On the edge-reconstruction number of disconnected graphs

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Abstract

The edge-reconstruction number of a graph G is the minimum number of edge-deleted subgraphs which are required to determine the isomorphism type of G. Molina has shown that a disconnected graph whose components are not all isomorphic has edge-reconstruction number at most three. He also showed that under certain conditions, including the property that at least one component has a cycle, the edge-reconstruction number is 2. In this paper we give an alternative proof of Molina's main result, characterise those disconnected graphs which have the largest possible edge-reconstruction number, and we also investigate what properties can force a forest to have edge-reconstruction number 2.

1 Introduction

The *edge-reconstruction number* of a graph G, denoted by $ern(G)$ is equal to the least number of edge-deleted subgraphs (also called *edge-cards*) which alone can determine G up to isomorphism. The *(vertex-)reconstruction number*, denoted by $rn(G)$ is similarly defined as the least number of vertex-deleted subgraphs (called *vertex-cards*) of G which alone can determine G. Other graph theoretic definitions not given here can be found in [3] or [5]. In this paper we shall study the edge-reconstruction number of a disconnected graph G. Unless specifically stated otherwise, we will always assume that G has at least two non-trivial components and at least four edges. We often say that G is the union of its components. In [7] the following result is proved.

Theorem 1.1 *Let G be a disconnected graph with at least three edges and at least two non-trivial components. Then*

1. *If not all components of G are isomorphic, then $ern(G) \leq 3$;*
2. *If all components are isomorphic then $ern(G) \leq t+2$ where t is the number of edges of a single component;*
3. *If there are two non-isomorphic components at least one of which has a cycle and G does not have any components isomorphic either to K_3 or $K_{1,3}$, then $ern(G) \leq 2$.*

In this paper we shall give a short proof of the first part of Theorem 1.1 based on earlier work by Myrvold and Molina himself on the vertex-reconstruction number of disconnected graphs, we shall characterise those graphs in the second part of the theorem which have edge-reconstruction number equal to $t+2$ and we shall investigate forests, which are not covered by the third part, which have edge-reconstruction number equal to 2.

We shall need the following result, first presented in [8] with the proof corrected in [6], and the second appearing in [2].

Theorem 1.2 *Let G be a disconnected graph. Then,*

1. *If not all components are isomorphic then $rn(G) = 3$, and one of the three cards can always be chosen by deleting a non-cutvertex from a component with a maximal number of vertices;*
2. *If all components are isomorphic and each has c vertices then $rn(G) \leq c+2$.*

Theorem 1.3 *Let G be a disconnected graph in which all components are isomorphic and each has c vertices. If $rn(G) \geq c+1$, then G must be a union of complete graphs K_c. In particular, no graph with all components isomorphic and having c vertices can have reconstruction number equal to $c+1$.*

We shall also need the concept of a *line graph.* The line graph $L(G)$ of a graph G is the graph whose vertices are the edges of G and such that two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. If e is an edge in G we often refer to the corresponding vertex in $L(G)$ as e^*. We shall say that G is L-invertible if it can be uniquely determined by its line graph. In that case, if $G' = L(G)$ then we sometimes denote G by $L^{-1}(G')$. It is well known that the only connected graphs which are not L-invertible are K_3 and $K_{1,3}$, since both have K_3 as line graph; as in [7] we shall denote the set containing these two graphs by

S. Therefore a disconnected graph is L-invertible if and only if none of its components is isomorphic to K_3 or $K_{1,3}$ or is an isolated vertex.

It is clear that if e is an edge of G and e^* is the corresponding vertex of $L(G)$, then $L(G-e) = L(G) - e^*$. From this observation we obtain the following result which is a simple variation of a result of Greenwell and Hemminger [4].

Lemma 1.1 *For any L-invertible graph G, $ern(G) \leq rn(L(G))$.*

Proof. Let $rn(L(G)) = p$. Therefore there exist vertices $e_1^*, \ldots, e_p^*$ such that $L(G)$ is uniquely determined by $L(G) - e_1^*, \ldots, L(G) - e_p^*$. We claim that $G - e_1, \ldots, G - e_p$ reconstruct G uniquely, therefore $ern(G) \leq p$.

