$\binom{[n]}{r} := \{A \subseteq [n] : |A| = r\}$, $\quad 2^{[n]} := \{A : A \subseteq [n]\}$

For a family $\mathcal{F}$, $\mathcal{F}^{(r)} := \{F \in \mathcal{F} : |F| = r\}$

(So $\binom{[n]}{r} = \mathcal{F}^{(r)}$ with $\mathcal{F} = 2^{[n]}$)

A family $\mathcal{A}$ is s.t.b. <u>t-intersecting</u> if $|A \cap B| \geq t$ for any $A, B \in \mathcal{A}$. A t-intersecting family $\mathcal{A}$ is said to be <u>trivial</u> if the sets in $\mathcal{A}$ have a common subset of size t, i.e. $\bigcap_{A \in \mathcal{A}} A \geq t$; otherwise, $\mathcal{A}$ is s.t.b. non-trivial.

E.g. The family $\{A \in \binom{[n]}{r} : [t] \subseteq A\}$ is a trivial t-int.^{g} sub-family of $\binom{[n]}{r}$ $(t \leq r)$, whereas the family $\{A \in \binom{[n]}{r} : |A \cap [t+2]| \geq t+1\}$ is a non-trivial t-int.^{g} family

<u>Erdös-Ko-Rado Theorem (1961)</u>: (1) For any $t \leq r$, there exists an integer $n_0(r,t)$ such that, for all $n \geq n_0(r,t)$, the size of a t-int^g sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ is at most $\binom{n-t}{r-t}$ (and if $n > n_0(r,t)$, then the bound is attained ~~on~~ iff $\mathcal{A}$ is isomorphic to $\{A \in \binom{[n]}{r} : [t] \subseteq A\}$
(2) For the special case $t = 1$, the smallest $n_0(r,t)$ is $2r$.

In fact, ~~if n < 2r then the whole of~~ $\binom{[n]}{r}$ is intersecting iff $n < 2r$. If $n = 2r$ then, since $|\{A, [2r] \setminus A\} \cap \mathcal{A}| \leq 1$, we trivially get $|\mathcal{A}| \leq \cancel{\binom{n-1}{r-1}} \frac{1}{2}\binom{2r}{r} = \binom{2r-1}{r-1} = \binom{n-1}{r-1}$.

<u>Frankl (1978)</u>: For $t \geq 15$, the smallest $n_0(r,t)$ is $(r-t+1)(t+1)$.

<u>Wilson (1984)</u>: For all t, " " " " "

<u>Frankl Conjecture (1978)</u>: Let $t \leq r \leq t$, and let $\mathcal{A}_i := \{A \in \binom{[n]}{r} : |A \cap [t+2i]| \geq t+i\}$, $i = 0, 1, \ldots, r-t$. Then, ~~for any~~ at least one of the families $\mathcal{A}_i$ is a largest t-int^g sub-family of $\binom{[n]}{r}$.

Ahlswede and Khachatrian (1997): Frankl's Conj is true!
They also determined exactly when each of the families $\mathcal{A}_i$ is largest!

t-int^g sub-families of $2^{[n]}$:

EKR (1961): If $\mathcal{A}$ is a 1-intersecting sub-family of $2^{[n]}$, then
$$|\mathcal{A}| \leq 2^{n-1} = |\{A \in 2^{[n]} : 1 \in A\}|.$$

Proof: If $A \in \mathcal{A}$, then $[n] \setminus A \notin \mathcal{A}$ because $|([n]\setminus A) \cap A| = 0$.
So $\mathcal{A}$ can have at most $\frac{1}{2}(2^n) = 2^{n-1}$ members.

EKR asked: what is the size of a largest t-int^g sub-family of $2^{[n]}$

Katona (1964): Let $n \geq t \geq 2$, and let $\mathcal{A}$ be a largest t-int^g sub-family of $2^{[n]}$
(i) If $n+t = 2l$ then $\mathcal{A} = \{A \subseteq [n] : |A| \geq l\}$.
(ii) If $n+t = 2l+1$ then $\mathcal{A}$ ~~is~~ $\simeq \{A \subseteq [n] : |A \cap ([n-1])| \geq l\}$.

