(1) If \( a = 4 \mathbf{i} + \mathbf{j} - \mathbf{k} \), \( b = 3 \mathbf{i} - 2 \mathbf{j} + 2 \mathbf{k} \) and \( c = -\mathbf{i} - 2 \mathbf{j} + \mathbf{k} \), calculate:

(i) \( a + b + c \);  \([\text{Ans: } 6 \mathbf{i} - 3 \mathbf{j} + 2 \mathbf{k}]\)

(ii) \( |a - b - c|\);  \([\text{Ans: } 3 \sqrt{5}]\)

(iii) a unit vector parallel to \( 2 \mathbf{a} - \mathbf{b} - \mathbf{c} \) but in the opposite direction.  \([\text{Ans: } -\frac{1}{97} (6 \mathbf{i} + 6 \mathbf{j} - 5 \mathbf{k})]\)

(2) Prove that the vectors \( \mathbf{i} - \mathbf{k}, -\mathbf{i} + \mathbf{j} + 2 \mathbf{k} \) and \( 2 \mathbf{i} - \mathbf{j} - 3 \mathbf{k} \) can form the sides of a triangle.

(3) Show that the three vectors \( \mathbf{b} + \mathbf{c} - 2 \mathbf{a}, \mathbf{c} + \mathbf{a} - 2 \mathbf{b} \) and \( \mathbf{a} + \mathbf{b} - 2 \mathbf{c} \) are linearly dependent.

(4) Let \( \mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB} \) and \( \mathbf{c} = \overrightarrow{OC} \). Then, if \( \mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b} \) with \( \lambda + \mu = 1 \):

(a) show that the terminal points \( A, B \) and \( C \) lie on the same straight line.

(b) find the values of \( \lambda \) and \( \mu \) if:

(i) \( C \) is the midpoint of \( AB \);  \([\text{Ans: } \lambda = \mu = \frac{1}{2}]\)

(ii) \( A \) is the midpoint of \( CB \);  \([\text{Ans: } \lambda = 2, \mu = -1]\)

(iii) \( C \) is one-third the distance from \( A \) to \( B \).  \([\text{Ans: } \lambda = \frac{2}{3}, \mu = \frac{1}{3}]\)

(5) Points \( D, E \) and \( F \) are the mid-points of the sides \( BC, CA \) and \( AB \), respectively , of a triangle. If \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are the position vectors of the points \( A, B \) and \( C \), respectively, show that:

(a) the sum of the vectors \( \overrightarrow{AD}, \overrightarrow{BE} \) and \( \overrightarrow{CF} \) is zero; and

(b) the medians have a common point \( P \) of trisection.

Deduce the position vector of point \( P \).  \([\text{Ans: } \frac{1}{3} (\mathbf{a} + \mathbf{b} + \mathbf{c})]\)

(6) (a) In a parallelogram \( ABCD \), \( X \) is the midpoint of \( AB \) and the line \( DX \) cuts the diagonal \( AC \) at \( P \). Writing \( \overrightarrow{AB} = \mathbf{a}, \overrightarrow{AD} = \mathbf{b}, \overrightarrow{AP} = \lambda \overrightarrow{AC} \) and \( \overrightarrow{DP} = \mu \overrightarrow{DX} \), express \( \overrightarrow{AP} \):

(i) in terms of \( \lambda, \mathbf{a} \) and \( \mathbf{b} \);

(ii) in terms of \( \mu, \mathbf{a} \) and \( \mathbf{b} \).

Deduce that \( P \) is a point of trisection of both \( AC \) and \( DX \).
(b) The resultant of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is perpendicular to \( \mathbf{a} \). If \( | \mathbf{b} | = \sqrt{2} | \mathbf{a} | \), show that the resultant of \( 2 \mathbf{a} \) and \( \mathbf{b} \) is perpendicular to \( \mathbf{b} \).

(7) Two points \( A \) and \( B \) have position vectors \( \mathbf{a} \) and \( \mathbf{b} \) respectively relative to the origin \( O \).
Show that the position vector \( \mathbf{d} \) of the point \( D \), which divides the line \( AB \) internally in the ratio \( AD : DB \) as \( \lambda : \mu \), is given by:
\[
\mathbf{d} = \frac{\lambda \mathbf{b} + \mu \mathbf{a}}{\lambda + \mu}.
\]

(8) Obtain the acute angle between two diagonals of a cube.
\( \text{(N.B.: It does not matter which two diagonals you take).} \) \[\text{Ans: 70.5°}\]