Take the line graphs of $G - e_1, \ldots, G - e_p$. By the observation above, these are $L(G) - e_1^*, \ldots, L(G) - e_p^*$, and therefore $L(G)$ is determined. But G is L-invertible, therefore $L(G)$ gives us G, uniquely. □

2 Not all components are isomorphic

By Lemma 1.1 and the first part of Theorem 1.2, what we now need to do in order to prove the first part of Theorem 1.1 is to tackle the case when the disconnected graph is not L-invertible.

The following would appear to be a simple way to proceed. Let G be a disconnected graph with at least two non-isomorphic components and not all components coming from **S**. By Theorem 1.2, $L(G)$ can be reconstructed from three cards. Analogously to Lemma 1.1 we obtain G_0 which is G except for possibly a number of components which are triangles in $L(G)$ and which we do not know if they are K_3 or $K_{1,3}$ in G. But we know that there are, say, k of them. Let $\mathcal{A}$ be the components of G not from **S**. From G_0 we know the total number p of vertices in the components in $\mathcal{A}$. From any one of the subgraphs $G-e$ we also know the total number q of vertices of G. Therefore the number of vertices in the non-L-invertible components is $q-p$. So, let s and t be the number of components in G isomorphic to $K_{1,3}$ and K_3, respectively. Therefore $s+t=k$ and $4s+3t=q-p$, but we know k and $q-p$, therefore we can find s and t and hence G.

However, this approach fails because G can have isolated vertices. Since these are lost when taking line graphs, although we do know the right values of p and q, we do not know whether the value of $q-p$ is equal to the number of vertices in the non-L-invertible components or whether it also contains a number of isolated vertices. Therefore solving the above two equations for s and t does give us the correct total number of components from **S** in G but not the correct mix of stars and triangles. We therefore need to proceed more carefully.

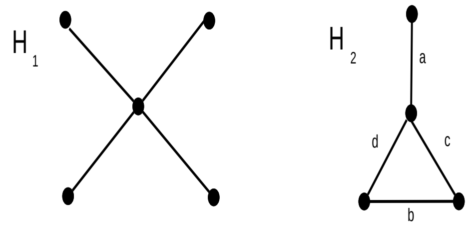

Figure 1: The two graphs with a member of **S** in an edge-card

As above, we first assume that G is a disconnected graph with at least two non-isomorphic components and at least one of them is not in **S**. We again let $\mathcal{A}$ be the collection of components of G none of which is isomorphic to a graph in **S**, that is, the L-invertible components, and let $\mathcal{B}$ the components of G isomorphic to one or the other of the graphs in **S** ($\mathcal{B}$ could be empty). Using line graphs of three appropriately (according to Theorem 1.2) chosen edge-cards we reconstruct the L-invertible components of G, the number k of non-invertible components in $\mathcal{B}$, but not the number of isolated vertices. Let $G - e$ be the edge-card corresponding to the vertex-card of $L(G)$ with maximal order (as per Theorem 1.2). If in $G - e$ we see the right number k of components from $\mathcal{B}$ then we are done. The problem arises if in $G - e$ there are more than the required k components from **S**. Since, by Theorem 1.2 e^* is not a cutvertex in $L(G)$, this can only happen if the edge e is taken from a component C of G which is like either one of the two in Figure 1. This way, in $G - e$ there is one more component from **S** than the correct number k.

Now if C is the first graph H_1 shown in Figure 1, then $G - e$ has one less component isomorphic to H_1 than there are in $\mathcal{A}$ (remember that we know $\mathcal{A}$) and also an extra component than there should be in $\mathcal{B}$. But then there is only one way to add an edge to $G - e$ such that a member of **S** is transformed into H_1. Therefore we have unique reconstruction in this case. This argument however does not work when C is H_2, because in this case both K_3 (plus an isolated vertex) and $K_{1,3}$ can be changed into H_2. But we can dispose of this case very easily without resort to line-graphs.

