For each copy of $P(7)$ in $\mathcal{O}$ and for each vertex x of $P(7)$ we get the induced subgraph $T(x)$, generated by the out-neighbors of x. Clearly, $T(x)$ is a directed triangle. Denote by $\mathcal{B}$ the collection of all such triangles, $T(x)$. We are ready to present our first construction.

Let us identify a tuple (x, y) in Ω (Ω is defined as above) with vertex y of the copy of $P(7)$ that has x as an isolated point. Also, consider the incidence structure $\mathfrak{S} = (\Omega, \mathcal{B})$ with inclusion in the role of incidence relation.

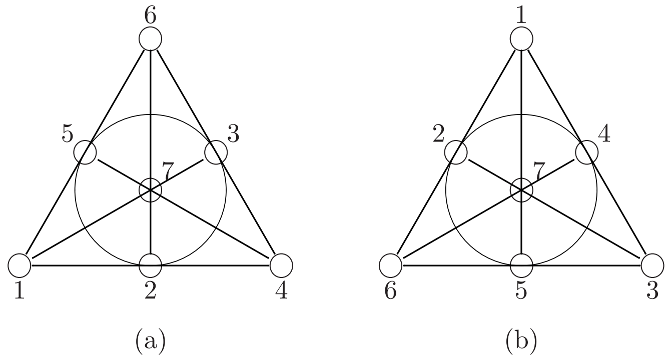


Fig. 4. Two models of Fano plane

Proposition 10. *(i)* $|\mathcal{O}| = 8$, $|\mathcal{B}| = 56$, $\mathfrak{S}$ *is a symmetric* 56_3 *configuration without repeated blocks.*
(ii) $Aut(\mathfrak{S}) = G$.
(iii) The Levi graph of the configuration $\mathfrak{S}$ *is isomorphic to* $\mathcal{L}$.

Proof. The proof of (i) is a trivial consequence of the 2-transitivity of (G, F_8).

For the proof of the remaining parts the reader is referred to the extended version of this paper [Klin et al. (Preprint)]. □

9. More auxiliary structures

We are now in a position to present a new way to glance at all possible embeddings of the Ljubljana graph $\mathcal{L}$ into the Nikolaev graph $\mathcal{N}$. With this aim in mind, let us consider some extra concepts and related combinatorial structures.

An arbitrary permutation group (H, X), according to [Betten (1977)], is called a *geometric* group if it appears as the full automorphism group of a graph or a hypergraph with the vertex set X. Here, a *hypergraph* is a collection of subsets of the set X (*hyperedges*) together with the entire (vertex) set X. In other words, a geometric permutation group can be interpreted as the group of all symmetries of a suitable incidence structure.

According to [Klin et al. (In preparation)], let us call a permutation group (H, X) a *geometric group of the second order* if (H, X) is the full automorphism group of a suitable collection of graphs or hypergraphs.

Let us now consider again the group $G = A\Gamma L(1, 8)$. This group, of course, is not the full automorphism group of any graph. Counting orbits of (G, Ω) on the 3− and 4−subsets of F_8, and comparing the obtained numbers with the similar ones for an overgroup $AGL(3, 2)$ of (G, F_8) in $Sym(F_8)$ (cf. e.g. [Sims (1970)]) it is easy to reveal that (G, Ω) is not a geometric group (for details see, e.g., [Klin et al. (In preparation)]).

At this point let us consider again one more famous structure, namely the projective plane $\mathcal{F} = PG(2, 2)$, commonly known under the name of Fano plane. The classical picture in Figure 4(a) depicts a difference set model for $\mathcal{F}$.

Indeed, the seven lines of this model are obtained from the line $\{1, 2, 4\}$ via consecutive cyclic shift with the aid of the permutation g_1 (see Section 6).

In a similar manner one more model (depicted in part (b) of the same Figure) appears from the line $\{3,5,6\}$. Let us call these two models of $\mathcal{F}$ the *standard model* and the *non-standard model* respectively. Clearly, the two models are isomorphic and have disjoint sets of lines. Both models are invariant with respect to the same cyclic group $\langle g_1 \rangle$ of order 7.

According to [Colbourn and Rosa (1999)], an *overlarge set* $\mathcal{O}(v,k)$ of Steiner triple systems $S(2,3,v)$ is a partition of the set of all 3-element subsets of a $(v+1)$-element set into $v+1$ disjoint Steiner systems, each of type $S(2,3,v)$. A similar definition may be formulated for the case of Steiner systems $S(k-1,k,v)$. A pioneering paper, in which such overlarge sets were investigated is [Sharry and Street (1988)] (though the name itself was coined later on).

It was proved in [Sharry and Street (1988)] that up to isomorphism, there exist 11 different $\mathcal{O}(7,3)$ on 8 points, having groups of order 1344, 168, 96, 64, 48, 24, 24, 8, 8, 8, 6. Of course, all these groups provide particular examples of geometric groups. The two most symmetric models for $\mathcal{O}(7,3)$ are of a particular interest in this text. Below we briefly repeat the representations of these models, as they are described in [Klin et al. (In preparation)].

For this purpose, first we introduce a copy E_8 of an elementary Abelian group of order 8 acting regularly on the set $F_8 = [0,7]$. E_8 contains exactly 7 involutions t_i as follows: $t_1 = (0,1)(2,4)(3,7)(5,6)$, $t_2 = (0,2)(1,4)(3,6)(5,7)$, $t_3 = (0,3)(1,7)(2,6)(4,5)$, $t_4 = (0,4)(1,2)(3,5)(6,7)$, $t_5 = (0,5)(1,6)(2,7)(3,4)$, $t_6 = (0,6)(1,5)(2,3)(4,7)$, $t_7 = (0,7)(1,3)(2,5)(4,6)$.

Note that $t_1 = g_2$; it is easy to check that the group $AGL(1,8) = \langle g_1, t_7 \rangle$ has order 56, acts sharply 2-transitively on F_8 and is a subgroup of index 3 in our group G.

