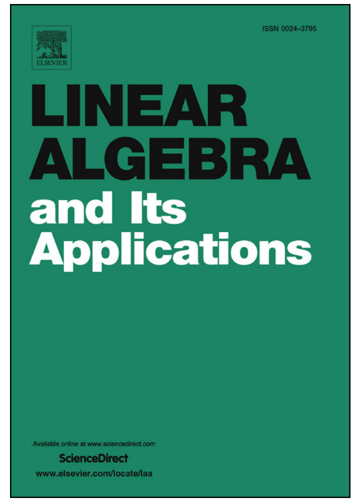

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PII: S0024-3795(16)30270-1
DOI: http://dx.doi.org/10.1016/j.laa.2016.07.008
Reference: LAA 13787

To appear in: *Linear Algebra and its Applications*

Received date: 25 October 2015
Accepted date: 11 July 2016

On eigenspaces of some compound signed graphs

Francesco Belardo [a,b], Irene Sciriha [c], Slobodan K. Simić [d,e]

[a] Department of Mathematics and Applications, University of Naples "Federico II", Italy
[b] University of Primorska – FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
[c] Department of Mathematics, University of Malta, Malta
[d] State University of Novi Pazar, Vuka Karadžića bb, 36 300 Novi Pazar, Serbia
[e] Mathematical Institute SANU, P.O. Box 367, 11001 Belgrade, Serbia

Dedicated to the memory of Lucia Gionfriddo (1973–2008).

Abstract

In the theory of (simple) graphs the concepts of the line and subdivision graph (as compound graphs) are well-known. It is possible to consider them also in the context of (edge) signed graphs. Some relations between the Laplacian spectrum of signed graphs and adjacency spectra of their associated compound (signed) graphs have been recently established in the literature. In this paper, we study the relations between the corresponding eigenspaces.

Keywords: signed graph, line graph, subdivision graph, adjacency matrix, Laplacian, eigenvalues, eigenspaces.
AMS Classification: 05C50, 05C22.

1 Introduction

Let $G = (V(G), E(G))$ be a graph (simple, unless otherwise stated) of order $n = |V(G)|$ and size $m = |E(G)|$, and let $\sigma : E(G) \to \{+, -\}$ be a mapping defined on the edge set of G. Then $\Gamma = (G, \sigma)$ is a *signed graph* (or *sigraph*) and G is its *underlying graph*, while σ its *sign function* (or *signature*). Furthermore, it is common to interpret the signs as the integers ± 1. Hence, sometimes signed graphs are treated as weighted graphs, whose (edge) weights belong to $\{+1, -1\}$. An edge e is *positive* (*negative*) if $\sigma(e) = +$ (resp. $\sigma(e) = -$). If all edges in Γ are positive (negative), then Γ is denoted by $(G, +)$ (resp. $(G, -)$).

Most of the concepts defined for graphs are directly extended to signed graphs. For example, the degree of a vertex v in G, denoted by $deg(v)$, is also its degree in Γ. Furthermore, if some subgraph of the underlying graph is observed, then the sign function for the subgraph is the restriction of the original one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$ (note, all edges incident to v are deleted). If $U \subset V(G)$ then $\Gamma[U]$ (with underlying graph $G[U]$) denotes the (signed) induced subgraph arising from U, while $\Gamma - U = \Gamma[V(G) \setminus U]$. We also write $\Gamma - \Gamma[U]$ instead of $\Gamma - U$.

A cycle of Γ is said to be *balanced* (or *positive*) if it contains an even number of negative edges. A signed graph is said to be *balanced* if all its cycles are balanced; otherwise, it is *unbalanced*. For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$ (namely, $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) *switching equivalent* to Γ, and the corresponding relation is an equivalence relation. So switching equivalent signed graphs can be considered as (switching) isomorphic graphs and their signatures are said to be equivalent. Observe also that switching equivalent graphs have the same set of positive cycles. For other notation or definitions not given here the reader is referred to [20].

