

# CLASSICAL MECHANICS

an introduction to

## Newtonian and Lagrangian Methods

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C J Camilleri

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### *Motion of a Rigid Body about a Fixed Point*

Inertial Frame  $X \equiv (Ox_i, \mathbf{e}_i)$

Rest Frame  $X' \equiv (Ox'_j, \mathbf{e}'_j)$

Euler's Angles  $(\psi, \theta, \phi)$

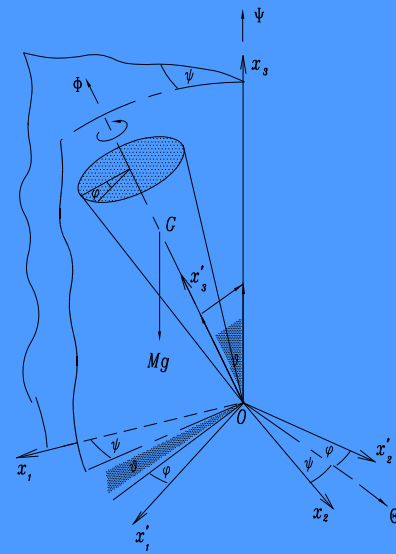
Principal Moments of Inertia  $(A, A, C)$

Angular velocity

$$\begin{aligned}\boldsymbol{\omega} = & (\dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi) \mathbf{e}'_1 \\ & + (\dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi) \mathbf{e}'_2 \\ & + (\dot{\phi} + \dot{\psi} \cos \theta) \mathbf{e}'_3\end{aligned}$$

Lagrangian

$$\mathcal{L} = \frac{1}{2}A(\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta) + \frac{1}{2}C(\dot{\phi} + \dot{\psi} \cos \theta)^2 - Mgh \cos \theta$$



The Spinning Top

**Electronic  
Portable Document Format (PDF)**

**copy to**

*G M Borg*

*of*

*Msida, Malta*

*With best wishes*

A handwritten signature in black ink, appearing to read 'G. Camilleri', with a horizontal line underneath.

# **CLASSICAL MECHANICS**

an introduction to  
Newtonian and Lagrangian Methods

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**University of Malta**

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*To my beloved wife Kate, who makes life a joy,  
and my children  
Anthony, Susan, David, Christopher*

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# Contents

<b>Preface</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Basic Concepts . . . . .	1
1.2 Motion with One Degree of Freedom . . . . .	2
1.3 Motion in a Plane . . . . .	3
1.4 Motion in Three-Dimensional Space . . . . .	4
1.5 Tensor Notation . . . . .	5
Exercises 1.1 . . . . .	8
<b>2 The Nature of Force</b>	<b>9</b>
2.1 Newton's First Law . . . . .	9
2.2 Newton's Second Law . . . . .	9
2.3 Newton's Third Law . . . . .	10
2.4 Newton's Law of Gravity . . . . .	10
2.5 Internal and External Forces . . . . .	11
Exercises 2.1 . . . . .	15
<b>3 Frame of Reference</b>	<b>17</b>
3.1 Rest Frame . . . . .	17
3.2 Inertial Frame . . . . .	17
3.3 Rotating Frame . . . . .	18
3.3.1 Rate of Change of $\mathbf{e}_j$ relative $\tilde{\mathbf{e}}_i$ . . . . .	19
3.4 Time-Derivative of a Vector . . . . .	20
3.4.1 Non-Rotating Frame . . . . .	20
3.4.2 Rotating Frame - Coriolis Theorem . . . . .	21
3.5 Equivalent Frames . . . . .	23
Exercises 3.1 . . . . .	25