Lemma 2.1 *Let G be a disconnected graph whose components are not all isomorphic and one of which is the graph H_2 shown in Figure 1. Then G is uniquely reconstructible from the two edge-cards $G - c$ and $G - b$.*

Proof. Let G be a disconnected graph whose components are not all isomorphic but one of which is isomorphic to H_2. Let b and c be two edges of H_2 as shown in the figure. Therefore the edge-card $C_1 = G - b$ has a component isomorphic to $K_{1,3}$ and the edge-card $C_2 = G - c$ has a component isomorphic to the path P_4. Let e' be the new edge added to C_1. If e' is made to join two vertices from a component isomorphic to $K_{1,3}$ then there is only one way to do this, up to isomorphism. This would give G.

We must therefore consider what happens if either the edge e' joins two vertices of any two components or it joins two vertices in the same component but which is not isomorphic to $K_{1,3}$ to form a new component H. In this case, in order to obtain the edge-card C_2, one has to remove an edge from the newly formed component H or one isomorphic to it, with the result that one cannot obtain the necessary number of P_4 components present in card C_2. Therefore this possibility leads to a contradiction, hence the two edge-cards C_1, C_2 reconstruct G uniquely. $\square$

The last case we now need to tackle is when all the components of G are from $\mathbf{S}$. Note that Theorem 1.1 leaves open the possibility $ern = 2$ in this case. Here we show that this is not possible. We obviously cannot take line-graphs here since the line-graph of G would have all of its components isomorphic. So we need to proceed directly, but this is not difficult.

Lemma 2.2 *Let G be a disconnected graph whose non-trivial components are not all isomorphic but all of which come from $\mathbf{S}$. Then $ern(G) = 3$*

Proof. Suppose G has exactly two types of non-trivial components, namely p copies of $K_{1,3}$ and q copies of K_3. Therefore the edge-deck of G consists of two different edge-cards C_1 and C_2. The edge-card C_1 is the graph $G - e_1$ where e_1 is an edge in K_3 and therefore C_1 consists of p copies of $K_{1,3}$, $(q-1)$ copies of K_3, one copy of P_3 and r isolated vertices, where $r \geq 0$. The edge-card C_2 is the graph $G - e_2$ where e_2 is an edge in $K_{1,3}$ and therefore C_2 consists of $(p-1)$ copies of $K_{1,3}$, q copies of K_3, one P_3 and $r+1$ isolated vertices. Suppose

- G_1 is a disconnected graph whose components are P_3, H_2, $(p-1)$ copies of $K_{1,3}$ and $q-1$ copies of K_3 and r isolated vertices;
- G_2 is a disconnected graph whose components are K_3, P_4, $(p-1)$ copies of $K_{1,3}$, $(q-1)$ copies of K_3 and r isolated vertices;
- G_3 is a disconnected graph whose components are Z, made up of a vertex joined to the two centres of two copies of $K_{1,2}$, $(p-1)$ copies of $K_{1,3}$, $(q-1)$ copies of K_3, and r isolated vertices.

It is easy to check that whether we are given the edge-cards C_1 and C_2, or two copies of C_1 or two copies of C_2, one of G_1, G_2, G_3 has the two edge-cards in its edge-deck. Therefore two cards are not sufficient to reconstruct G uniquely.

To show that three cards are sufficient we reproduce, for completeness' sake, Molina's proof. Let e_1 be an edge in K_3 and e_2, e_3 two edges in $K_{1,3}$. Assume that there is a graph in which $G - e_i \simeq H - e_i'$ for $i = 1, 2, 3$. Now, e_1 is not a cut-edge, so the replacing edge e must join two vertices in the same component of $G - e_1$. If e forms a component with four edges, then G and H can have only two cards in common, which contradicts the above assumption. Therefore e must join two vertices in the component P_3, that is, $G \simeq H$. □

This therefore gives the first part of Theorem 1.1.

3 All components isomorphic

Our aim here is to prove a result similar to Theorem 1.3 to describe exactly which are the graphs in Theorem 1.1 which have the highest possible edge-reconstruction number $t + 2$, where t is the number of edges of a single component. This will be easy to do by taking line-graphs and using Theorem 1.3.