Signed graphs, as the unsigned ones, can be studied by using matrix theory. If M ($= M(\Gamma)$) is a real and symmetric matrix associated with Γ, then $\det(xI - M)$ is the characteristic polynomial (or M-polynomial) of Γ with respect to M; it is denoted by $\phi_M(x; \Gamma)$. The eigenvalues of M, or equivalently the roots of $\phi_M(x; \Gamma)$, are also called the *eigenvalues of* Γ

with respect to M. They are real, since M is real and symmetric. Together with their multiplicities, they comprise the spectrum of Γ (with respect to M) which is denoted by $\hat{\sigma}_M(\Gamma)$. Note that algebraic and geometric multiplicities of any eigenvalue of Γ are the same (since M is real and symmetric). The multiplicity of the eigenvalue μ is denoted by $\mathrm{mult}(\mu;\Gamma)$. A non-zero vector $\mathbf{x}$ satisfying the equation $M\mathbf{x} = \mu\mathbf{x}$, i.e. an *eigenvalue equation*, is the eigenvector (or μ-eigenvector) of M, and also of Γ if it is considered as a vertex labelled signed graph. The eigenspace of M for $\mu \in \hat{\sigma}_M(\Gamma)$ is the set $\mathcal{E}_M(\mu;\Gamma) = \{\mathbf{x} : M\mathbf{x} = \mu\mathbf{x}\}$; it is also an eigenspace of Γ for μ with respect to M. Finally, a basis of $\mathcal{E}_M(\mu;\Gamma)$ is called an *M-basis* of Γ for μ; note, its size is equal to $mult(\mu;\Gamma)$.

In this paper, we assume that M is one of the following matrices:

- A – the *adjacency matrix* (recall, $A = (a_{ij})$, where $a_{ij} = \sigma(ij)$ if vertices i and j are adjacent, and 0 otherwise);
- L ($= D - A$) – the *Laplacian matrix*, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of vertex degrees.

Accordingly, this gives rise to two possibilities to develop relevant spectral (signed) graph theory – namely, the *A-theory* and the *L-theory*, respectively. In addition, the study of their interrelations becomes a very important issue.

At this stage, it is worth mentioning that switching equivalent signed graphs have similar adjacency and Laplacian matrices. In fact, any switching arising from vertex subset U can be described by a diagonal matrix $S_U = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ with $s_i = +1$ for each $i \in U$, and $s_i = -1$ otherwise. The matrix S_U is sometimes called the *state matrix*. Hence, $A(\Gamma) = S_U A(\Gamma^U) S_U$ and $L(\Gamma) = S_U L(\Gamma^U) S_U$. A similar effect features with eigenvectors. When we consider a signed graph Γ, from a spectral viewpoint, we are in fact considering its switching isomorphism class $[\Gamma]$. For more details see Section 4 below.

In this paper we will consider both, the A-polynomial the L-polynomial of signed graphs. For the sake of readability, we denote by

$$\alpha(x;\Gamma) := \phi_A(x;\Gamma) = \det(xI - A(\Gamma)) \quad \text{and} \quad \lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$$

the adjacency characteristic polynomial and the adjacency eigenvalues, respectively. Similarly, for the Laplacian matrix, we have the following notation:

$$\beta(x,\Gamma) := \phi_L(x;\Gamma) = \det(xI - L(\Gamma)) \quad \text{and} \quad \mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) \geq 0;$$

the last inequality holds since the Laplacian matrix is positive semidefinite (see the next section).

Recently, the signed graphs have been considered in [3] where the authors considered formulas for computing the coefficients of the Laplacian polynomial. In the same paper the authors defined the notion of signed line graph and signed subdivision graph, and they obtained some formulas which relate the Laplacian polynomial of signed graphs to the adjacency polynomials of their compound graphs. In fact, such formulas for signed graphs are a generalization of those given for unsigned graphs, which are well-known in the literature (see, for example, [8]). In this paper, we focus our attention on the corresponding eigenspaces and generalize the results given in [13]; namely, we investigate how the eigenspaces are related between the original signed graphs and their compound derivates. As a byproduct, we generalize a result of Sachs (see [12]) which features only for regular unsigned graphs (for more details, see [4], Theorem 3.36).

The remainder of the paper is organized as follows: in Section 2 we introduce some basic facts needed in Section 3, which itself covers our main results. In Section 4 we discuss the effects of switching and orientation on the eigenspaces of the observed compound (signed) graphs. Finally, Section 5 contains some concluding remarks.

2 Preliminaries

The definition of a signed line graph and a signed subdivision graph used here are given in [3]. Here we reproduce them in order to make this paper more self-contained. It is worth mentioning that in the literature different definitions of signed line graphs can be found. We refer the reader to [17, 14, 18, 10] for more results on signed line graphs and their spectra.