<b>4</b>	<b>Motion of a Particle</b>	<b>27</b>
4.1	Momentum . . . . .	27
4.1.1	Newton's Equation of motion . . . . .	27
4.2	Moment of a Force . . . . .	28
4.3	Conservation of Momentum . . . . .	29
4.4	Kinetic Energy, Work and Power . . . . .	29
4.5	Conservative Forces and Potential Energy . . . . .	31
4.6	Impulsive Forces . . . . .	34
4.6.1	Collision of Particles . . . . .	35
	Exercises 4.1 . . . . .	41
4.7	Motion Relative to a Non-inertial Frame . . . . .	46
4.7.1	Motion near the earth's surface . . . . .	48
4.7.2	Foucault's Pendulum . . . . .	50
	Exercises 4.2 . . . . .	51
<b>5</b>	<b>Motion of a System of Particles</b>	<b>55</b>
5.1	Mass Centre or Centre of Inertia . . . . .	55
5.2	Origin at the Mass Centre . . . . .	56
5.3	Linear Momentum of System . . . . .	56
5.4	Angular Momentum of System . . . . .	57
5.5	Internal and External Forces . . . . .	57
5.6	Linear Motion of System . . . . .	59
5.7	Angular Motion of System About a Fixed Point . . . . .	59
5.8	Angular Motion of System About the Mass Centre . . . . .	60
5.9	Motion of System Relative to a Moving Frame . . . . .	60
5.9.1	Frame with Linear Motion . . . . .	61
5.9.2	Frame with Linear and Angular Motion . . . . .	62
5.9.3	Linear Motion of System . . . . .	62
5.9.4	Angular Motion of System . . . . .	63
5.10	Kinetic Energy of System . . . . .	63
5.11	Impulsive Motion . . . . .	64
	Exercises 5.1 . . . . .	67
<b>6</b>	<b>Moments and Products of Inertia</b>	<b>71</b>
6.1	Definitions . . . . .	71
6.2	Parallel Axes Theorem . . . . .	72
6.3	Two-Dimensional Systems – Plane Lamina . . . . .	73
6.3.1	Perpendicular Axes Theorem . . . . .	73
6.3.2	Moment of Inertia of a Lamina About a Line $Ox'$ in its Plane . . . . .	73

6.3.3	Product of Inertia of a Lamina About Perpendicular Lines $Ox', Oy'$ in its Plane . . . . .	74
6.3.4	Principal Axes of Inertia of a Lamina. . . . .	74
6.3.5	Momental Ellipse at a Point $O$ of a Lamina . . . . .	75
6.3.6	Parallel Axes Theorem for Products of Inertia of Lamina	76
6.4	Three Dimensional Systems . . . . .	77
6.4.1	Moment of Inertia About a Variable Line $ON$ . . . . .	77
6.4.2	Momental Ellipsoid at a Point $O$ . . . . .	77
6.4.3	Determination of Principal Moments of Inertia . . . . .	79
6.5	Continuous Distributions of Mass – Uniform Solids . . . . .	80
6.6	Tensor Notation – Inertia Tensor . . . . .	84
6.6.1	Parallel Axes Theorem . . . . .	84
6.6.2	Rotation of Coordinate Axes . . . . .	85
6.6.3	Principal Axes and Moments of Inertia . . . . .	86
6.7	Equimomental Systems . . . . .	87
6.8	Symmetrical Solids – Routh’s Formula . . . . .	88
	Exercises 6.1 . . . . .	91
<b>7</b>	<b>Motion of Rigid Body About Fixed Point</b>	<b>95</b>
7.1	Boscovitch’s Hypothesis . . . . .	95
7.2	Instantaneous Motion of Body . . . . .	96
7.3	Motion About a Fixed Point $O$ . . . . .	96
7.4	Angular Momentum About $O$ . . . . .	97
7.5	Euler’s Equations of Motion About $O$ . . . . .	98
7.6	Kinetic Energy About $O$ . . . . .	100
7.7	Conservation of Angular Momentum . . . . .	101
7.8	Motion of a Spinning Top . . . . .	102
7.8.1	Integral of the Axial Spin . . . . .	104
7.8.2	Energy Integral . . . . .	104
7.8.3	Angular Momentum Integral . . . . .	104
7.8.4	Steady Precessional Motion . . . . .	105
7.8.5	An Alternative Method . . . . .	106
	Exercises 7.1 . . . . .	108
<b>8</b>	<b>General Motion of Rigid Body</b>	<b>111</b>
8.1	Linear Motion of $G$ . . . . .	111
8.2	Angular Motion about $G$ . . . . .	111
8.3	Kinetic Energy . . . . .	112
8.4	Screw Motion – Central Axis . . . . .	115
8.5	Pure Rolling Condition . . . . .	117



