VECTOR ANALYSIS

with a gentle introduction to

Cartesian Tensors

C J Camilleri

Differential Vector Operators

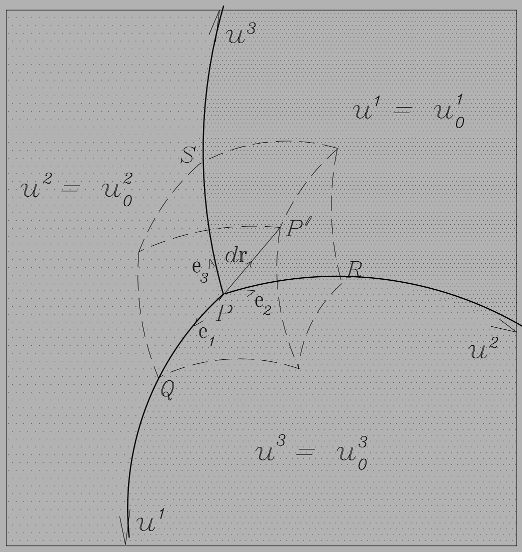
$$h_I = \left|\frac{\partial \mathbf{r}}{\partial u^I}\right| \qquad \mathbf{e}_i = \frac{1}{h_I}\frac{\partial \mathbf{r}}{\partial u^i}$$

$$\operatorname{grad}\phi = \frac{1}{h_I}\frac{\partial \phi}{\partial u^i}\,\mathbf{e}_i$$

$$\operatorname{div}\mathbf{F} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u^i}\left(\frac{h_1h_2h_3}{h_I}F^i\right)$$

$$\operatorname{curl}\mathbf{F} = \frac{h_I}{h_1h_2h_3}\epsilon_{ijk}\frac{\partial(h_K F^k)}{\partial u^j}\,\mathbf{e}_i$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u^i}\left(\frac{h_1h_2h_3}{h_Ih_I}\frac{\partial \phi}{\partial u^i}\right)$$



Orthogonal Curvilinear Coordinates

Summation is understood over repeated *lower-case* suffixes. No components or summation are implied by *upper-case* suffixes – they are just labels. At each stage an *upper-case* suffix ‘follows’ the value of the corresponding *lower-case* suffix.

Electronic

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copy to

J R Jones

of

Swansea, Wales

With best wishes

VECTOR ANALYSIS

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Cartesian Tensors

C J Camilleri
Department of Mathematics
University of Malta

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To my beloved wife Kate, who makes life a joy,
and my children
Anthony, Susan, David, Christopher

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Preface

This book is based on lectures delivered to undergraduate students reading for a degree in Mathematics, Science or Engineering. The text is primarily intended to cover a course in vector algebra and differentiation of vectors (Chapter 0–4) and a course in vector field theory, including integral theorems and orthogonal curvilinear coordinates (Chapter 5–9). However, the original lecture notes have been drastically revised so that the book may serve a variety of curricula.

The approach adapted here is rather different to that found in most popular textbooks on vector analysis. Vectors in $\mathbb{R}^3$ are treated directly in terms of their Cartesian components using the tensor suffix notation and employing Einstein's summation convention. In fact vectors are considered as Cartesian tensors of order one and defined in terms of appropriate rules of transformation under rotation of coordinate axes.

This approach has been successfully employed by the author for almost thirty years and offers several advantages over traditional methods based on the definition of a vector as a directed line segment.

- The use of the summation convention economizes on the amount of written material.

- The rules of vector algebra and differentiation are immediately understood since componentwise they are basically identical to those of ordinary algebra and calculus.

- This simplicity is carried over to scalar and vector fields since the gradient, divergence, curl and vector operators are also defined in terms of Cartesian components making the manipulation of vector identities easier.

- Having been exposed to this direct treatment of vectors defined via transformation rules of their components, the student is ready to move to tensors of any order with very little mental adjustment.

In this edition a new Chapter 0 is added. It introduces the two ways of looking at vectors – their geometrical representation as directed line segments and the component representation through the use of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along Cartesian coordinate directions. This preliminary chapter should help the students, particularly those encountering vectors for the first time, to visualize the concept of vectors through the more physical geometrical description, and to appreciate the advantages offered by the rather nonphysical component description. These advantages will become more obvious when the tensor notation is employed in the rest of the book.

In Chapter 1 the suffix notation and summation convention are introduced and the properties of real and dummy suffixes are illustrated through various examples. The substitution and alternating tensors are defined and relevant identities involving them are established and used to obtain the basic properties of the determinant of a (3×3) matrix.

