In questions 1-4, you may identify (if you find it helpful) a point \((x, y, z)\) in \(\mathbb{R}^3\) (or \((x, y)\) in \(\mathbb{R}^2\)) with the corresponding vector \(xi + yj + zk\) (or \(xi + yj\)).

(1) Determine whether or not the following transformations are linear:
(a) the transformation \(F\) from \(\mathbb{R}^3\) onto \(\mathbb{R}\) defined by \(F(x, y, z) = 2x - 3y + 4z\).
(b) the transformation \(F\) from \(\mathbb{R}^2\) onto \(\mathbb{R}^3\) defined by \(F(x, y) = (x + 1, 2y, x + y)\).

\([\text{Ans: (a) linear; (b) not linear}]\)

(2) Let the linear transformation \(F\) in the plane be defined by:
\[F(i + 2j) = 2i + 3j\] and \(F(j) = i + 4j\).

Find a formula for \(F\), that is, find \(F(ai + bj)\).

\([\text{Ans: } F(ai + bj) = bi + (5a - b)j]\)

[Hint: \(i + 2j, \ j\) form a basis in the plane.]

(3) Determine whether or not each linear transformations in the plane is nonsingular. If the transformation is singular, find \(a, b\) (not both 0) such that \(F(a, b) = (0, 0)\).
(a) \(F\) defined by \(F(x, y) = (x - y, x - 2y)\)

\([\text{Ans: nonsingular}]\)

(b) \(G\) defined by \(G(x, y) = (2x - 4y, 3x - 6y)\)

\([\text{Ans: singular; eg. } y = 2, 1]\)

(4) Let the transformation \(H\) of \(\mathbb{R}^3\) be defined by:
\[H(x, y, z) = (x + y - 2z, x + 2y + z, 2x + 2y - 3z)\].

Show that \(H\) is nonsingular.
Consider a change of coordinates from a first frame of reference (coordinate system) $O e_1 e_2 e_3$ to a second frame $O e_1' e_2' e_3'$, where:

\[
\begin{align*}
e_1' &= \alpha_{11} e_1 + \alpha_{12} e_2 + \alpha_{13} e_3, \\
e_2' &= \alpha_{21} e_1 + \alpha_{22} e_2 + \alpha_{23} e_3, \\
e_3' &= \alpha_{31} e_1 + \alpha_{32} e_2 + \alpha_{33} e_3.
\end{align*}
\]

Let $M$ be the point having coordinates $(x, y, z)$ and $(x', y', z')$ relative to $O e_1 e_2 e_3$ and $O e_1' e_2' e_3'$, respectively. Show that the relation:

\[
\begin{align*}
x &= \alpha_{11} x' + \alpha_{21} y' + \alpha_{31} z', \\
y &= \alpha_{12} x' + \alpha_{22} y' + \alpha_{32} z', \\
z &= \alpha_{13} x' + \alpha_{23} y' + \alpha_{33} z',
\end{align*}
\]

holds. Assuming now that the first coordinate system is a rectangular Cartesian system ($e_1, e_2, e_3$ are mutually orthogonal unit vectors), and that $e_1', e_2', e_3'$ are unit vectors, show that:

\[
\alpha_{rs} = \cos(\theta_{rs}), \quad r, s = 1, 2, 3
\]

where $\theta_{rs}$ denotes the angle between $e_r'$ and $e_s$. Assuming furthermore that $e_1', e_2', e_3'$ are mutually orthogonal (so that the second frame of reference is also rectangular), deduce that:

\[
\begin{align*}
x' &= \alpha_{11} x + \alpha_{12} y + \alpha_{13} z, \\
y' &= \alpha_{21} x + \alpha_{22} y + \alpha_{23} z, \\
z' &= \alpha_{31} x + \alpha_{32} y + \alpha_{33} z.
\end{align*}
\]

(Hint: look at your notes.)

Assuming that the frames of reference are both rectangular Cartesian systems (the triples $(e_1, e_2, e_3)$, $(e_1', e_2', e_3')$) each consist of mutually orthogonal unit vectors) show that the $3 \times 3$ matrix $(\alpha_{rs})_{r,s}$ is orthogonal.

Recall that, as long as the frames of reference are rectangular Cartesian systems, the quantities $x^2 + y^2 + z^2$ and $(x')^2 + (y')^2 + (z')^2$ both give the length of the segment $OM$. **Without** using this, using instead part (b) above, show that:

\[
x^2 + y^2 + z^2 = (x')^2 + (y')^2 + (z')^2.
\]

Suppose that the rectangular Cartesian system $O e_1' e_2' e_3'$ is obtained from the rectangular Cartesian system $O e_1 e_2 e_3$ by an anticlockwise rotation of an angle $\theta$ about the $O e_2$ axis. Find the transformation matrix $A = (\alpha_{rs})_{r,s}$ (which expresses $x'$, $y'$, $z'$ as functions of $x$, $y$, $z$ as above) in terms of the angle $\theta$.

**Ans:**

\[
A = \begin{pmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{pmatrix}
\]
The matrix of a transformation representing the change of coordinates from the rectangular Cartesian system \( O e_1 e_2 e_3 \) to a new rectangular Cartesian system \( O e_1' e_2' e_3' \) is:

\[
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Describe by a diagram how the position of the two set of axes are related.

Show that the equations:

\[
\begin{align*}
x' &= x \sin \theta \cos \phi + y \sin \theta \sin \phi + z \cos \theta, \\
y' &= x \cos \theta \cos \phi + y \cos \theta \sin \phi - z \sin \theta, \\
z' &= -x \sin \phi + y \cos \phi,
\end{align*}
\]

represent an orthogonal transformation.

Fix once and for all three mutually orthogonal unit vectors \( i, j, k \), and think of vectors in terms of their coordinates. Consider an orthogonal transformation \( F \) of vectors defined by:

\[
F(a_1 i + a_2 j + a_3 k) = a_1' i + a_2' j + a_3' k \quad \text{where}
\]

\[
\begin{pmatrix}
a_1' \\
a_2' \\
a_3'
\end{pmatrix} = A 
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\]

for a fixed orthogonal matrix \( A \). (If you find it helpful, you may also fix an origin in space, and therefore think of the vectors as position vectors, and of \( F \) as a linear transformation in space. You may also assume the triple \((i, j, k)\) is right-handed.)

Show that for any two vectors \( v, w \):

(a) \( F(v) F(w) = v \cdot w \)

(b) \( F(v) \wedge F(w) = \det(A) F(v \wedge w) \)