In [3], *bi-directed graphs* are used to represent signed graphs. Bi-directed graphs first appeared in [9], while their role with the theory of signed graphs was recognized in [16]. A *bi-directed signed graph* is an ordered pair $\Gamma_\eta = (\Gamma, \eta)$, where

$$\eta : V(G) \times E(G) \to \{+1, -1, 0\} \tag{1}$$

is an orientation satisfying the following three conditions:

(*i*) $\eta(u, vw) = 0$ whenever $u \neq v, w$;

(*ii*) $\eta(v, vw) = +1$ (or -1) if an arrow at v is going into (resp. out of) v (cf. Fig. 1);

(*iii*) $\eta(v, vw)\eta(w, vw) = -\sigma(vw)$.

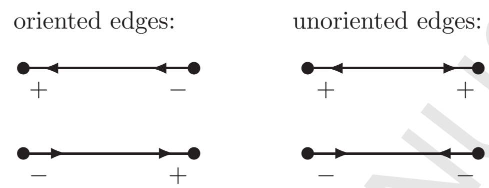

Fig. 1: Bi-directed edges.

Edges in each pair of oriented and unoriented edges from Fig. 1 are said to be mutually *doubly-inverted edges*. So we have that positive edges are oriented edges, while negative unoriented (see also Fig. 1). Thus each bi-directed graph gives rise to a signed graph. The converse is also true, but then one arrow (at any end) can be taken arbitrarily, while not the other arrow (in view of (iii) from above). Note, two bi-directed graphs lead to the same signed graph whenever they differ only in some doubly-inverted edges.

The *incidence matrix* of Γ_η, denoted by $B(\Gamma_\eta)$ or B_η, is an $n \times m$ matrix (b_{ij}), where $b_{ij} = \eta(v_i, e_j)$ for each $v_i \in V(G)$ and $e_j \in E(G)$. Usually, only Γ is given, and then η is determined as explained above. Any row of the incidence matrix corresponding to vertex v_i contains $deg(v_i)$ non-zero entries, each equal to $+1$ or -1. On the other hand, each column of the incidence matrix corresponding to edge e_j contains two non-zero entries, each equal to $+1$ or -1. Therefore, even in the case that multiple edges exist, we easily obtain that

$$B_\eta B_\eta^\top = D(G) - A(\Gamma_\eta) = L(\Gamma_\eta), \tag{2}$$

where $D(G)$ is the diagonal matrix of vertex degrees of G. In particular, if $\Gamma = (G, +)$ (or $\Gamma = (G, -)$) then we obtain the (standard) Laplacian (resp. signless Laplacian) matrix of G (see also [8]). Needless to add, multiple edges, but not loops, if they exist in the underlying graph are treated as all other edges. Observe also that in view of (2) the matrix $L(\Gamma_\eta)$ is positive semidefinite, as already noted in the previous section.

It is also easy to see that

$$B_\eta^\top B_\eta = 2I + A(\mathcal{L}(\Gamma_\eta)), \tag{3}$$

where $\mathcal{L}(\Gamma_\eta)$ is line graph of an oriented signed graph. It is noteworthy to say here that $\mathcal{L}(\Gamma_\eta)$ has $\mathcal{L}(G)$ as its underlying graph, while the sign of the edge $ef \in \mathcal{L}(\Gamma)$ $(e, f \in E(G))$ in the resulting signed graph is equal to $\sigma_l(ef) = \eta(w, e)\eta(wf)$ if w is the unique common vertex of the edges e and f in G; if the edges e and f have two vertices in common (i.e., G is a multigraph) then the signs are summed up leading to either a zero sign edge (so no edge), or two parallel edges. Of course, this is rather a matrix than combinatorial definition of line graphs of signed graphs (tailored for the spectral graph theory). For more details concerning line graphs of unsigned graphs and also Hoffman's generalized line graphs the reader is referred to [3, 2, 6]. It is noteworthy that generalized line graphs, as unsigned graphs, can be seen also as line graphs (in the above sense) of some special class of bi-directed graphs with hanging double edges (see [3, 2] for more details).

