The purpose of this project is to investigate line graphs of trees and determine an eigenvector for the eigenvalue zero for those that are singular.

A graph $G(V,E)$ has a non-empty vertex set $V$ and an edge set $E$ consisting of pairs of distinct vertices. A tree is a graph without cycles. For the line graph $L_G$ of a graph $G$, the vertex set $V(L_G) = |E(G)|$ and two vertices in $L_G$ are adjacent if and only if the edges in $G$ have a common vertex. Thus the line graph of the star $K_{1,n-1}$ on $n$ vertices is the complete graph $K_{n-1}$.

Figure 1: A star and its Line Graph

Also the line graph of a tree consists of adjacent complete graphs (cliques) and two cliques share at most one vertex.

For a graph $G$ on $n$ vertices, the adjacency matrix $A(G) = A = (a_{ij})$ of $G$ is the $n \times n$ symmetric matrix such that

$$a_{ij} = \begin{cases} 
1 & \text{if } \{i, j\} \text{ is an edge of } G \\
0 & \text{otherwise}
\end{cases}$$

Figure 2 shows four trees:

Figure 2: Trees
1. Label the vertices of each tree.

2. Call the Mathematica package \texttt{<<DiscreteMath\'Combinatorica'}}

3. Write down the adjacency lists of the trees and transform to the line graphs.

4. Determine the determinant \( \det(L_T - v_i) \) of the adjacency matrix of each one-vertex deleted subgraph of \( L_T' \).

5. By using the \'Drop \' command or otherwise, find the adjugate \( \text{adjug}(L_T) \) of \( L_T \).

It is established in [1] that the nullity of the line graph of a tree is at most one.

1. Why are the columns of \( \text{adjug}(L_T) \) eigenvectors of \( A(L_T) \) for a singular line graph of a tree?

2. If each one-vertex deleted subgraph of the line graph of a tree is singular, why is \( L_T \) non-singular?

With reference to the four trees \( T_1, T_2, T_3, T_4 \) above, discuss the relevance of the one-vertex deleted subgraphs of the line graphs \( \{L_T - v_i\} \), to the nullspace of the adjacency matrix of \( L_T \).

Reference

[1]