Construction 9.1. Let us now regard our standard (non-standard) copy of $\mathcal{F}$ as a copy $\mathcal{F}_0$ carrying an extra isolated point 0. Then we define 7 new copies of $\mathcal{F}$ as the respective images $\mathcal{F}_i := \mathcal{F}_0^{t_i}$. Finally, we denote by $\mathcal{O}_S(7,3)$ (or $\mathcal{O}_N(7,3)$) the collection of 8 Fano planes $\{\mathcal{F}_0, \ldots, \mathcal{F}_7\}$, depending on which model of $\mathcal{F}$ (standard or non-standard) is used for the initial copy $\mathcal{F}_0$.

Proposition 11. *(i) $\mathcal{O}_S = \mathcal{O}_S(7,3)$ is an overlarge set with the automorphism group $Aut(\mathcal{O}_S)$ isomorphic to $AGL(3,2)$, a group of order 1344.*

(ii) $\mathcal{O}_N = \mathcal{O}_N(7,3)$ is an overlarge set with the automorphism group $Aut(\mathcal{O}_N) = G = A\Gamma L(1,8)$ of order 168.

Proof. A computer free proof is presented in [Klin et al. (In preparation)], though of course, the reader can easily confirm it with the aid of a computer, or even by routine hand computations. □

Corollary. The group $G = (A\Gamma L(1,8), \Omega)$ is a geometric group of the second order.

Remark. A mysterious (at first sight) distinction between the standard and non-standard models of $\mathcal{F}$ relies on the different role of the selected copy of E_8 with respect to the two prescribed models of the Fano plane.

Using these ideas we can now present the following.

Construction 9.2. Start with the non-standard model $\mathcal{F}_N$ of the Fano plane and consider the orbit $\mathcal{P}$ of $\mathcal{F}_N$ under the action of A_8. We get $|\mathcal{P}| = \frac{|A_8|}{|PSL(2,7)|} = \frac{8!}{2 \cdot 168} = 120$.

Consider also orbits $\mathcal{O}_N^{A_8}$ and $(\mathcal{O}_S^z)^{A_8}$ of non-standard and "skew" standard overlarge sets respectively under the action of A_8. Check that $|\mathcal{O}_N^{A_8}| = 120$, $|(\mathcal{O}_S^z)^{A_8}| = 15$. Define $L = \mathcal{O}_N^{A_8} \cup (\mathcal{O}_S^z)^{A_8}$ and consider the incidence structure $(\mathcal{P}, L)$ with incidence defined by inclusion.

Remark. In this construction, we need a shifted (skew) copy of the standard overlarge set with the aid of an odd permutation. So, for example, we here take z to be the permutation $(2,6)(3,7)(4,5)$, which makes $\mathcal{O}_S^z$ another isomorphic copy of $\mathcal{O}_S(7,3)$.

Proposition 12. *(i) The incidence structure $(\mathcal{P}, L)$ provides a model of a partial geometry $PG(8,9,4)$.*

(ii) The automorphism group of this partial geometry is isomorphic to the alternating group A_8.

Proof. A proof which is computer-free but relies on a number of combinatorial and group theoretical arguments is available in [Klin et al. (In preparation)]. □

In the next section we will try to benefit, at least implicitly, from the consideration of the presented A_8-geometry.

10. Embeddings of $\mathcal{L}$ into $\mathcal{N}$ revisited

Here we are interested in investigating once more all embeddings of $\mathcal{L}$ into a prescribed copy of $\mathcal{N}$, provided certain natural requirements are satisfied. These requirements will be formulated in group theoretical terms.

First, we start from the action of the group S_8 on the set V as it appears in Section 4. It is easy to understand that there are two copies of the graph $\mathcal{N}$ which are invariant with respect to (A_8, V): the one with the edge set E (as in Section 4) and the one with the edge set $E' = \{\{(x,a), \{a,b,c\}\} | x \notin \{a,b,c\}\}$. Both copies $\mathcal{N}$ and $\mathcal{N}'$ have the same group $Aut(\mathcal{N}) = Aut(\mathcal{N}') \cong S_8$. Moreover, these two copies are interchanged with the aid of the involution τ, which transposes pairs (a,b) and (b,a) from V_1 and fixes each element of V_2.

Thus, in principle, one may consider the group $S_8 \times S_2$, acting on the set V, and classify all embeddings into either $\mathcal{N}$ or $\mathcal{N}'$. We nevertheless prefer to fix a concrete *master copy* of the Nikolaev graph, say $\mathcal{N}$. In this fashion, the group $S_8 = Aut(\mathcal{N})$ still remains our "universal" group.

Let us now consider a concrete copy of the group $G = A\Gamma L(1,8)$ as a subgroup of S_8, and let us investigate all copies of the graph $\mathcal{L}$, which are invariant with respect to this selected group G and which are spanning subgraphs of the master graph $\mathcal{N}$. As we may easily deduce from the analysis of the master association scheme $\mathfrak{M}$, there exist exactly 8 different copies of $\mathcal{L}$ which are invariant with respect to (G, V). However computer analysis shows that only two of these copies are spanning subgraphs of the same copy of $\mathcal{N}$.

Proposition 13. *(1) For a given copy $\mathcal{N}$ of the Nikolaev graph and the group $Aut(\mathcal{N}) = S_8$ there exists exactly 1920 copies of the graph $\mathcal{L}$, which are invariant with respect to a suitable subgroup of S_8, isomorphic to G.*

(2) *480 copies of $\mathcal{L}$ in addition are spanning subgraphs of the master copy $\mathcal{N}$ and form two orbits with respect to S_8.*
(3) *Each of the above two orbits of embeddings of $\mathcal{N}$ splits into two of length 120 with respect to the group (A_8, V).*

Now we wish to define a representative $\mathcal{L}$ of one of four orbits of A_8. For this purpose first we need to use an alternative construction of $O_N(7,3)$.