Exercises 8.1 . . . . .	126
<b>9 Static Systems – Virtual Work</b>	<b>129</b>
9.1 Virtual Work for a Single Particle . . . . .	129
9.2 Virtual Work for a System – Internal Forces . . . . .	131
9.3 Virtual Work for a System – External Forces . . . . .	132
Exercises 9.1 . . . . .	139
<b>10 Lagrangian Methods – Rudiments</b>	<b>143</b>
10.1 Generalized Coordinates . . . . .	143
10.2 Degrees of Freedom . . . . .	144
10.2.1 Single particle moving in space . . . . .	145
10.2.2 Rigid body moving in space . . . . .	145
10.2.3 Cylinder rolling without slip on a rough horizontal plane	145
10.2.4 A disc rolling vertically on a rough horizontal plane . .	146
10.3 Holonomic and Non-holonomic Systems . . . . .	147
10.4 Moving Constraints . . . . .	147
10.5 Motion of Holonomic Systems – Generalized Velocities . . . . .	148
10.6 Kinetic Energy of the System . . . . .	149
10.7 Virtual Work – Generalized Forces . . . . .	150
10.8 Lagrange’s Equations of Motion . . . . .	150
10.8.1 Points to remember . . . . .	151
Exercises 10.1 . . . . .	156
10.9 Transformation of Generalized Coordinates . . . . .	159
Exercises 10.2 . . . . .	162
<b>11 Lagrangian Methods – Ramifications</b>	<b>165</b>
11.1 Conservative Forces . . . . .	165
11.2 Generalized Potential . . . . .	169
Exercises 11.1 . . . . .	171
11.3 Generalized Momentum . . . . .	174
11.4 Generalized Energy . . . . .	176
11.5 Ignorable or Cyclic Coordinates . . . . .	178
11.5.1 The Routhian Function . . . . .	179
11.6 Hamilton’s Equations . . . . .	181
11.7 Configuration Space and Phase Space . . . . .	184
11.7.1 Liouville’s Theorem . . . . .	184
11.7.2 Hamilton’s Principle . . . . .	186
Exercises 11.2 . . . . .	187
11.8 Generalized Impulse . . . . .	190

11.8.1 Impulsive Virtual Work . . . . .	190
11.8.2 Lagrange's Equations for Impulsive Motion . . . . .	191
Exercises 11.3 . . . . .	195
11.9 Small Oscillations - Normal Modes . . . . .	197
11.9.1 Normal modes of oscillation . . . . .	198
Exercises 11.4 . . . . .	203
<b>A Vectors</b>	<b>205</b>
A.1 Free and Localized Vectors . . . . .	205
A.2 Moment of a Vector about a Point . . . . .	206
A.3 Moment of a Vector about a Line . . . . .	207
A.4 Moment of a Couple . . . . .	208
A.5 Equation of Line and Plane . . . . .	209
A.6 Equation of Sphere . . . . .	210
<b>B Moments of Inertia</b>	<b>211</b>
<b>References</b>	<b>213</b>
<b>Answers and Notes</b>	<b>215</b>



# Preface

This book is based on lectures delivered to undergraduate students reading Mathematics, Physics or Engineering and is intended to cover a two-credit introductory course in Classical Mechanics. The motion of particles, rigid bodies and systems thereof are discussed. Both the Newtonian formulation with reference to Cartesian frames and the Lagrangian formulation with reference to generalized coordinates are discussed with a view to compare the two methods of approach.

These are alternative and equivalent formulations of the equations of motion both leading to differential equations which, when solved, describe the nature of the motion. In the Newtonian approach we consider the acceleration of the system under the action of the applied forces caused by external agencies. The resulting accelerated motion is governed by Newton's second law. In the Lagrangian approach we consider the kinetic and potential energies of the system, both of which are scalar functions and are therefore invariant under coordinate transformations. These transformations may be from Cartesian coordinates to generalized coordinates which may be chosen in one of many alternative ways that make the problem simpler to solve. The advantages of Lagrange's over Newton's equations are noted, particularly their independence on the choice of generalized coordinates. The book may be used as an introduction to any one of the two methods or both.

The student is required to have basic knowledge of vector algebra, matrix algebra and the calculus of ordinary functions of several variables including the chain rule for partial differentiation. He/She is also expected to be familiar with the suffix notation and the summation convention which are used wherever they result in more elegance and compactness. Besides it makes it easier for the student to make the transition to general relativity and mechanics of continua.

In rectangular Cartesian coordinates ( $x_i$ ) a Latin suffix such as  $i$  has a range of values ( $i = 1, 2, 3$ ) and summation over this range of values is understood when a (dummy) suffix is repeated in any term. The Cartesian tensor

and matrix notations are also employed whenever it is felt that they help to simplify the analysis as, for example, in rotating frames of reference, in moments of inertia and in Euler's equations.

