TENSOR ANALYSIS

with applications to

Geometry and Continuum Mechanics

C J Camilleri

Physical Components of Tensor Derivatives

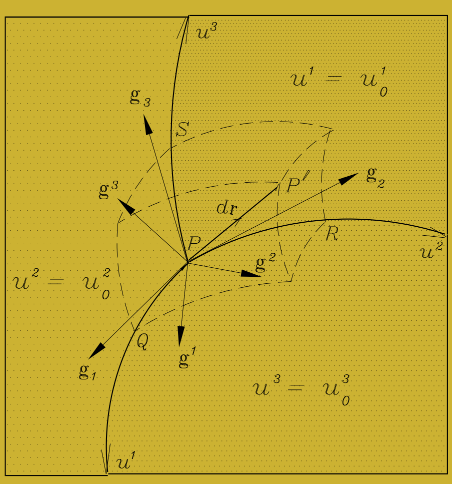
$$\widetilde{\nabla_m} b_{ik} = \frac{1}{h_M}\frac{\partial \tilde{b}_{ik}}{\partial u^m} + \frac{\tilde{b}_{ik}}{h_I h_K h_M}\frac{\partial h_I h_K}{\partial u^m} - \sum \frac{h_L}{h_I h_M}\left\{ {l \atop im} \right\}\tilde{b}_{lk}$$

$$\frac{\widetilde{\delta b}_{ik\ldots}(u,t)}{\delta t} = \frac{\partial \tilde{b}_{ik\ldots}}{\partial t} + \tilde{v}_m \widetilde{\nabla_m} b_{ik\ldots,m}$$

$$\frac{\widetilde{Db}_{ik\ldots}(u,t)}{Dt} = \frac{\partial \tilde{b}_{ik\ldots}}{\partial t} + \tilde{v}_m \widetilde{\nabla_m} b_{ik\ldots} + \sum \tilde{\omega}_{im}\tilde{b}_{mk\ldots}$$

$$\frac{\widetilde{\mathcal{V}b}{}^{\,..k..}_{\,.i\ldots}(u,t)}{\mathcal{V}t} = \frac{\widetilde{Db}_{\,.ik..}}{Dt} + \sum \tilde{e}_{mi}\tilde{b}_{\,.mk..} - \sum{}' \tilde{e}_{km}\tilde{b}_{\,.im..}$$

$$\widetilde{\pounds_v\, b}{}_i^{\,.k} = \tilde{v}_m \widetilde{\nabla_m} b_{ik} + \tilde{\omega}_{im}\tilde{b}_{mk} + \tilde{\omega}_{km}\tilde{b}_{im} + \tilde{e}_{im}\tilde{b}_{mk} - \tilde{e}_{km}\tilde{b}_{im}$$



Curvilinear Coordinates

Summation is understood over repeated *lower-case* indices. No components or summation are implied by *upper-case* suffixes of the scale factors – they are just labels. At each stage an *upper-case* suffix *'follows'* the value of the corresponding lower-case index. The $\sum$ denotes the sum of all such terms, one for each *similar* index of the tensor.

Electronic

Portable Document Format (PDF)

copy to

M J Roberts

of

Portsmouth, England

With best wishes

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Second Edition

July 16, 2004

First Edition published in 1999 by
Malta University Press, Tal-Qroqq, Msida, Malta

Second Edition in electronic-pdf format published in 2004 by
C J Camilleri, Department of Mathematics
University of Malta, Tal-Qroqq, Msida, Malta

Typesetting and Page Layout Produced C J Camilleri.

The text was prepared by the author using LaTeX2.09 under Windows XP.
The diagrams were drawn by the author using AutoCad.

To my beloved wife Kate, who makes life a joy,
and my children
Anthony, Susan, David, Christopher

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Preface

This book is based on a course of lectures given by the author at the University of Malta, and its main objective is to present an introductory course in tensor methods adapted to the needs of students in the physical and engineering sciences or in applied mathematics. The classical, component, local approach[1] is employed and the organization of the book is intended to firstly familiarize the student with index operations, through the use of the summation convention and the permutation symbols, and then to use these methods in the algebra and calculus of tensors which are defined in terms of appropriate transformation rules for their components denoted by indexed systems. Application to geometry and mechanics is made by including related exercises throughout the text, particularly in the last two chapters where tensors are applied to differential geometry and mechanics of continuous material.

Prerequisites include an elementary knowledge of the calculus of ordinary functions of several variables including the chain rule for partial differentiation, and a basic knowledge of vector algebra such as that found in the author's book on vector analysis [1] to which the present work is a follow up.

The emphasis of this book is on analytical techniques as revealed by the large number of exercises it contains. These exercises are to be taken as an integral part of the text as suggested by their positioning within the chapters. Those of the problem-type are supplied with answers and should provide practice in manipulation, the rest are meant to encourage the student to derive some of the important standard results for himself, as extensions of the preceding sections, which are often required and quoted later. At times these are accompanied by instructive remarks.