Construction 10.1. An alternative construction of $\mathcal{O}_N(7,3)$. Start with a prescribed copy of $S(3,4,8)$ and with its group $AGL(3,2)$ of order 1344. This group has 8 conjugate subgroups of order 168, each isomorphic to our group G. Select a copy of such a group G, and observe that the group G acts transitively on the set X of 56 4-subsets which is complementary to the block set of our Steiner design $S(3,4,8)$. Note that for each point x the stabilizer G_x of order 21 contains the unique cyclic subgroup $\mathbb{Z}_7$ of order 7. The 28 elements from X which contain x split into two orbits of G_x of the length 7 and 21. Consider the orbit of length 7 and remove x from each 4-subset. Obtain a copy of the Fano plane with the point set $X \setminus \{x\}$. In this way we obtain eight copies of the Fano plane (with all possible isolated points) which form a copy of $\mathcal{O}_N(7,3)$.

For the reader's convenience, in Supplement A we provide a list of all 8 Fano planes in the resulting copy of the same $\mathcal{O} = \mathcal{O}_N(7,3)$.

Now we are ready to describe a copy of the graph $\mathcal{L}$. It has again vertex set $V = V_1 \cup V_2$, exactly like graph $\mathcal{N}$ in Section 4. Consider vertex (a,b) from V_1. Find a copy F_a of Fano plane in our overlarge set O which does not contain vertex a. Find in this copy F_a three lines through the point b. Substitute b by a in each of the three lines. Get the three neighbours of (a,b) in our copy $\mathcal{L}$. For example, for the pair $(0,1)$, according our procedure, we obtain triples $\{0,2,6\}$, $\{0,3,4\}$, $\{0,5,7\}$. It is clear that the resulted graph $\mathcal{L}$ is indeed a spanning subgraph of $\mathcal{N}$ as it appears in Section 4.

In what follows we will call the above copy of $\mathcal{L} = (V,E)$ the *canonical Ljubljana subgraph* of the master copy $\mathcal{N}$ (with respect to a prescribed copy G of the group $A\Gamma L(1,8)$). The orbit of A_8 on the 120 copies of the canonical copy $\mathcal{L}$ should be called the *canonical set* $\mathcal{E}$ of the embeddings of $\mathcal{L}$ into the master copy of $\mathcal{N}$. This canonical set $\mathcal{E}$ is, in fact, the subject of our further investigations.

We consider transitive permutation group $(A_8, \mathcal{E})$ and investigate it with the use of COCO. Let $\mathcal{X} = (\mathcal{E}, 2 - orb(A_8, \mathcal{E}))$ be the Schurian association scheme formed by the 2-orbits of $(A_8, \mathcal{E})$.

Proposition 14. *(1) The group $(A_8, \mathcal{E})$ has rank 5; valencies of non-reflexive 2-orbits R_i, $1 \leq i \leq 4$, are 42, 14, 56, 7; all the orbits are symmetric.*
(2) *$\mathcal{X}$ has 6 non-trivial mergings $\mathcal{X}_1 = (1,2)$, $\mathcal{X}_2 = (1,3)$, $\mathcal{X}_3 = (3,4)$, $\mathcal{X}_4 = (1,2)(3,4)$, $\mathcal{X}_5 = (1,3,4)$, $X_6 = (1,2,3)$ (we use here brief COCO notation for the mergings).*
(3) *The strongly regular graphs corresponding to the mergings X_4, X_5, X_6, have parameters $(120,56,28)$, $(120,14,13)$, $(120,7,6)$ respectively.*
(4) *$|Aut(\mathcal{X}_4)| = 1290240 = 2^5 \cdot 8!$.*

Additional analysis shows that the strongly regular graph generating the rank 3 scheme $\mathcal{X}_4$ is isomorphic to the graph which was described in [Brouwer et al. (1989)].

This gives extra interest to the association scheme $\mathcal{X}$. In addition we observe that the basic graph Γ_3 of valency 56 is Deza graph (in the sense of [Deza and Deza (1994)], [Erickson et al. (1999)]). We again summarize our computer aided discoveries.

Proposition 15. *The rank 5 Schurian association scheme $\mathcal{X}$ is generated by the Deza graph Γ_3.*

The elaboration of a computer free proof of the theorem deserves special attention, though it is out of the scope of the current paper.

According to the terminology introduced in [Klin et al. (2009)], the scheme $\mathcal{X}$ provides an example of a Schurian Deza family in a Higmanian house. To best of our knowledge, this fact is new. Moreover, it seems that after the Schurian example on 40 points described in [Klin et al. (2009)], the current example on 120 points is the second non-trivial one which appears in the literature.

11. The Dejter approach to the Ljubljana graph

An alternative approach to the Ljubljana graph was developed by I. Dejter et al in a sequence of papers [Dejter and Guan (1991)], [Dejter and Weichsel (1993)], [Brouwer et al. (1993)], [Dejter (1994)], [Dejter and Pujol (1995)], [Borges and Dejter (1996)], [Dejter (1997)]. This approach is quite original and it strictly differs from the other ways already presented. We shall only give a brief summary of his method here.

Denote by $\mathcal{E}_n$ the full automorphism group $Aut(Q_n)$ of the n-cube Q_n. It is well-known that $|\mathcal{E}_n| = 2^n \cdot n!$; the group $\mathcal{E}_n$ (as an abstract group) is isomorphic to the wreath product of the symmetric groups of degree n and 2.

Let us now consider the Hamming code H_3 as the induced subgraph of Q_7. It is easy to check that it forms a coclique of Q_7 and thus each vertex from H_3 has all 7 neighbors in the set $V(Q_7) \setminus H_3$, which in this section will be denoted by Ω. The proposition below describes all the information that is required at this stage about the stabilizer F of H_3 in $E_7 = Aut(Q_7)$. Though this proposition was also obtained with the aid of a computer, it may be proved using computer free arguments.