Note: In this case, $\mathcal{A}$ is not trivial!

t-int^g sub-families of hereditary families:

A family $\mathcal{H}$ is s.t.b. hereditary if ~~any subset of any set~~
$A \subset B \in \mathcal{H} \Rightarrow A \in \mathcal{H}$. E.g. $2^{[n]}$ is hereditary; if $S_1, \dots, S_m$ are sets then $2^{S_1} \cup 2^{S_2} \cup \dots \cup 2^{S_m}$ is hereditary.
A set M is s.t.b. $\mathcal{H}$-maximal if $M \not\subset H$ $\forall H \in \mathcal{H}$.

Simple Fact: Let $M_1, \dots, M_p$ be the $\mathcal{H}$-max. sets in her. $\mathcal{H}$. Then $\mathcal{H} = 2^{M_1} \cup \dots \cup 2^{M_p}$

Chvátal Conj (1974): If $\mathcal{H}$ is her., then one of the largest 1-int^g sub-famili. of $\mathcal{H}$ ~~the~~ is trivial.

Note: (1) By EKR result, this is true if $\mathcal{H} = 2^{M_1}$ $(p=1)$.
(2) By Katona result, this conj. does not generalise for $t > 1$.

What about t-int^g sub-families of $\mathcal{H}^{(r)} := \{H \in \mathcal{H} : |H| = r\}$?

Let $\mu(\mathcal{H})$ be the ~~min~~ size of a smallest $\mathcal{H}$-maximal set in $\mathcal{H}$.

Holroyd-Talbot (2005), Borg (2006) Conj: If ~~$\mu(\mathcal{H})$~~ $\mathcal{H}$ is her. and $\mu(\mathcal{H}) \geq 2r$, then one of the largest 1-int^g sub-families of $\mathcal{H}^{(r)}$ is trivial.

~~Borg (2007):~~ Theorem (B, 2007): Let $t \leq r$, and $n_0(r,t) := \dots$

If $\mathcal{H}$ is her. and $\mu(\mathcal{H}) \geq n_0(r,t) \left(:= (r-t)\binom{3r-2t+1}{t+1}+1\right)$, then the largest t-int^g sub-families of $\mathcal{H}^{(r)}$ ~~is~~ are trivial.

Note: - The EKR Thm. is the case $\mathcal{H} = 2^{[n]}$.

- The H-T-B Conj is true if $\mu(\mathcal{H}) \geq n_0(r,1)$.

Key Lemma: If $\mathcal{H}$ is hereditary and $r < s \leq \mu(\mathcal{H}) - r$, then

$$|\mathcal{H}^{(s)}| \geq \frac{\binom{\mu(\mathcal{H})-r}{s-r}}{\binom{s}{s-r}} |\mathcal{H}^{(r)}|.$$

Proof: For any $A \in \mathcal{H}$, let M_A be some $\mathcal{H}$-maximal set in $\mathcal{H}$ such that $A \subseteq M_A$. So $|M_A| \geq \mu(\mathcal{H})$, and $\binom{M_A}{s} \subseteq \mathcal{H}^{(s)}$ as $\mathcal{H}$ is hereditary. Therefore,

$$\binom{\mu(\mathcal{H})-r}{s-r}|\mathcal{H}^{(r)}| \leq \sum_{A \in \mathcal{H}^{(r)}} \binom{|M_A|-r}{s-r} = \sum_{A \in \mathcal{H}^{(r)}} \left|\{B \in \mathcal{H}^{(s)} : A \subset B\} \cap \binom{M_A}{s}\right|$$

$$\leq \sum_{A \in \mathcal{H}^{(r)}} |\{B \in \mathcal{H}^{(s)} : A \subset B\}| = \sum_{B \in \mathcal{H}^{(s)}} |\{A \in \mathcal{H}^{(r)} : A \subset B\}|$$

$$= \binom{s}{s-r}|\mathcal{H}^{(s)}|. \qquad \square$$