In Chapter 2 rectangular Cartesian coordinate systems and their rotation are considered. The orthonormality relations and other identities obeyed by the transformation matrix are obtained. The transformation rules for the coordinates of a point in terms of the transformation matrix and the concept of invariance with respect to rotation of axes complete the basic requirements for the stated method of approach.

A quick look at the contents of the rest of the chapters will not reveal anything strikingly unfamiliar. The range of topics is central and traditional and includes titles which are normally covered in an introductory course on the subject. However, the treatment which is different to that normally adopted, leads to the advantages listed above.

Chapter 3 is dedicated to vector algebra. The usual properties of vector operations and standard algebraic identities are obtained directly from the tensor notation definitions of the basic operations.

In Chapter 4 the differentiation and integration of vector functions of a scalar variable are defined in terms of those of their Cartesian components for which the basic rules of ordinary calculus apply[1].

Chapter 5 deals with scalar and vector fields and differential vector functions. Here the advantages of using tensor notation definitions stand out when proving standard identities involving vector functions.

Line, surface and volume integrals of vector functions are covered in Chapter 6 and are followed by the integral theorems of vector field theory in Chapter 7. Again, the component form of basic definitions is here used to great

[1]The student is expected to be familiar with the calculus of ordinary functions of several variables including the chain rule for partial differentiation.

advantage, enabling one to give rigorous proofs of these theorems which are nevertheless within the grasp of the average student.

The suffix notation and summation convention are normally of limited use when it comes to general orthogonal curvilinear coordinates. However, a modified convention[2] has been used in Chapter 8 to overcome these difficulties thus making it possible to establish compact expressions for vector fields, gradient, divergence, curl and the Laplacian in general orthogonal curvilinear coordinates. This avoids the alternative, more complicated, proofs based on the definitions of vector functions involving limits of integrals.

Chapter 9 deals with some applications of vector analysis to physical and engineering fields.

There are many worked examples within the text and the exercises at the end of each section are intended to clarify the basic concepts introduced in that section. In a few cases an exercise may pose the extension of a concept which is left for the student to develop and some hints are given in the 'Answers and Notes' section at the end of the book. Starred questions are somewhat harder.

I have tried to provide as complete an index as possible and in some cases the list of pages indicated is rather long. An <u>underlined</u> page number indicates the most important reference to whatever is being indexed, its definition or its first appearance in the text shown in bold type.

I would like to record my thanks to the following reviewers whose constructive comments and observations were extremely helpful in writing the first edition of this book: Dr J Muscat, Dr D Buhagiar and Professors A P Calleja, J P Gauci, J Lauri. My thanks also go to Mr F Ciantar and Professor S Fiorini for a number of valuable suggestions.

I would also like to express my gratitude to the students who followed my lectures with great interest and whose enthusiastic reaction to the first edition inspired me to embark on the second.

Last but certainly not least, I wish to thank my wife for her unlimited patience, tolerance and continuous encouragement.

The book's faults, errors, and other imperfections are entirely my own particularly since I have typed the text and typeset it myself using LaTeX.

C J Camilleri

Department of Mathematics,
University of Malta
July, 2004

[2]This was first introduced by the author in 1965 and used to write compact formulae for the physical components of Oldroyd's *material* and *convected* time derivatives of general tensors which arise in the mechanics of elastico-viscous liquids.

Published by C J Camilleri

This book provides a concise but complete treatment of vector analysis. It should meet the needs of most undergraduate students of mathematics, science and engineering.

The approach adapted here is rather different to that found in most popular textbooks on vector analysis. Vectors in $\mathbb{R}^3$ are treated directly in terms of their Cartesian components using the tensor suffix notation and employing Einstein's summation convention.

This approach offers several advantages over traditional methods. Particularly, the component form of the basic definitions of vector functions makes the manipulation of vector identities almost trivial, enabling one to give rigorous proofs of the integral theorems which are nevertheless within the grasp of the average student.

The suffix notation and summation convention are normally of limited use when it comes to general orthogonal curvilinear coordinates. However, a novel modified convention is used to overcome these difficulties. This makes it possible to establish compact expressions for the gradient, the divergence, the curl and the Laplacian, thus avoiding the alternative, more complicated, derivations based on the definitions of vector functions involving limits of integrals.

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THE AUTHOR

C J CAMILLERI is professor emeritus in the Department of Mathematics at the University of Malta.