Proposition 16. *(i) $F \cong AGL(3,2) \times S_2$, has order 2688 and acts transitively and faithfully on the subset H_3 of the set $V(Q_7)$.*

(ii) The group F acts transitively and faithfully on the set Ω.

(iii) An equitable partition of Q_7 formed by the two orbits of size 16 and 112 of the group F has matrix $\begin{pmatrix} 0 & 7 \\ 1 & 6 \end{pmatrix}$.

(iv) The subgraph D of Q_7, induced by the vertex subset Ω, has valency 6 and is vertex- and edge-transitive.

(v) $Aut(D) \cong F = AGL(3,2) \times S_2$.

In what follows we will call the graph D the *Dejter graph*. One would wish here to provide for the graph D a self contained computer free description, which allows also us to figure out easily its full automorphism group. Such a task requires considering more combinatorial structures. Although this proof is not difficult, we omit it here because it consists of a long list of case-by-case analysis. (Details are available in [Klin et al. (Preprint)].)

The main reason for our interest in the Dejter graph in the context of this paper was the fact, originally observed by Dejter et al, that D splits into two copies of the graph $\mathcal{L}$. The proof of the existence of such a split depends on the selected model of the Dejter graph. An outline of such a proof for the model D, that is induced subgraph of Q_7, was provided in [Brouwer et al. (1993)]. (See also [Klin et al. (Preprint)].)

Finally we mention a few new interesting non-Schurian association schemes which were discovered in the course of the investigation of the centralizer algebra of the group $F = Aut(D)$.

The centralizer algebra W of the group F has rank 16. In fact, it is a direct product of two centralizer algebras of orders 56 and 2 and ranks 8 and 2 respectively. With the aid of COCO we investigated all coherent subalgebras of W. There are 81 such (non-trivial) subalgebras, the rank of which varies between 3 and 11. It turns out that a few of these algebras are non-Schurian: a quite rare occurrence for the direct product of two Schurian algebras such that each of them contains only Schurian subalgebras. Of a special interest is a commutative subalgebra of rank 7 and valencies $1, 6, 7, 7, 21, 28, 42$, having the same group F of order 2688. There is also a pair of isomorphic rank 5 non-Schurian algebras with valencies $1, 7, 28, 28, 48$, which belongs to the intersection of classes I and II in the sense of [Higman (1995)]. (We refer to the paper [Klin et al. (2009)] where the problem of investigation of association schemes of rank 5, as it was formulated in [Higman (1995)], is considered in the flavor of computer algebra experimentation.) The automorphism group in this case is isomorphic to $S_2 \times (E_{64} : PSL(3, 2))$, has order 21504 and rank 10. In our view, these non-Schurian coherent algebras deserve special attention in the future.

12. Some association schemes on 56 points

We now come back to the consideration of all merging schemes of our master association scheme $\mathfrak{M}$ on 56 points as it was presented in Section 6. Although we discovered this scheme in our attempts to understand the graphs $\mathcal{L}$, $\mathcal{NL}$ and $\mathcal{N}$, in this section we shall see that this scheme and its mergings take on a life of their own and are worthy of an independent study. In fact, this association scheme and its mergings turns out to be one of the most important by-products of our investigations.

Recall that there are altogether exactly 50 merging schemes, which split into 43 isomorphism classes under the action of the group $CAut(\mathfrak{M})$. The isomorphism classes are as follows (only those consisting of two schemes are listed; the remaining ones form classes consisting of a single scheme): $\{2, 3\}$, $\{18, 19\}$, $\{22, 33\}$, $\{35, 39\}$, $\{37, 41\}$, $\{38, 40\}$, $\{46, 47\}$ (see Supplement B).

Among the 50 merging schemes not all are of equal interest. The most important ones are the eight non-Schurian schemes, two of rank 6 and six of rank 5. A detailed analysis of these eight non-Schurian schemes can be found in the preprint [Klin et al. (Preprint)]. The results of the investigation which we present in [Klin et al. (Preprint)] is work in progress and will be published elsewhere together with a wider panorama of the entire collection of all mergings of $\mathfrak{M}$, both Schurian and non-Schurian. For the reader's convenience in [Klin et al. (Preprint)] we also provide the Hasse diagram for all (up to isomorphism) 43 merging association schemes of the scheme $\mathfrak{M}$.

13. Two-fold isomorphisms and related concepts

The modified Iofinova-Ivanov criterion, presented in Section 3, gives a necessary condition for $\Gamma = IDC(\Delta)$ to be a semisymmetric graph. We know that if, in addition, the permutation group (G, Ω) is primitive, then this necessary condition also becomes sufficient.

But, as soon as (G, Ω) is imprimitive this condition is not sufficient. Indeed, relations R_i, $i = 3, 4, 8, 12$ (see Table 6.1) provide counterexamples although all properties in the criterion are satisfied.

In our view, the observed phenomenon is one of the most significant by-products of this paper. Further clarification of this phenomenon is necessary. One of the helpful concepts to be considered for such a purpose is two-fold automorphisms and two-fold 2-orbits, as they were recently introduced in [Lauri et al. (2004)], [Lauri et al. (2011)].

Thus, let G be a subgroup of $Sym(V)^2$ and let Γ be a graph or digraph on the vertex set V. For $(a, b) \in G$ and (x, y) an arc of Γ, let $(a, b) : (x, y) \mapsto (x^a, y^b)$. When the following condition holds: "(x, y) is an arc of Γ if and only if (x^a, y^b) is also an arc", then (a, b) is said to be a *TF-automorphism* of Γ.

We believe that TF-automorhisms provide a natural language whenever there is the posibility of considering simultaneously the same matrix $A(\Gamma)$ as the incidence matrix $I(S)$ of an incidence structure S. Our first attempts at analysing the symmetries of graphical matrices from this point of view indicate that TF-automorhisms can create the suitable context for bringing out the relationship between the combinatorial and the formal algebraic understanding of these symmetries.

