UNIVERSITY OF MALTA
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
MAT 1802 Mathematics for Engineers II
Problem Sheet 1

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(1) If $\mathbf{a} = 4\,\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$, calculate:

(i) $\mathbf{a} + \mathbf{b} + \mathbf{c}$; *{Ans: $6\,\boldsymbol{i} - 3\,\boldsymbol{j} + 2\,\boldsymbol{k}$}*

(ii) $|\,\mathbf{a} - \mathbf{b} - \mathbf{c}\,|$; *{Ans: $3\sqrt{5}$ }*

(iii) a unit vector parallel to $2\,\mathbf{a} - \mathbf{b} - \mathbf{c}$ but in the opposite direction. *{Ans: $-\frac{1}{\sqrt{97}}\,(6\,\boldsymbol{i} + 6\,\boldsymbol{j} - 5\,\boldsymbol{k})$}*

(2) Prove that the vectors $\mathbf{i} - \mathbf{k}$, $-\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$ and $2\,\mathbf{i} - \mathbf{j} - 3\,\mathbf{k}$ can form the sides of a triangle.

(3) Show that the three vectors $\mathbf{b} + \mathbf{c} - 2\,\mathbf{a}$, $\mathbf{c} + \mathbf{a} - 2\,\mathbf{b}$ and $\mathbf{a} + \mathbf{b} - 2\,\mathbf{c}$ are linearly dependent.

(4) Let $\mathbf{a} = \overrightarrow{\mathbf{OA}}$, $\mathbf{b} = \overrightarrow{\mathbf{OB}}$ and $\mathbf{c} = \overrightarrow{\mathbf{OC}}$. Then, if $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ with $\lambda + \mu = 1$:

(a) show that the terminal points *A*, *B* and *C* lie on the same straight line.

(b) find the values of λ and μ if:

(i) *C* is the midpoint of *AB*; *{Ans: $\lambda = \mu = \frac{1}{2}$}*

(ii) *A* is the midpoint of *CB*; *{Ans: $\lambda = 2$, $\mu = -1$}*

(iii) *C* is one-third the distance from *A* to *B*. *{Ans: $\lambda = \frac{2}{3}$, $\mu = \frac{1}{3}$}*

(5) Points *D*, *E* and *F* are the mid-points of the sides *BC*, *CA* and *AB*, respectively , of a triangle. If **a**, **b** and **c** are the position vectors of the points *A*, *B* and *C*, respectively, show that:

(a) the sum of the vectors $\overrightarrow{\mathbf{AD}}$, $\overrightarrow{\mathbf{BE}}$ and $\overrightarrow{\mathbf{CF}}$ is zero; and

(b) the medians have a common point *P* of trisection.

Deduce the position vector of point *P*. *{Ans: $\frac{1}{3}\,(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$}*

(6) **(a)** In a parallelogram *ABCD*, *X* is the midpoint of *AB* and the line *DX* cuts the diagonal *AC* at *P*. Writing $\overrightarrow{\mathbf{AB}} = \mathbf{a}$, $\overrightarrow{\mathbf{AD}} = \mathbf{b}$, $\overrightarrow{\mathbf{AP}} = \lambda\,\overrightarrow{\mathbf{AC}}$ and $\overrightarrow{\mathbf{DP}} = \mu\,\overrightarrow{\mathbf{DX}}$, express $\overrightarrow{\mathbf{AP}}$:

(i) in terms of λ, **a** and **b**;

(ii) in terms of μ, **a** and **b**.

Deduce that *P* is a point of trisection of both *AC* and *DX*.

(b) The resultant of two vectors **a** and **b** is perpendicular to **a**. If $|\mathbf{b}| = \sqrt{2}\,|\mathbf{a}|$, show that the resultant of $2\,\mathbf{a}$ and $\mathbf{b}$ is perpendicular to **b**.

(7) Two points *A* and *B* have position vectors **a** and **b** respectively relative to the origin *O*. Show that the position vector **d** of the point *D*, which divides the line *AB* internally in the ratio $AD : DB$ as $\lambda : \mu$, is given by:

$$\mathbf{d} = \frac{\lambda\mathbf{b} + \mu\mathbf{a}}{\lambda + \mu}.$$

(8) Obtain the acute angle between two diagonals of a cube.
(N.B.: It does not matter which two diagonals you take). *{Ans: 70.5 °}*

(9) Given $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{b} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$:

(a) show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are mutually orthogonal;

(b) calculate the acute angle between the vectors $2\,\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + 2\,\mathbf{b}$; *{Ans: 51.7 °}*

(c) obtain a unit vector orthogonal to both **a** and **b**. *{Ans:* $\pm\frac{1}{\sqrt{171}}(\boldsymbol{i} - 11\,\boldsymbol{j} - 7\,\boldsymbol{k})$*}*

(10) Given that $\mathbf{a} = 2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{k}$ and $\mathbf{c} = 2\,\mathbf{j} - \mathbf{k}$:

(a) calculate in terms of **i**, **j** and **k**:

(i) $\mathbf{a} \wedge \mathbf{b}$ *{Ans:* $-3(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$*}*