Theorem 3.1 *Let G be a disconnected graph with at least three edges and such that all components are isomorphic to a graph H. Then*

1. *If H is isomorphic to K_3, then $ern(G) = 2$;*
2. *If H is isomorphic to $K_{1,3}$, then $ern(G) = 5$;*
3. *If H is not isomorphic to K_3 or $K_{1,3}$ then $ern(G) \leq t + 2$, where t is the number of edges in H. Moreover, if $ern(G) \geq t + 1$ then $H \simeq K_{1,t}$.*

Proof. 1. Let G consist of p copies of K_3 and let the graph T be obtained from any $G - e$ by adding an edge e', $e' \neq e$. Then there is at most one other card in $\mathcal{ED}(G)$ that is also in $\mathcal{ED}(T)$, therefore $ern(G) = 2$.

2. Let G consist of p copies of $K_{1,3}$. Let $Q = P_3 \cup K_{1,4} \cup (p-2)K_{1,3}$. Then $ern(G) = 5$ because four edge-cards are also in $\mathcal{ED}(Q)$, whereas there is no graph with five edge-cards in common with G.

3. In this case G is L-invertible, and $L(G)$ also has all its components isomorphic. Therefore Theorem 1.2 gives that $rn(L(G)) \leq t + 2$ from which it follows, using Lemma 1.1, that $ern(L(G)) \leq t + 2$.

Now suppose $ern(G) \geq t + 1$, therefore $rn(L(G)) \geq t + 1$. Therefore, by Theorem 1.3, $L(G)$ is the union of components isomorphic to K_t. But since $t > 3$, this gives that G is the union of components isomorphic to $K_{1,t}$, as required. □

The gap between $ern = 3$ and $ern = t+1$ is crying out to be investigated. We single out this general question. (See also [1].)

Question 3.1 *Let G be a disconnected graph with all components isomorphic to H and where H has t edges. What is the largest number N such that $ern(G) = N$ and H is not isomorphic to $K_{1,t}$? Is N a function of t or is there a constant N such that whenever $ern(G) > N$ then $H \simeq K_{1,t}$?*

We shall come back to this question at the end of the paper.

4 Forests

We shall now try to investigate conditions which force or do not allow $ern(G)$ to be equal to 2. In view of Theorem 1.1 we shall focus our attention on the case when G is a forest and not all of its components are isomorphic.

There is, in general, no straightforward relationship between the edge-reconstruction number of G and that of its components. For example, $ern(K_{1,t}) = 2$ for $t > 3$, but $ern(K_{1,r} \cup K_{1,t}) = 3$, where $t \neq r$ or $(r + 1)$. Another simple example: $ern(P_3) = 1$ but $ern(P_3 \cup P_5) = 3$.

The opposite effect can also be seen, namely, the edge-reconstruction number can go down from 3 to 2 when another component is added. For example, $ern(P_n) = 3$ for $n > 4$ but $ern(P_n \cup K_{1,3}) = 2$. Another simple example: $ern(C_n) = 3$ but $ern(C_n \cup K_{1,3})$ and $ern(C_n \cup K_3)$ are both equal to 2.

Moreover, since almost every graph has edge-reconstruction number equal to 2 [5] it is not feasible to try and characterise all disconnected graphs with $ern = 2$. We shall therefore try to obtain some conditions which are sufficient or necessary to guarantee $ern = 2$.

4.1 A few general results

We first start with a result which is very helpful in order to eliminate possibilities when trying to show that $ern(G) = 2$.

Lemma 4.1 *Suppose G has two components H and K such that H has a cut-edge e and K has a cut-edge f such that a component of $H - e$ is isomorphic to a component of $K - f$. Then $G - e$ and $G - f$ do not reconstruct G.*

Proof. Let $G = H \cup K \cup \mathcal{A}$ where $\mathcal{A}$ consists of a number of other components (possibly empty). Suppose that H consists of two components X and Y joined by the edge e, and K consists of two components X and Z joined by the edge f. That is, X is the component of H isomorphic to a component of K.