We should also mention here that it seems that Bohdan Zelinka (1940-2005), whose contribution to AGT is probably underrated, was actually the first mathematician who studied TF-automorphisms (see [Zelinka (1972)], [Zelinka (1982)] and [Zelinka (1983)], for example).

14. Further work

Computer algebra experimentation in AGT is the main subject of this paper. Our approach was to start from the consideration of two concrete graphs, $\mathcal{L}$ and $\mathcal{N}$, both on 112 vertices and from there to follow simply, in a logical and natural way, the path of computer aided investigation of association schemes and diverse structures which were naturally linked to the starting graphs. The discovery of a number of new, quite interesting, association schemes on 56, 112 and 120 points confirmed that our methodology was natural: it detected links of the investigated graphs with many other structures considered in AGT which could otherwise have remained hidden from sight.

Several interesting additional links between the objects considered by us are discussed in [Klin et al. (Preprint)] within the wider context of AGT, in much more detail and with more references.

Now, to end this paper we wish to suggest a few concrete formulations of tasks for further research which are, in our view, closely related to the area of scientific computation and AGT.

Problem 1. To carry out the constructive enumeration of all (up to isomorphism) small semisymmetric graphs (thus extending the results in [Ivanov (1987)]), paying a special attention to the graphs of non-parabolic type. It seems that relying in particular on catalogs of transitive permutation groups currently available in GAP, this task may be fulfilled at least to $n = 86$ vertices.

Problem 2. To arrange an extensive computer aided experimentation in order to measure the efficiency of the Iofinova-Ivanov criterion, as it is modified in Theorem 4. The goals would be to find new nice examples of the cases where it does not work, and/or to reach further strengthening of the criterion, relying on the properties of the revealed examples.

Problem 3. To apply the techniques of two-fold automorphisms to the study of the Doyle-Holt graph on 27 vertices and its automorphism group of order 54 (see [Holt (1981)], [Weisstein (2010)], [Klin et al. (2010)] and references in it). The double cover of this graph (as well as of the corresponding antisymmetric 2-orbit) may provide a worthwhile training ground for a better understanding of the similarities and distinctions in the behavior of the IDC for directed and undirected graphs.

Acknowledgements

It is a great pleasure for author MK to thank officially (after more than 30 years!) Laci Babai for his assistance in the publication of [Klin (1981)] at a time when it was difficult for MK to attend conferences outside the Soviet Union. The details and circumstances of this communication with Laci are presented in [Babai (1999)] (see also [Klin et al. (Preprint)]). Conversations with Tomo Pisanski in 2003 about the graph $\mathcal{L}$ were very helpful. We thank Italo Dejter, Marston Conder and Misha Muzychuk for helpful material, discussions and communication. The authors were very fortunate to have participated in diverse projects jointly with Sven Reichard, Raffaele Scapellato, Russell Mizzi and Andy Woldar. These efforts were very influential in the writing of this paper. Finally, we wish to thank Viktor Levandovskyy and Dušan Pagon for their kind and constructive attention to this project as well as anonymous referees for supportive comments.

Supplement A. List of $\mathcal{O}_N(7,3)$

0	$\{1,2,6\},\{3,5,6\},\{2,3,7\},\{4,6,7\},\{1,3,4\},\{1,5,7\},\{2,4,5\}$
1	$\{0,2,7\},\{0,3,6\},\{2,4,6\},\{0,4,5\},\{5,6,7\},\{3,4,7\},\{2,3,5\}$
2	$\{0,1,3\},\{0,4,7\},\{3,5,7\},\{0,5,6\},\{1,6,7\},\{1,4,5\},\{3,4,6\}$
3	$\{0,1,5\},\{1,4,6\},\{0,6,7\},\{1,2,7\},\{2,5,6\},\{4,5,7\},\{0,2,4\}$
4	$\{0,1,7\},\{1,2,3\},\{3,6,7\},\{1,5,6\},\{0,3,5\},\{0,2,6\},\{2,5,7\}$
5	$\{0,1,2\},\{2,3,4\},\{1,4,7\},\{2,6,7\},\{0,4,6\},\{0,3,7\},\{1,3,6\}$
6	$\{0,1,4\},\{2,4,7\},\{0,2,3\},\{3,4,5\},\{1,2,5\},\{1,3,7\},\{0,5,7\}$