(ii) $\mathbf{b} \wedge \mathbf{c}$ *{Ans:* $-(2\,\boldsymbol{i} + \boldsymbol{j} + 2\,\boldsymbol{k})$*}*

(iii) $\mathbf{c} \wedge \mathbf{a}$ *{Ans:* $-(\boldsymbol{i} + 2\,\boldsymbol{j} + 4\,\boldsymbol{k})$*}*

(b) obtain a unit vector orthogonal to both **b** and **c**; *{Ans:* $\pm\frac{1}{3}(2\,\boldsymbol{i} + \boldsymbol{j} + 2\,\boldsymbol{k})$*}*

(11) The vectors **a**, **b** and **c** lie along the sides of the triangle *ABC* such that $\mathbf{a} \equiv \overrightarrow{\mathbf{BC}}$, $\mathbf{b} \equiv \overrightarrow{\mathbf{CA}}$ and $\mathbf{c} \equiv \overrightarrow{\mathbf{AB}}$. Show that:

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{c} = \mathbf{c} \wedge \mathbf{a},$$

and hence obtain the *sine* and the *cosine rules* for the triangle *ABC*.

(12) Prove that for any vectors **a**, **b** and **c**:

(i) $(\mathbf{a} \wedge \mathbf{b}) \,.\, (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{a} \,.\, \mathbf{a})(\mathbf{b} \,.\, \mathbf{b}) - (\mathbf{a} \,.\, \mathbf{b})^2$;

(ii) $\mathbf{a} \,.\, (\mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{a})) = (\mathbf{a} \,.\, \mathbf{c})(\mathbf{b} \,.\, \mathbf{a}) - (\mathbf{a} \,.\, \mathbf{a})(\mathbf{b} \,.\, \mathbf{c})$;

(iii) $\mathbf{a} \wedge (\mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{a})) = (\mathbf{a} \,.\, \mathbf{b})(\mathbf{a} \wedge \mathbf{c})$.

(13) If **a**, **b** and **c** are othogonal vectors, show that:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 = \mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2, \quad \text{where} \quad \mathbf{v}^2 = \mathbf{v} \,.\, \mathbf{v}.$$

(14) Show that for any vectors **a**, **b** and **c**:

(i) $(\mathbf{a} + \mathbf{b}) \,.\, \{(\mathbf{b} + \mathbf{c}) \wedge (\mathbf{c} + \mathbf{a})\} = 2[\mathbf{a}, \mathbf{b}, \mathbf{c}]$;

(ii) $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) + \mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{a}) + \mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b}) = \mathbf{0}$;

(iii) $[\mathbf{b} \wedge \mathbf{c}, \mathbf{c} \wedge \mathbf{a}, \mathbf{a} \wedge \mathbf{b}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2$.

(15) The velocity $\mathbf{v}$ of a particle is related to its angular velocity ω and its space position vector $\mathbf{r}$ by $\mathbf{v} = \omega \wedge \mathbf{r}$. If the particle has mass m, show that its kinetic energy T, given by $T = \frac{1}{2} m \mathbf{v}^2$, may be expressed in the form:

$$T = \frac{1}{2} m(\omega^2 \mathbf{r}^2 - (\omega \cdot \mathbf{r})^2).$$

(16) Prove that for any vectors **a**, **b**, **c**, **d**, **e** and **f** we have:

(a) $$(\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}))\,(\mathbf{d} \cdot (\mathbf{e} \wedge \mathbf{f})) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{e} & \mathbf{a}\cdot\mathbf{f} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{f} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{f} \end{vmatrix}$$

(b) $$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \wedge \mathbf{b} \cdot \mathbf{d})\,\mathbf{c} - (\mathbf{a} \wedge \mathbf{b} \cdot \mathbf{c})\,\mathbf{d}.$$

(17) The vectors **a**, **b**, **c** and **d** lie along the sides of the quadrilateral $ABCD$ such that $\mathbf{a} \equiv \overrightarrow{\mathbf{DA}}$, $\mathbf{b} \equiv \overrightarrow{\mathbf{AB}}$, $\mathbf{c} \equiv \overrightarrow{\mathbf{BC}}$ and $\mathbf{d} \equiv \overrightarrow{\mathbf{CD}}$. Show that:

$$\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2 + 2\,\mathbf{b} \cdot \mathbf{c} + 2\,\mathbf{c} \cdot \mathbf{d} + 2\,\mathbf{d} \cdot \mathbf{b}$$

and, hence, show that, if the figure is coplanar (i.e. A, B, C, D lie in the same plane), then:

$$AD^2 = AB^2 + BC^2 + CD^2 - 2\,AB.BC \cos B - 2\,BC.CD \cos C + 2\,AB.CD \cos(A + D).$$

If the figure is skew, i.e. the sides AB and CD are not in the same plane, show that the angle θ between AB and CD is given by:

$$\cos\theta = \frac{DA^2 - AB^2 - BC^2 - CD^2 + 2\,AB.BC \cos B + 2\,BC.CD \cos C}{2\,AB.CD}.$$