(9) Given \( \mathbf{a} = \mathbf{i} + 2 \mathbf{j} - 3 \mathbf{k} \) and \( \mathbf{b} = 3 \mathbf{i} - \mathbf{j} + 2 \mathbf{k} \):
(a) show that the vectors \( \mathbf{a} + \mathbf{b} \) and \( \mathbf{a} - \mathbf{b} \) are mutually orthogonal;
(b) calculate the acute angle between the vectors \( 2 \mathbf{a} + \mathbf{b} \) and \( \mathbf{a} + 2 \mathbf{b} \); \[\text{Ans: 51.7°}\]
(c) obtain a unit vector orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \). \[\text{Ans: } \pm \frac{1}{\sqrt{117}} (i - 11 j - 7 k)\]

(10) Given that \( \mathbf{a} = 2 \mathbf{i} - 3 \mathbf{j} + \mathbf{k} \), \( \mathbf{b} = -\mathbf{i} + \mathbf{k} \) and \( \mathbf{c} = 2 \mathbf{j} - \mathbf{k} \):
(a) calculate in terms of \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \):
\[ \begin{align*}
(i) & \quad \mathbf{a} \cdot \mathbf{b} & \text{(Ans: } -3 (i + j + k)) \\
(ii) & \quad \mathbf{b} \cdot \mathbf{c} & \text{(Ans: } -2 (i + j + 2 k)) \\
(iii) & \quad \mathbf{c} \cdot \mathbf{a} & \text{(Ans: } -i + 2 j + 4 k) \\
\end{align*} \]
(b) obtain a unit vector orthogonal to both \( \mathbf{b} \) and \( \mathbf{c} \); \[\text{Ans: } \pm \frac{1}{7} (2 i + j + 2 k)\]

(11) The vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) lie along the sides of the triangle \( ABC \) such that \( \mathbf{a} \equiv \overrightarrow{BC}, \mathbf{b} \equiv \overrightarrow{CA} \) and \( \mathbf{c} \equiv \overrightarrow{AB} \). Show that:
\[ \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}, \]
and hence obtain the sine and the cosine rules for the triangle \( ABC \).

(12) Prove that for any vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \):
\[ \begin{align*}
(i) & \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2; \\
(ii) & \quad \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{a})) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}); \\
(iii) & \quad \mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{a})) = (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{c}).
\end{align*} \]

(13) If \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are orthogonal vectors, show that:
\[ [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 = \mathbf{a}^2 \mathbf{b}^2 \mathbf{c}^2, \text{ where } \mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v}. \]

(14) Show that for any vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \):
\[ \begin{align*}
(i) & \quad (\mathbf{a} + \mathbf{b}) \cdot ((\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})) = 2[\mathbf{a}, \mathbf{b}, \mathbf{c}]; \\
(ii) & \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0; \\
(iii) & \quad [\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.
\end{align*} \]
The velocity \( v \) of a particle is related to its angular velocity \( \omega \) and its space position vector \( r \) by \( v = \omega \times r \). If the particle has mass \( m \), show that its kinetic energy \( T \), given by \( T = \frac{1}{2} m v^2 \), may be expressed in the form:
\[
T = \frac{1}{2} m (\omega^2 \cdot r^2 - (\omega \cdot r)^2).
\]

Prove that for any vectors \( a, b, c, d, e \) and \( f \) we have:

(a) \( (a \cdot (b \times c)) (d \cdot (e \times f)) = \begin{vmatrix} a \cdot d & a \cdot e & a \cdot f \\ b \cdot d & b \cdot e & b \cdot f \\ c \cdot d & c \cdot e & c \cdot f \end{vmatrix} \)

(b) \( (a \times b) \wedge (c \times d) = (a \times b) \cdot d - (a \times b) \cdot c \).

The vectors \( a, b, c \) and \( d \) lie along the sides of the quadrilateral \( ABCD \) such that \( a = \overrightarrow{DA}, \ b = \overrightarrow{AB}, \ c = \overrightarrow{BC} \) and \( d = \overrightarrow{CD} \). Show that:
\[
a^2 = b^2 + c^2 + d^2 + 2b \cdot c + 2c \cdot d + 2d \cdot b
\]
and, hence, show that, if the figure is coplanar (i.e. \( A, B, C, D \) lie in the same plane), then:
\[
AD^2 = AB^2 + BC^2 + CD^2 - 2AB \cdot BC \cos B - 2BC \cdot CD \cos C + 2AB \cdot CD \cos (A + D).
\]

If the figure is skew, i.e. the sides \( AB \) and \( CD \) are not in the same plane, show that the angle \( \theta \) between \( AB \) and \( CD \) is given by:
\[
\cos \theta = \frac{DA^2 - AB^2 - BC^2 - CD^2 + 2AB \cdot BC \cos B + 2BC \cdot CD \cos C}{2AB \cdot CD}.
\]