Supplement B. List of mergings of association scheme $\mathfrak{M}$

No.	rank	merging	unmerged	$\lvert Aut \rvert$
1	12	(1,7)(5,9)(3,4)(12,8)(6,14)(17,15)(10,16)(11,13)	2,18,19	336
2	8	(1,5)(6,17,10,11,13,16,14,15)(7,9)(18,19)(3,8)(12,4)	2	1344
3	8	(2,19)(4,8,6,17)(7,9)(10,11)(1,12,13,15)(5,3,16,14)	1	1344
4	8	(1,3,4,7)(5,12,8,9)(6,11,13,14)(17,10,16,15)	2,18,19	21504
5	8	(1,3,6,11)(5,12,17,10)(4,7,13,14)(8,9,16,15)	2,18,19	672
6	8	(1,3,13,14)(5,12,16,15)(4,6,7,11)(8,17,9,10)	2,18,19	1344
7	8	(1,4,11,14)(5,8,10,15)(3,6,7,13)(12,17,9,16)	2,18,19	1344
8	8	(1,6,7,14)(5,17,9,15)(3,4,11,13)(12,8,10,16)	2,18,19	2688
9	8	(1,5)(2,19)(3,12,14,15)(13,16)(4,6,9,10)(8,17,7,11)	2	1344
10	7	(1,5)(6,17,10,11,13,16,14,15,18,19)(7,9)(3,8)(12,4)	2	40320
11	7	(18,19)(1,3,4,7)(5,12,8,9)(6,11,13,14)(17,10,16,15)	2	$2^{28} \cdot 168$
12	6	(6,17,10,11,13,16,14,15)(18,19)(1,12,8,7)(5,3,4,9)	2	336
13	6	(1,5,3,12,4,8,7,9)(6,17,14,15)(10,11,13,16)(18,19)	2	2688
14	6	(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	2,18,19	$2^{28} \cdot 3^8 \cdot 7$
15	6	(1,5,3,12,4,8,7,9)(18,19)(6,10,16,14)(17,11,13,15)	2	336
16	6	(1,5,6,17,7,9,14,15)(3,12,4,8,10,11,13,16)	2,18,19	21504
17	6	(1,5,7,9)(3,12,4,8)(6,17,10,11,13,16,14,15)(18,19)	2	2688
18	6	(2,18,19)(1,3,4,6,11,13,14)(5,12,8,17,10,16,15)	7,9	846720
19	6	(2,18,19)(3,4,6,7,11,13,14)(12,8,17,9,10,16,15)	1,5	846720
20	6	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15)	2,18,19	172032
21	5	(2,19)(1,3,8,17,9,10,13,14)(5,12,4,6,7,11,16,15)	18	1344
22	5	(2,19)(1,12,4,17,7,10,16,14)(5,3,8,6,9,11,13,15)	18	168
23	5	(2,19)(1,12,4,17,9,11,16,14)(5,3,8,6,7,10,13,15)	18	10752
24	5	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)	2,18,19	$2^{32} \cdot 3^9 \cdot 5 \cdot 7$
25	5	(6,17,10,11,13,16,14,15,18,19)(1,12,8,7)(5,3,4,9)	2	336
26	5	(18,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	2	$2^{49} \cdot 3^8 \cdot 7$
27	5	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15)(18,19)	2	$2^{31} \cdot 168$
28	5	(1,5,7,9)(3,12,4,8)(6,17,10,11,13,16,14,15,18,19)	2	80640
29	5	(1,5,3,12,6,17,10,11)(2,19)(4,8,7,9,13,16,14,15)	18	672
30	5	(1,5,3,12,13,16,14,15)(2,19)(4,8,6,17,7,9,10,11)	18	10752
31	5	(1,5,4,8,10,11,14,15)(2,19)(3,12,6,17,7,9,13,16)	18	10752
32	5	(2,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	18	$2^{49} \cdot 3^{15} \cdot 7$
33	5	(2,19)(1,3,8,6,7,10,16,15)(5,12,4,17,9,11,13,14)	18	168
34	5	(2,19)(1,3,8,6,9,11,16,15)(5,12,4,17,7,10,13,14)	18	672
35	4	(2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)	1,5	$2^7 \cdot 3^{10} \cdot 5 \cdot 7^9$
36	4	(2,18,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)		$21 \cdot (8!)^7$
37	4	(1,5,3,12,4,8,6,17,10,11,13,16,14,15)(2,18,19)(7,9)		$8 \cdot (7!)^2$
38	4	(1,5,3,12,7,9,13,16,14,15)(2,10,11,19)(4,8,6,17,18)		40320
39	4	(1,5,2,3,12,4,8,6,17,10,11,13,16,14,15,18,19)	7,9	$2^7 \cdot 3^{10} \cdot 5 \cdot 7^9$
40	4	(1,5,4,8,6,17,7,9,10,11)(2,13,16,19)(3,12,14,15,18)		40320
41	4	(1,5)(2,18,19)(3,12,4,8,6,17,7,9,10,11,13,16,14,15)		$8! \cdot (7!)^2$
42	4	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15,18,19)	2	$2^{28} \cdot 8!$
43	4	(1,5,6,17,7,9,14,15,19)(3,12,4,8,10,11,13,16,18)	2	$8 \cdot 9!$
44	4	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(2,19)	18	$7! \cdot (2 \cdot 24^2)^7$
45	4	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(18,19)	2	$7! \cdot (2^4 \cdot 4!)^7$
46	3	(1,5)(2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)		$8! \cdot (7!)^8$
47	3	(1,5,2,3,12,4,8,6,17,10,11,13,16,14,15,18,19)(7,9)		$8! \cdot (7!)^8$
48	3	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)	2	$2^{28} \cdot 28!$
49	3	(1,5,2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,19)	18	$(4!)^{14} \cdot 14!$
50	3	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(2,18,19)		$7! \cdot (8!)^7$

References

[Babai (1999)] Babai, L., 1999. The forbidden sidetrip. In: People & ideas in theoretical computer science. Springer Ser. Discrete Math. Theor. Comput. Sci. Springer, Singapore, pp. 1–31.

[Bannai and Ito (1984)] Bannai, E., Ito, T., 1984. Algebraic combinatorics. I. The Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA, association schemes.

[Betten (1977)] Betten, D., 1977. Geometrische Permutationsgruppen. Mitt. Math. Gesellsch. Hamburg 10 (5), 317–324.

[Biggs (1993)] Biggs, N., 1993. Algebraic graph theory, 2nd Edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge.

[Borges and Dejter (1996)] Borges, J., Dejter, I. J., 1996. On perfect dominating sets in hypercubes and their complements. J. Combin. Math. Combin. Comput. 20, 161–173.

[Bouwer (1968)] Bouwer, I. Z., 1968. An edge but not vertex transitive cubic graph. Canad. Math. Bull. 11, 533–535.

[Brouwer et al. (1989)] Brouwer, A. E., Cohen, A. M., Neumaier, A., 1989a. Distance-regular graphs. Vol. 18 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin.