We now consider the subdivision graphs. As with line graphs, we will now extend to signed graphs the well-known matrix representation of the adjacency matrix of subdivision graphs, which, in block form, now reads

$$A(\mathcal{S}(\Gamma_\eta)) = \begin{pmatrix} O_n & B_\eta \\ B_\eta^\top & O_m \end{pmatrix}, \tag{4}$$

where O_t is the $t \times t$ zero matrix. It is easy to see that the underlying graph of $\mathcal{S}(\Gamma_\eta)$ is $\mathcal{S}(G)$, while the signature σ_s is defined by $\sigma_s(v_i e_j) = \eta(v_i, e_j)$ (note that $V(S(G)) = V(G) \cup E(G)$). An example of the line and subdivision graphs of a signed graph are depicted in Fig. 2. Here and thereafter we denote positive edges by bold lines, and negative edges by dotted lines.

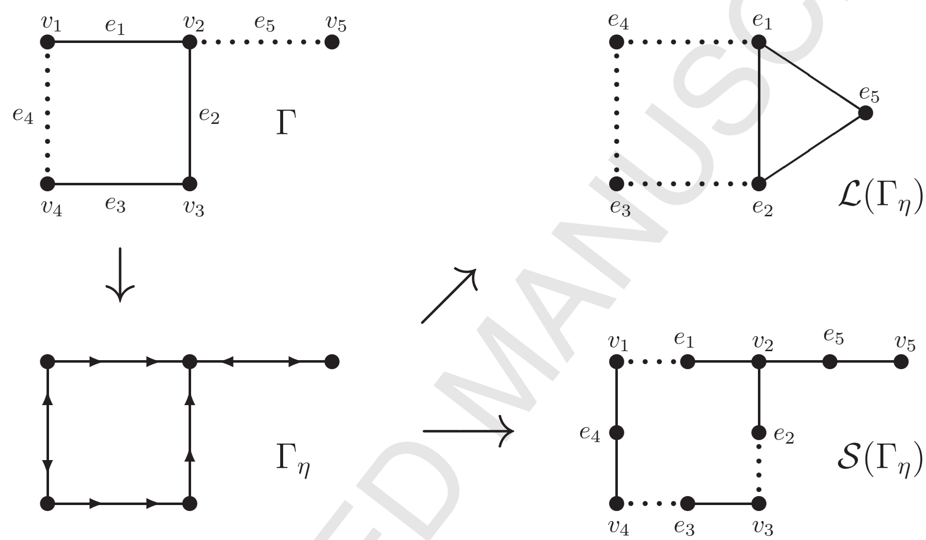

Fig. 2: A signed graph and the corresponding signed line and subdivision graphs.

Remark 2.1. *It is important to observe that any (random) orientation η to the edges of Γ gives rise to the same matrix $L(\Gamma_\eta) = L(\Gamma)$, while the matrices $A(\mathcal{L}(\Gamma_\eta))$ and $A(\mathcal{S}(\Gamma_\eta))$ do depend on η. It is not difficult to see that a different orientation η' leads to matrices $A(\mathcal{L}(\Gamma_{\eta'}))$ and $A(\mathcal{S}(\Gamma_{\eta'}))$, which are switching equivalent to $A(\mathcal{L}(\Gamma_\eta))$ and $A(\mathcal{S}(\Gamma_\eta))$, respectively. From now on, the index η will be no longer specified.*

Since $BB^\top$ and $B^\top B$ share the non-zero eigenvalues, from (3) and (4) we get:

Theorem 2.2 ([3])**.** *Let Γ be a signed graph of order n and size m, and let α and β be its adjacency and Laplacian characteristic polynomials, respectively. Then the following relations hold*

1^o $\alpha(x; \mathcal{L}(\Gamma)) = (x+2)^{m-n}\beta(x+2; \Gamma)$,

2^o $\alpha(x; \mathcal{S}(\Gamma)) = x^{m-n}\beta(x^2; \Gamma)$.

Remark 2.3. *From the above theorem, if Γ is a connected signed graph (of order n and size m) we obtain that* $\mathrm{mult}(-2; \mathcal{L}(\Gamma)) = m-n+1$ *if Γ is balanced; otherwise* $\mathrm{mult}(-2; \mathcal{L}) = m-n$. *Moreover, $\lambda_i(\mathcal{L}(\Gamma)) = \mu_i(\Gamma)-2$ whenever $\mu_i(\Gamma) \neq 0$; otherwise $\lambda_i(\mathcal{L}(\Gamma)) = -2$. Similarly, we obtain that* $\mathrm{mult}(0; \mathcal{S}(\Gamma)) = m-n+2$ *if Γ is balanced; otherwise* $\mathrm{mult}(0; \mathcal{S}(\Gamma)) = m-n$, *and moreover that $\lambda_i(\mathcal{S}(\Gamma)) = \sqrt{\mu_i(\Gamma)}$, and $\lambda_{m+n+1-i}(\mathcal{S}(\Gamma)) = -\sqrt{\mu_i(\Gamma)}$ whenever $\mu_i(\Gamma) \neq 0$; otherwise $\lambda_i(\mathcal{S}(\Gamma)) = 0$.*

