In questions 1-4, you may identify (if you find it helpful) a point \((x,y,z)\) in \(\mathbb{R}^3\) (or \((x,y)\) in \(\mathbb{R}^2\)) with the corresponding vector \(x \mathbf{i} + y \mathbf{j} + z \mathbf{k}\) (or \(x \mathbf{i} + y \mathbf{j}\)).

(1) Determine whether or not the following transformations are linear:

(a) the transformation \(F\) from \(\mathbb{R}^3\) onto \(\mathbb{R}\) defined by \(F(x, y, z) = 2x - 3y + 4z\).

(b) the transformation \(G\) from \(\mathbb{R}^2\) into \(\mathbb{R}^3\) defined by \(G(x, y) = (x + 1, 2y, x + y)\).

\{Ans: (a) linear; (b) not linear\}

(2) Let the linear transformation \(F\) of the plane be defined by

\[F(i + 2j) = 2i + 3j\quad \text{and}\quad F(j) = i + 4j.\]

Find a formula for \(F\), that is, find \(F(a \mathbf{i} + b \mathbf{j})\).

(Hint: \(i + 2j, j\) form a basis in the plane.)

\{Ans: \(F(a, b) = (b, -5a + 4b)\}\}

(3) Determine whether or not each of the following linear transformations in the plane is nonsingular. If the transformation is singular, find \(a, b\) (not both 0) such that \(F(a,b)=(0,0)\).

(a) \(F\) defined by \(F(x, y) = (x - y, x - 2y)\)

\{Ans: nonsingular\}

(b) \(G\) defined by \(G(x, y) = (2x - 4y, 3x - 6y)\)

\{Ans: singular; e.g. \((2, I)\)\}

(4) Let the transformation \(H\) of \(\mathbb{R}^3\) be defined by:

\[H(x, y, z) = (x + y - 2z, x + 2y + z, 2x + 2y - 3z)\]

Show that \(H\) is nonsingular.
Consider a change of coordinates from a first frame of reference (coordinate system) \( O e_1 e_2 e_3 \) to a second frame \( O e_1' e_2' e_3' \), where:

\[
\begin{align*}
e_1' &= a_{11} e_1 + a_{12} e_2 + a_{13} e_3, \\
e_2' &= a_{21} e_1 + a_{22} e_2 + a_{23} e_3, \\
e_3' &= a_{31} e_1 + a_{32} e_2 + a_{33} e_3.
\end{align*}
\]

Let \( M \) be the point having coordinates \((x, y, z)\) and \((x', y', z')\) relative to \( O e_1 e_2 e_3 \) and \( O e_1' e_2' e_3' \), respectively. Show that the relation:

\[
\begin{align*}
x &= a_{11} x' + a_{12} y' + a_{13} z', \\
y &= a_{21} x' + a_{22} y' + a_{23} z', \\
z &= a_{31} x' + a_{32} y' + a_{33} z',
\end{align*}
\]

holds. Assuming now that the first coordinate system is a rectangular Cartesian system (\( e_1, e_2, e_3 \) are mutually orthogonal unit vectors), and that \( e_1', e_2', e_3' \) are unit vectors, show that

\[
\alpha_{rs} = \cos(\theta_{rs}) \quad r, s = 1, 2, 3
\]

where \( \theta_{rs} \) denotes the angle between \( e_r' \) and \( e_s \). Assuming furthermore that \( e_1', e_2', e_3' \) are mutually orthogonal (so that the second frame of reference is also rectangular), deduce that:

\[
\begin{align*}
x' &= a_{11} x + a_{12} y + a_{13} z, \\
y' &= a_{21} x + a_{22} y + a_{23} z, \\
z' &= a_{31} x + a_{32} y + a_{33} z.
\end{align*}
\]

(Hint: look at your notes)

(b) Assuming that the frames of reference are both rectangular Cartesian systems (the triples \((e_1, e_2, e_3), (e_1', e_2', e_3')\) each consist of mutually orthogonal unit vectors) show that the 3x3 matrix \((\alpha_{rs})_{r,s}\) is orthogonal.

(c) Recall that, as long as the frames of reference are rectangular Cartesian systems, the quantities \( x^2 + y^2 + z^2 \) and \((x')^2 + (y')^2 + (z')^2 \) both give the length of the segment OM. Without using this, using instead part (b) above, show that

\[
x^2 + y^2 + z^2 = (x')^2 + (y')^2 + (z')^2
\]

(d) Suppose that the rectangular Cartesian system \( O e_1'e_2'e_3' \) is obtained from the rectangular Cartesian system \( O e_1 e_2 e_3 \) by an anticlockwise rotation of an angle \( \theta \) about the \( O e_2 \) axis. Find, the transformation matrix \( A=(\alpha_{rs})_{r,s} \) (which expresses \( x', y', z' \) as functions of \( x, y, z \), as above) in terms of the angle \( \theta \).

\[\text{[Ans: } A = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \}\]
The matrix of a transformation representing the change of coordinates from the rectangular Cartesian system \( Oe_1 e_2 e_3 \) to a new rectangular Cartesian system \( Oe_1' e_2' e_3' \) is:

\[
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Describe by a diagram how the position of the two set of axes are related.

Show that the equations:

\[
x' = x \sin \theta \cos \phi + y \sin \theta \sin \phi + z \cos \theta,
\]
\[
y' = x \cos \theta \cos \phi + y \cos \theta \sin \phi - z \sin \theta,
\]
\[
z' = -x \sin \phi + y \cos \phi,
\]

represent an orthogonal transformation.

Fix once and for all three mutually orthogonal unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \), and think of vectors in terms of their coordinates. Consider an orthogonal transformation \( F \) of vectors defined by

\[
F( a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} ) = a_1' \mathbf{i} + a_2' \mathbf{j} + a_3' \mathbf{k}
\]

where

\[
\begin{pmatrix}
a_1' \\
a_2' \\
a_3'
\end{pmatrix} = A
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\]

for a fixed orthogonal matrix \( A \). (If you find it helpful, you may also fix an origin in space, and therefore think of the vectors as position vectors, and of \( F \) as a linear transformation in space. You may also assume the triple \( (\mathbf{i}, \mathbf{j}, \mathbf{k}) \) is right-handed). Show that for any two vectors \( \mathbf{v}, \mathbf{w} \)

a) \( F(\mathbf{v}) \cdot F(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w} \)

b) \( F(\mathbf{v}) \wedge F(\mathbf{w}) = \det(A) F(\mathbf{v} \wedge \mathbf{w}) \)