[Brouwer et al. (1993)] Brouwer, A. E., Dejter, I. J., Thomassen, C., 1993. Highly symmetric subgraphs of hypercubes. J. Algebraic Combin. 2 (1), 25–29.
URL `http://dx.doi.org/10.1023/A:1022472513494`

[Brouwer et al. (1989)] Brouwer, A. E., Ivanov, A. V., Klin, M. H., 1989b. Some new strongly regular graphs. Combinatorica 9 (4), 339–344.
URL `http://dx.doi.org/10.1007/BF02125346`

[Cameron (1999)] Cameron, P. J., 1999. Permutation groups. Vol. 45 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge.
URL `http://dx.doi.org/10.1017/CBO9780511623677`

[Colbourn and Rosa (1999)] Colbourn, C. J., Rosa, A., 1999. Triple systems. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York.

[Conder et al. (2005)] Conder, M., Malnič, A., Marušič, D., Pisanski, T., Potočnik, P., 2005. The edge-transitive but not vertex-transitive cubic graph on 112 vertices. J. Graph Theory 50 (1), 25–42.
URL `http://dx.doi.org/10.1002/jgt.20089`

[Conder et al. (2006)] Conder, M., Malnič, A., Marušič, D., Potočnik, P., 2006. A census of semisymmetric cubic graphs on up to 768 vertices. J. Algebraic Combin. 23 (3), 255–294.
URL `http://dx.doi.org/10.1007/s10801-006-7397-3`

[Coxeter (1950)] Coxeter, H. S. M., 1950. Self-dual configurations and regular graphs. Bull. Amer. Math. Soc. 56, 413–455.

[Dejter (1994)] Dejter, I. J., 1994. On symmetric subgraphs of the 7-cube: an overview. Discrete Math. 124 (1-3), 55–66, graphs and combinatorics (Qawra, 1990).
URL `http://dx.doi.org/10.1016/0012-365X(92)00128-E`

[Dejter (1997)] Dejter, I. J., 1997. Symmetry of factors of the 7-cube Hamming shell. J. Combin. Des. 5 (4), 301–309.
URL `http://dx.doi.org/10.1002/(SICI)1520-6610(1997)5:4<301::AID-JCD5>3.3.CO;2-B`

[Dejter and Guan (1991)] Dejter, I. J., Guan, P., 1991. Square-blocking edge subsets in hypercubes and vertex avoidance. In: Graph theory, combinatorics, algorithms, and applications (San Francisco, CA, 1989). SIAM, Philadelphia, PA, pp. 162–174.

[Dejter and Pujol (1995)] Dejter, I. J., Pujol, J., 1995. Perfect domination and symmetry in hypercubes. In: Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995). Vol. 111. pp. 18–32.

[Dejter and Weichsel (1993)] Dejter, I. J., Weichsel, P. M., 1993. Twisted perfect dominating subgraphs of hypercubes. In: Proceedings of the Twenty-fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1993). Vol. 94. pp. 67–78.

[Deza and Deza (1994)] Deza, A., Deza, M., 1994. The ridge graph of the metric polytope and some relatives. In: Polytopes: abstract, convex and computational (Scarborough, ON, 1993). Vol. 440 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, pp. 359–372.

[Dixon and Mortimer (1996)] Dixon, J. D., Mortimer, B., 1996. Permutation groups. Vol. 163 of Graduate Texts in Mathematics. Springer-Verlag, New York.

[Erickson et al. (1999)] Erickson, M., Fernando, S., Haemers, W. H., Hardy, D., Hemmeter, J., 1999. Deza graphs: a generalization of strongly regular graphs. J. Combin. Des. 7 (6), 395–405.
URL `http://dx.doi.org/10.1002/(SICI)1520-6610(1999)7:6<395::AID-JCD1>3.3.CO;2-L`

[Faradžev et al. (1990)] Faradžev, I. A., Ivanov, A. A., Klin, M. H., 1990. Galois correspondence between permutation groups and cellular rings (association schemes). Graphs Combin. 6 (4), 303–332.
URL http://dx.doi.org/10.1007/BF01787700

[Faradžev and Klin (1991)] Faradžev, I. A., Klin, M. H., 1991. Computer package for computations with coherent configurations. In: Proc. ISSAC. ACM Press, Bonn, pp. 219–223.

[Faradžev et al. (1994)] Faradžev, I. A., Klin, M. H., Muzichuk, M. E., 1994. Cellular rings and groups of automorphisms of graphs. In: Investigations in algebraic theory of combinatorial objects. Vol. 84 of Math. Appl. (Soviet Ser.). Kluwer Acad. Publ., Dordrecht, pp. 1–152.

[Folkman (1967)] Folkman, J., 1967. Regular line-symmetric graphs. J. Combinatorial Theory 3, 215–232.

[GAP (2008)] GAP, 2008. GAP – Groups, Algorithms, and Programming, Version 4.4.12. The GAP Group.
URL http://www.gap-system.org

[Godsil and Royle (2001)] Godsil, C., Royle, G., 2001. Algebraic graph theory. Vol. 207 of Graduate Texts in Mathematics. Springer-Verlag, New York.

[Harary (1969)] Harary, F., 1969. Graph theory. Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London.

[Higman (1970)] Higman, D. G., 1970. Coherent configurations. I. Rend. Sem. Mat. Univ. Padova 44, 1–25.

[Higman (1987)] Higman, D. G., 1987. Coherent algebras. Linear Algebra Appl. 93, 209–239.
URL http://dx.doi.org/10.1016/S0024-3795(87)90326-0

[Higman (1995)] Higman, D. G., 1995. Rank 5 association schemes and triality. Linear Algebra Appl. 226/228, 197–222.
URL http://dx.doi.org/10.1016/0024-3795(95)00102-W

[Holt (1981)] Holt, D. F., 1981. A graph which is edge transitive but not arc transitive. J. Graph Theory 5 (2), 201–204.
URL http://dx.doi.org/10.1002/jgt.3190050210

[Ivanov (1994)] Ivanov, A. A., 1994. Computation of lengths of orbits of a subgroup in a transitive permutation group. In: Investigations in algebraic theory of combinatorial objects. Vol. 84 of Math. Appl. (Soviet Ser.). Kluwer Acad. Publ., Dordrecht, pp. 275–282.