Let G' be a graph made up of the components $\mathcal{A}$ and two components X and W, where W is the component X joined by two edges: one edge e'

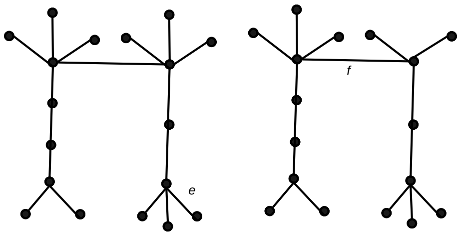

Figure 2: Counterexample to converse of Lemma 4.1: the graph G

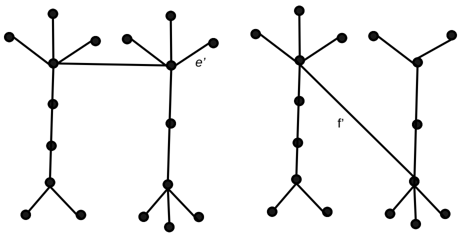

Figure 3: Counterexample to converse of Lemma 4.1: the graph G'

joins X to component Y and another edge f' joins X to Z, and such that $W - e' \simeq Y \cup K$ and $W - f' \simeq H \cup Z$.

Then it is clear that G' is not isomorphic to G but $G' - e' \simeq G - e$ and $G' - f' \simeq G - f$. Therefore $G - a$ and $G - b$ are not sufficient to reconstruct G. □

One would like the converse of the conditions of Lemma 4.1 to be sufficient for reconstruction, however the converse of this Lemma is, unfortunately, not true, as the following example shows.

Example 4.1 *Let G consist of the two components shown in Figure 2, and let G' be the graph whose two components are as shown in Figure 3. Then no component of $G - e$ is isomorphic to a component of $G - f$, but $G - e \simeq G' - e'$ and $G - f \simeq G' - f'$, with $G \not\simeq G'$. Therefore $G - e$ and $G - f$ do not reconstruct G.*

Even if e and f are not cut-edges, the converse of Lemma 4.1 still does not hold, as Figure 4 shows.

Example 4.2 *In Figure 4, $H - e$ is not isomorphic to $K - f$, $G - e \simeq G' - e'$ and $G - f \simeq G' - f'$ but $G \not\simeq G'$.*

But, by adding appropriate extra conditions we can avoid such counterexamples to obtain partial converses of Lemma 4.1 such as the following.

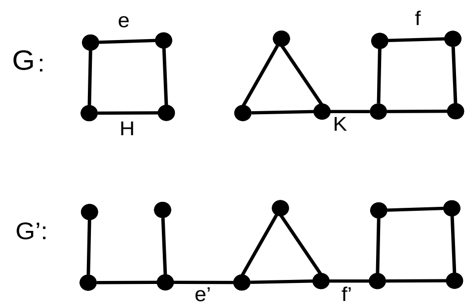

Figure 4: Counterexample to converse of Lemma 4.1 when e and f are not cut-edges

Lemma 4.2 *Let G be a disconnected graph with two non-isomorphic components H and K. Suppose that H has a cut-edge e and K has a cut-edge f such that,*

1. *No component of $H - e$ is isomorphic to a component of $K - f$; and*
2. *There exists no edge $f' \notin E(K - f)$ such that $H' = K - f + f'$ is isomorphic to a component of $G - e$.*

Then $G - e$ and $G - f$ reconstruct G.

Proof. Let C_1 be the edge-card $G-f$ which consists of the components H, A, B, and let C_2 be the edge-card $G-e$ which consists of the components K, C, D. Therefore component H is not isomorphic to any of K, C, D and component K is not isomorphic to any of H, A, B. Without loss of generality let $|E(H)| \geq |E(K)|$. Suppose that a reconstruction $G = G - f + f'$ of G is obtained by adding a new edge f' to the edge-card C_1. We now require that $G - f + f'$ contains the edge-card C_2 in its edge-deck.

Case 1: The edge f' joins two vertices of H. In order to obtain the edge-card C_2 from this reconstruction we must remove an edge from the component $H + f'$, with the result that there remain two components A and B which are not in the edge-card C_2.

Case 2: The edge f' either joins two vertices in A or two vertices in B. Without loss of generality we can suppose that f' joins two vertices in A. Since component H does not appear in the edge-card C_2, to obtain this edge-card from G' we must remove an edge from H. But this leaves the component B which is not in C_2.

Case 3: The edge f' joins a vertex in H to a vertex in A or in B. This case also leads to a contradiction using similar arguments as in Case 2.