3 Main results

In the previous section (see Theorem 2.2) we mentioned the relations between the characteristic polynomials of signed graphs (with respect to the Laplacian matrix) and their compound signed graphs (with respect to the adjacency matrix). So it is natural to ask how the corresponding eigenspaces of these signed graphs are interrelated. In Subsection 3.1 we establish the connections between the L-eigenspaces of a signed graph Γ and the A-eigenspaces of its signed line graph $\mathcal{L}(\Gamma)$. In Subsection 3.2 we establish, for a given signed graph Γ, the connection between the following eigenspaces (i) the A-eigenspaces of $\mathcal{S}(\Gamma)$ and (ii) the L-eigenspaces of Γ and the A-eigenspaces of $\mathcal{L}(\Gamma)$.

Without loss of generality, we will assume in the sequel that Γ is connected (for disconnected signed graphs we can apply component-wise the results for connected case). Note also that if Γ is connected, then the same holds for both compound (signed) graphs considered in this paper. As already indicated, we will write, for the incidence matrix B_η, just B. Clearly, the signed subdivision graph and the signed line graph are both expressed in terms of the same incidence matrix. Orientation will be revisited in Section 4, where we analyze the effect of switching and orientation on the eigenspaces of compound graphs.

3.1 The relation between $\mathcal{E}_L(\mu;\Gamma)$ and $\mathcal{E}_A(\lambda;\mathcal{L}(\Gamma))$

Let $\Gamma = (G, \sigma)$ be a signed graph of order n and size m, and let $B (= B_\eta)$ be the $n \times m$ incidence matrix of Γ. Hence we have

$$BB^\top = D(G) - A(\Gamma) = L(\Gamma), \quad B^\top B = 2I + A(\mathcal{L}(\Gamma)). \tag{5}$$

Recall that the spectrum of $B^\top B$ can be obtained from the spectrum of $BB^\top$ by adding to (or subtracting from) it the number 0 repeated $m-n$ times if $m > n$ (or, respectively, $n-m$ times if $n > m$). Therefore, if

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0 \ \text{ and } \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq -2,$$

are the L-eigenvalues of Γ and the A-eigenvalues of $\mathcal{L}(\Gamma)$, respectively, then $\mu_i = \lambda_i + 2$ for $i = 1, 2, \ldots, \min\{m, n\}$; for $i > \min\{m, n\}$ then either $\mu_i = 0$ (if any) or $\lambda_i = -2$ (if any) (cf. also Remark 2.3).

To find relations between the eigenspaces corresponding to Γ and $\mathcal{L}(\Gamma)$, we consider their eigenvectors. For the sake of readability, we write $A_{\mathcal{L}}$ for $A(\mathcal{L}(\Gamma))$. We next distinguish two cases depending on λ and μ.

Case 1: $\mu = \lambda + 2 \neq 0$ (so $\lambda \neq -2$).

We first prove two claims:

Claim 1: If $\mathbf{x} \in \mathcal{E}_L(\mu;\Gamma) \setminus \{\mathbf{0}\}$ then $B^\top\mathbf{x} \in \mathcal{E}_A(\lambda;\mathcal{L}(\Gamma)) \setminus \{\mathbf{0}\}$.

Multiplying the first equality of (5) by $\mathbf{x}$, the μ-eigenvector of Γ, we obtain $BB^T\mathbf{x} = L(\Gamma)\mathbf{x} = \mu\mathbf{x}$. Putting $\mathbf{y} = B^\top\mathbf{x}$ we obtain $\mu\mathbf{x} = B\mathbf{y}$. Clearly, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, and both are non-zero vectors. Next we have that $B^\top B\mathbf{y} = B^\top(BB^\top\mathbf{x}) = B^\top L\mathbf{x} = \mu B^\top\mathbf{x} = \mu\mathbf{y}$. Therefore, by the second equality in (5), we obtain that $A_{\mathcal{L}}\mathbf{y} = (\mu - 2)\mathbf{y} = \lambda\mathbf{y}$. Hence, $\mathbf{y} = B^\top\mathbf{x} \neq \mathbf{0}$ is a λ-eigenvector of $\mathcal{L}(G)$, and the claim follows.