In the Lagrangian method the index (suffix) notation is reserved for generalized coordinates ( $q_i$ ) where a Latin index such as  $i$  has a range of values ( $i = 1, 2, 3, \dots, n$ ) and a repeated index in a term implies summation over this range of values. No distinction between covariant and contravariant indices is made as this is not necessary for the development of this work. Such distinction appears only in a few exercises which demonstrate the connection between Newton's and Lagrange's equations of motion. Here reference is made to relevant text on curvilinear coordinates and general tensors, but these exercises may be omitted unless the student is interested in these topics.

Chapters **1** and **2** deal with the basic concepts, definitions and laws used in classical mechanics to analyse the motion of a system relative to some frame of reference. Chapter **3** considers Cartesian frames of reference and the effect of their motion on the description of the motion of a system.

Chapter **4** is concerned with the mechanics of a single particle including its motion relative to a rotating frame. Lagrange's equations are introduced by a worked example and in some exercises by way of their verification for the case of the motion of a particle. These particular exercises could be omitted at this stage.

Chapter **5** considers the motion of a system of particles. This includes its motion relative to a frame which is itself translating and rotating with respect to an inertial frame. The two-particle system is discussed.

Chapter **6** discusses moments and products of inertia in preparation for the motion of rigid bodies which is covered by Newtonian formulation using vector methods in Chapters **7** and **8**.

In Chapter **9** we introduce the idea of virtual displacement and the method of virtual work to solve static systems. The motion of dynamic systems using Lagrange's and Hamilton's formulation is discussed in Chapters **10** and **11**, where generalized energy and momentum, ignorable coordinates and the Routhian function are defined. The configuration and phase spaces, Liouville's Theorem and Hamilton's Principle are also considered. The powerful methods of Lagrange are applied to impulsive motion and small oscillations.

There are many worked examples within the text and the exercises at the end of each section are intended to clarify the basic concepts introduced in that section. Moreover, some questions are intended to pose an extension of a concept which the student may develop for himself in preparation for the work that follows in later sections. In such cases hints or part solutions are given

in the ‘Answers and Notes’ section at the end of the book. Starred questions are somewhat harder.

I have tried to provide as complete an index as possible and in some cases the number of pages indicated is rather large. An underlined page number indicates the most important reference to whatever is being indexed, its definition or its first appearance in the text shown in bold type.

I am thankful to Professor A. Buhagiar, Dr D. Buhagiar and Dr J. Muscat of the Department of Mathematics, University of Malta, who read and scrutinized the first and final drafts, made valid criticisms and suggestions, and many encouraging remarks.

I would also like to express my gratitude to the students who suffered through the many versions of the classroom editions of the text and for their help in checking the exercises.

Last but certainly not least, I wish to thank my wife for her unlimited patience, tolerance and continuous encouragement.

The book’s faults, errors, and other imperfections are entirely my own particularly since I have typed the text and typeset it myself using L<sup>A</sup>T<sub>E</sub>X.

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This book provides a concise but complete treatment of classical mechanics.

It should meet the needs of undergraduate students in mathematics, science or engineering.

Both the Newtonian and the Lagrangian methods are discussed and compared. These are alternative and equivalent formulations of the equations of motion both leading to differential equations which, when solved, describe the nature of the motion. In the Newtonian approach we consider the acceleration of the system under the action of the applied forces caused by external agencies. The resulting accelerated motion is governed by Newton's second law. In the Lagrangian approach we consider the kinetic and potential energies of the system, both of which are scalar functions and are therefore invariant under coordinate transformations.

The suffix notation is used to denote Cartesian coordinates and the components of vectors and tensors in the 3-dimensional space. The notation is also used to denote generalized coordinates and components in the n-dimensional configuration space for a holonomic system with n degrees of freedom.

The employment Einstein's summation convention permits compact expressions and equations resulting in economy of written material by avoiding the repetition of similar terms.

## CONTENTS

Preface

1. Introduction
2. The Nature of Force
3. Frame of Reference
4. Motion of a Particle
5. Motion of a system of Particles
6. Moments and Products of Inertia
7. Motion of a Rigid Body About a Fixed Point
8. General Motion of a Rigid Body
9. Static Systems – Virtual Work
10. Lagrangian Methods – Rudiments
11. Lagrangian Methods – Ramifications

## APPENDICES

A. Vectors

B. Moments of Inertia - Table

References

Answers and Notes

Index



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