Case 4: The edge f' joins a vertex in A to a vertex in B. Suppose that the new component formed by adding the edge f' is H'. Suppose, for contradiction, that $H' \not\simeq K$. Since H does not appear in C_2, to obtain this edge-card from G' we must remove an edge from the component H in G'. Therefore H' appears as a component in C_2. Since we are assuming that $H' \not\simeq K$, then it must be isomorphic to C or D. But this means that $H' = K - f + f'$ is isomorphic to a component of $G - e$, which contradicts the second condition of the Lemma. □

These results give us a sufficient condition for a forest with two components to have $ern = 2$

Theorem 4.1 *Let G be a disconnected graph with two non-isomorphic trees H and K as components, such that both H and K are not stars and $|V(K)| < |V(H)| - 1$. Then $ern(G) = 2$.*

Proof. Suppose that an endvertex e is deleted from H to give one edge-card $G - e$ and a non-endvertex f is deleted from K to give the other edge-card $G - f$. Then $G - e$ and $G - f$ satisfy the conditions of Lemma 4.2 and therefore reconstruct G. □

In the cases considered so far, the two edge-cards which give the reconstruction arise by deleting edges from different components. The next results address this question. We have noted earlier that, even if $ern(H) = 2$ it could happen that $ern(H \cup K) > 2$. If two edge-cards from the same component reconstruct that component, what conditions can ensure that this property extend to the graph obtained by adding another component? First we need a definition.

A graph K is said to have a *replaceable edge* e if there exists and edge $e' \notin E(K)$ such that $K \simeq K - e + e'$.

Theorem 4.2 *Let the graph G consist of two connected and non-isomorphic components H and K such that $|E(H)| \neq |E(K)|+1$. Let e_1, e_2 be two edges in H. Suppose that either $H - e_1 \neq H - e_2$ or K does not have a replaceable edge. Let $C_1 = H - e_1 \cup K$ and $C_2 = H - e_2 \cup K$ and let $G' = (H - e_1) \cup K + e$, where e is any new edge added to K.*

Then the graph G' cannot contain the two edge-cards C_1, C_2 in its edge-deck

Proof. Suppose the lemma is false. Then card C_2 must be isomorphic to the edge-card $H - e_1 \cup K + e - e'$ of G', where e' is an edge in K. Now, $H - e_1$ cannot be isomorphic to K, by the condition on the number of edges of H and K. Therefore $H - e_1 \simeq H - e_2$ and $K + e - e' \simeq K$. But these possibilities cannot hold, by hypothesis. □

As a simple application of this result we give the following corollary. This shows that, under certain conditions, if G is a forest obtained by adding a star as a component to a tree with edge-reconstruction number equal to 2, then $ern(G)$ is still equal to 2.

Corollary 4.1 *Let T be a tree such that for some $e_1, e_2 \in E(T)$ the two subgraphs $T - e_1, T - e_2$ reconstruct T. Suppose that $T - e_1$ consists of the two components T_1 and T_2, while $T - e_2$ consists of the components T_3 and T_4, such that none of T_1 and T_2 is isomorphic to any of T_3 or T_4. Assume also that S is a star such that $|E(T)| \neq |E(S)| + 1$. Let $G = S \cup T$. Then $G - e_1$ and $G - e_2$ reconstruct G.*

Proof. All the conditions of the previous theorem are satisfied. Therefore if G' is a graph containing the two edge-cards $C_1 = G - e_1$ and $C_2 = G - e_2$, then G' cannot be obtained from any of these edge-cards by adding an edge to the component S. Therefore G' is obtained from $G - e_1$ either (i) by adding an edge joining vertices from the two components T_1 and T_2, creating a new component T'; or (ii) by adding an edge joining vertices from the component S and one of T_1 or T_2, say T_1, without loss of generality. Let the new component so created be S'.

In case (i), the edge-card C_2 can only be obtained from G' by removing an edge from T'. Therefore T' has the edge-cards $T_1 \cup T_2$ and $T_3 \cup T_4$. But these reconstruct T, therefore $T' \simeq T$.