[Ivanov and Iofinova (1985)] Ivanov, A. A., Iofinova, M. E., 1985. Biprimitive cubic graphs. In: Investigations in the algebraic theory of combinatorial objects (Russian). Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, pp. 123–134.

[Ivanov (1987)] Ivanov, A. V., 1987. On edge but not vertex transitive regular graphs. In: Combinatorial design theory. Vol. 149 of North-Holland Math. Stud. North-Holland, Amsterdam, pp. 273–285.
URL http://dx.doi.org/10.1016/S0304-0208(08)72893-7

[Klin (1974)] Klin, M., 1974. Investigation of relational algebras invariant under the action of certain classes of permutation groups (ph.d. thesis).

[Klin et al. (2007)] Klin, M., Muzychuk, M., Pech, C., Woldar, A., Zieschang, P.-H., 2007. Association schemes on 28 points as mergings of a half-homogeneous coherent configuration. European J. Combin. 28 (7), 1994–2025.
URL http://dx.doi.org/10.1016/j.ejc.2006.08.010

[Klin et al. (2009)] Klin, M., Muzychuk, M., Ziv-Av, M., 2009a. Higmanian rank-5 association schemes on 40 points. Michigan Math. J. 58 (1), 255–284.
URL http://dx.doi.org/10.1307/mmj/1242071692

[Klin et al. (2010)] Klin, M., Pech, C., Reichard, S., Woldar, A., Ziv-Av, M., 2010. Examples of computer experimentation in algebraic combinatorics. Ars Math. Contemp. 3, 237–258.

[Klin et al. (2009)] Klin, M., Reichard, S., Woldar, A., 2009b. Siamese combinatorial objects via computer algebra experimentation. In: Algorithmic algebraic combinatorics and Gröbner bases. Springer, Berlin, pp. 67–112.
URL http://dx.doi.org/10.1007/978-3-642-01960-9_2

[Klin et al. (In preparation)] Klin, M., Reichard, S., Woldar, A., In preparation. On a simple group of order 504, its automorphism group and related combinatorial structures.

[Klin et al. (1988)] Klin, M. C., Pöschel, R., Rosenbaum, K., 1988. Angewandte Algebra für Mathematiker und Informatiker. VEB Deutscher Verlag der Wissenschaften, Berlin, einführung in gruppentheoretisch-kombinatorische Methoden. [Introduction to group-theoretical combinatorial methods].

[Klin (1978)] Klin, M. H., 1978. Examination on the MIR-1 computer of exclusive cases in the problem of maximality of induced symmetric groups. In: Computations in algebra and combinatorial analysis (Russian). Akad. Nauk Ukrain. SSR Inst. Kibernet., Kiev, pp. 54–72.

[Klin (1981)] Klin, M. H., 1981. On edge but not vertex transitive graphs. In: Algebraic methods in graph theory, Vol. I, II (Szeged, 1978). Vol. 25 of Colloq. Math. Soc. János Bolyai. North-Holland, Amsterdam, pp. 399–403.

[Klin et al. (Preprint)] Klin, M. H., Lauri, J., Ziv-Av, M., Preprint. Links between two semisymmetric graphs on 112 vertices through the lens of association schemes.

[Lauri et al. (2004)] Lauri, J., Mizzi, R., Scapellato, R., 2004. Two-fold orbital digraphs and other constructions. Int. J. Pure and Appl. Math. Sci. 1, 63–93.

[Lauri et al. (2011)] Lauri, J., Mizzi, R., Scapellato, R., 2011. Two-fold automorphisms of graphs. The Australasian J. Combin. 49, 165–176.

[Lauri and Scapellato (2003)] Lauri, J., Scapellato, R., 2003. Topics in graph automorphisms and reconstruction. Vol. 54 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge.

[wikipedia (2010)] LJU, 2010. The ljubljana graph.
URL http://en.wikipedia.org/wiki/Ljubljana_graph

[Marušič and Pisanski (2000)] Marušič, D., Pisanski, T., 2000. The Gray graph revisited. J. Graph Theory 35 (1), 1–7.
URL http://dx.doi.org/10.1002/1097-0118(200009)35:1<1::AID-JGT1>3.0.CO;2-7

[Marušič et al. (1989)] Marušič, D., Scapellato, R., Zagaglia Salvi, N., 1989. A characterization of particular symmetric $(0, 1)$ matrices. Linear Algebra Appl. 119, 153–162.
URL http://dx.doi.org/10.1016/0024-3795(89)90075-X

[McKay (1990)] McKay, B. D., 1990. nauty user's guide (version 1.5).

[Sharry and Street (1988)] Sharry, M. J., Street, A. P., 1988. Partitioning sets of triples into designs. Ars Combin. 26 (B), 51–66.

[Sims (1970)] Sims, C. C., 1970. Computational methods in the study of permutation groups. In: Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967). Pergamon, Oxford, pp. 169–183.

[Soicher (1993)] Soicher, L. H., 1993. GRAPE: a system for computing with graphs and groups. In: Groups and computation (New Brunswick, NJ, 1991). Vol. 11 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, pp. 287–291.

[Weisstein (2010)] Weisstein, E. W., 2010. Doyle graph.
URL `http://mathworld.wolfram.com/DoyleGraph.html/`

[Wielandt (1969)] Wielandt, H., 1969. Permutation groups through invariant relations and invariant functions. Ohio State University, Columbus.

[Zelinka (1972)] Zelinka, B., 1972. Isotopy of digraphs. Czechoslovak Math. J. 22(97), 353–360.

[Zelinka (1982)] Zelinka, B., 1982. On double covers of graphs. Math. Slovaca 32 (1), 49–54.

[Zelinka (1983)] Zelinka, B., 1983. Double covers and logics of graphs. Czechoslovak Math. J. 33(108) (3), 354–360.