The outstanding difficulty in introducing tensor notation is the amount of indices involved and Chapter **1** may appear to be unnecessarily detailed and possibly overwhelming for first-time readers. However, it is my experience

[1]It is felt that students will have deeper understanding of the modern, noncomponent, global treatment of the subject after mastering the component approach; after all in solving real physical or engineering problems calculations are carried out on components.

that the student who takes the challenge of working through the exercises for himself right at the beginning of the course, will soon realize how much easier it is to write tensor notation than to read it and will appreciate the advantages it offers in writing complex expressions compactly. Moreover, it is hoped that the student also appreciates the simplification offered by the index notation if it supplants the traditional notation of vector analysis and matrix theory.

The treatment of vectors in terms of basis along rectilinear coordinates in Chapter **2** provides a gentle introduction of the Riemannian metric 3-space via local basis along curvilinear coordinate directions considered in Chapter **3**.

In Chapter **4** tensors are introduced in the Cartesian framework to be developed more generally in the next two chapters. This may appear to involve duplication, but in my opinion this is offset by a definite pedagogical value. Physics and engineering students will welcome this lucid introduction of tensors to what may normally be considered a complex aspect of mathematics.

The two middle chapters make up the heart of the book. Chapter **5** deals with the algebra of absolute and relative (weighted) tensors; considers how these are related and how associated tensors are related, addressing the importance of the *lateral* position of the indices unless the tensor is symmetric. Chapter **6** deals with the calculus of tensors in Riemannian 3-space where the metric is a function of position but not of time. It considers curvature and the condition for Euclidean 3-space.

Physical components of vectors and tensors referred to local curvilinear coordinate directions are shown to represent quantities having the physical dimensions of the field and therefore capable of immediate physical interpretation. In the case of orthogonal curvilinear coordinates, general formulae are derived in Chapter **7** for the physical components of tensors including the intrinsic and covariant derivatives.

Chapter **8** provides a brief introduction to local differential geometry of curves and surfaces in Riemannian 3-space. Again the student is encouraged to obtain most of the standard results for himself by applying the basic methods and concepts of tensors. This chapter may be read immediately after Chapter **6** or it may be omitted completely.

An introduction to the theory of deformation and flow of homogeneous continuous material – a subject otherwise known as 'rheology' – is given in Chapter **9** as a natural application of tensor methods. The method of keeping the material element 'fixed' by introducing a convected coordinate system (drawn in the material and deforming continuously with it) is discussed, leading to Oldroyd's material and convected time derivatives and the Lie derivative.

General formulae[2] for the physical components of these derived tensors referred to local orthogonal curvilinear coordinate directions are established which formulae should prove to be invaluable for research scientists. This chapter is supplemented by an overview of rheology in Appendix A.

I have tried to provide as complete an index as possible and in some cases the number of pages indicated is rather large. An <u>underlined</u> page number indicates the most important reference to whatever is being indexed, its definition or its first appearance in the text shown in bold type.

I am thankful to my colleagues for their support and encouragement, particularly Dr D. Buhagiar who read the first draft and Dr J. Muscat who scrutinized the first and final drafts, made valid criticisms and suggestions, and many encouraging remarks.

I would also like to express my gratitude to the students who suffered through the many versions of the classroom editions of the text and for their help in checking the exercises.

Last but certainly not least, I wish to thank my wife for her unlimited patience, tolerance and continuous encouragement.

The book's faults, errors, and other imperfections are entirely my own particularly since I have typed the text and typeset it myself using LaTeX.

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July, 2004.

[2]The novel notation used enhances compactness and eliminates the necessity to introduce *left* and *right* physical components in the case of non-orthogonal coordinates as suggested by Truesdell [10].

Published by C J Camilleri

This book provides a concise but complete treatment of tensor analysis. It should meet the needs of most undergraduate students in the physical and engineering sciences or in applied mathematics.
The classical, component, local approach is employed and the organization of the book is intended to firstly familiarize the student with index operations, through the use of the summation convention and the permutation symbols, and then to use these methods in the algebra and calculus of tensors. The emphasis of this book is on analytical techniques as revealed by the large number of exercises it contains. An introduction to the theory of deformation and flow of homogeneous continuous material is given as a natural application of tensor methods. The method of keeping the material element 'fixed' by introducing a convected coordinate system (drawn in the material and deforming continuously with it) is discussed, leading to Oldroyd's material and convected time derivatives and the Lie derivative. General formulæ for the physical components of these derived tensors referred to local orthogonal curvilinear coordinate directions are established which formulæ should prove to be invaluable for research scientists. This is supplemented by an overview of rheology.

CONTENTS

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