In case (ii), the edge-card C_2 can be obtained from G' by removing an edge from the component S'. But if $T_2 = S$, then again component S' contains the edge-cards $T_1 \cup T_2$ and $T_3 \cup T_4$, which means that $S' \simeq T$. If $T_2 \neq S$, then the component S must be obtained by deleting an edge from S'. Without loss of generality we can then assume that $T_2 \simeq T_4$, again contradicting one of the corollary's assumption. □

All conditions in the previous theorem are required as can be seen by the following example.

Example 4.3 *The tree T shown in Figure 5 is reconstructible from the two edge-cards $T - e_1$ and $T - e_2$. Let $S = K_{1,4}$ and $G = T \cup S$. Let $G' = T' \cup K_{1,2}$ where T' is also shown in Figure 5. Then $G - e_1$ and $G - e_2$ do not reconstruct G because they are isomorphic to $G' - e'_1$ and $G' - e'_2$, respectively, but G' is not isomoprhic to G.*

4.2 A few special cases

We now give a few special cases of forests with two non-isomorphic components and their edge-reconstruction number. We do not give the proofs

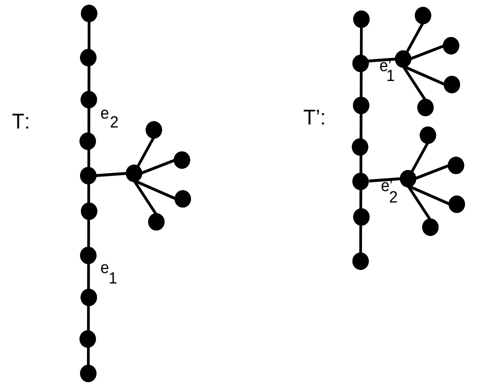

Figure 5: T is reconstructible from $T - e_1$ and $T - e_2$, but if $G = T \cup K_{1,4}$ and $G' = T' \cup K_{1,2}$ then $G - e_1$ and $G - e_2$ are isomorphic to $G' - e_1'$ and $G' - e_2'$, respectively, but G' is not isomorphic to G.

since they all follow easily from the results in the previous sub-section or can be easily checked directly.

Theorem 4.3 *The following forests with two components have edge-reconstruction numbers as given:*

1. $ern(K_{1,n} \cup K_{1,n+1}) = 2$, $n \geq 3$.
2. $ern(K_{1,n} \cup K_{1,m}) = 3$, $m \neq n$ *or* $n+1$ *and* $n > 2$.
3. $ern(P_n \cup K_{1,m}) = 2$, $n > 3, m > 1$.
4. $ern(P_2 \cup K_{1,m}) = 3$, $n > 2$.
5. *Let* $G = H \cup K$ *where* $H = K_{1,m}, m > 3$ *and* K *is formed by adding a vertex joining the centres of two copies of* $K_{1,m-1}$*. Then* $ern(G) = 2$.
6. *Let* $G = H \cup K$ *where* $H = K_{1,m}, m > 3$ *and* K *is formed by adding a vertex joining the centres of two copies of* $K_{1,m-2}$*. Then* $ern(G) = 3$.

5 Conclusion

In spite of [7] and our efforts in this paper, not everything is clear about the edge-reconstruction number of disconnected graphs. One would have liked to find the most general partial converse to Lemma 4.1 which could perhaps go a long way towards understanding when such graphs with non-isomorphic components have $ern > 2$ but, as we have seen with examples, simple converses of this lemma do not hold and partial converses such as Lemma 4.2 have to be investigated (another example is Lemma 3 in Section 4 of [7].)

But the most obvious gap in our knowledge of the edge-reconstruction number of disconnected graph is when all components are equal. We have already discussed this situation above in Question 3.1. We end here by answering that question with a conjecture. As empirical evidence for this conjecture we remark that out of more than a billion graphs on at most eleven vertices only seventy have edge-reconstruction number greater than 3 [9]. An examination of these graphs, made available to us by David Rivshin, shows that the only ones with all components isomorphic are unions of stars.

Conjecture 5.1 *Suppose that* $ern(G) > 3$ *for a disconnected graph all of whose components are isomorphic to* H*. Then* H *is isomorphic to a star* $K_{1,r}$*.*

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