Claim 2: If $\mathbf{y} \in \mathcal{E}_A(\lambda;\mathcal{L}(\Gamma)) \setminus \{\mathbf{0}\}$ then $B\mathbf{y} \in \mathcal{E}_L(\mu;\Gamma) \setminus \{\mathbf{0}\}$.

Multiplying the second equality in (5) with $\mathbf{y}$, the λ-eigenvector of $\mathcal{L}(G)$, we obtain $(A_{\mathcal{L}} + 2I)\mathbf{y} = B^\top B\mathbf{y} = (\lambda + 2)\mathbf{y}$. So $B(B^\top B\mathbf{y}) = (\lambda + 2)(B\mathbf{y})$. Putting $\mathbf{x} = B\mathbf{y}$, and using the first equality in (5), we obtain $L\mathbf{x} = (\lambda + 2)\mathbf{x} = \mu\mathbf{x}$. If $\mathbf{x} = \mathbf{0}$, then $B\mathbf{y} = \mathbf{0}$, and therefore $\lambda = -2$, a contradiction. So, $\mathbf{x} = B\mathbf{y} \neq \mathbf{0}$ is a μ-eigenvector of $L(\Gamma)$, and the claim follows.

From the above claims, if $\mu \neq 0$, or equivalently if $\lambda \neq -2$, we have that the above two vector spaces are isomorphic. Now, we say that two eigenvectors $\mathbf{x} \in \mathcal{E}_L(\mu;\Gamma)$ and $\mathbf{y} \in \mathcal{E}_A(\lambda;\mathcal{L}(\Gamma))$ are *μ-partners* if

$$\mu\mathbf{x} = B\mathbf{y} \ \text{ and } \ \mathbf{y} = B^\top\mathbf{x}. \tag{6}$$

Similarly, we say that two eigenvectors $\mathbf{x} \in \mathcal{E}_L(\mu;\Gamma)$ and $\mathbf{y} \in \mathcal{E}_A(\lambda;\mathcal{L}(\Gamma))$ are *λ-partners* if

$$\mathbf{x} = B\mathbf{y} \quad \text{and} \quad (\lambda+2)\mathbf{y} = B^\top \mathbf{x}. \tag{7}$$

Two eigenvectors $\mathbf{x}$ and $\mathbf{y}$ as above are *partners* if they are either *μ-partners* or *λ-partners*.

Case 2: $\mu = 0$ and $\lambda = -2$.

Recall first, as already adopted, that Γ is connected. It is well-known that $\mu = 0$ holds if and only if Γ is balanced (see, for example, [15]). On the other hand, $\Gamma = (G,\sigma)$ is balanced if and only if $V(G) = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, where negative edges have one end vertex in V_1 and the other one in V_2, and positive edges have both end vertices within the same V_i $(i = 1,2)$ [11]. In the latter case, we say that V_1 and V_2 are two color classes in Γ. The L-eigenvector $\mathbf{x}$ related to $\mu = 0$ has entries -1 for vertices from one color class, while $+1$ otherwise. But then the corresponding vector $\mathbf{y} = B^\top\mathbf{x}$ is equal to $\mathbf{0}$, and so $\mathbf{y}$ is not an eigenvector for $\mathcal{L}(\Gamma)$. In other words, $\mathbf{x}$ does not have partners.

The A-eigenspace of $\mathcal{L}(\Gamma)$ for $\lambda = -2$ is described in detail in [1], and we refer the readers to this paper for the details of the results described below. The eigenspace of -2 in signed line graphs can be directly obtained from a connected spanning signed subgraph Φ whose signed line graph does not have -2 as an eigenvalue (the so-called *signed foundation*). A foundation Φ is either a spanning tree whenever Γ is balanced, or it is a unbalanced unicyclic graph. From one-edge extensions of the foundation Φ, namely $\Phi+e$, we obtain three kinds of spanning subgraphs of Γ: those containing either a balanced cycle, or the *double-unbalanced infinite graph*, or the *double-unbalanced dumbbell* (i.e. graphs obtained from two unbalanced cycles joined by a path, possibly of length zero – see Fig. 3). By properly weighting the edges of $\Phi+e$ (that is a subgraph of Γ) we get a (-2)-eigenvector for $\mathcal{L}(\Gamma)$. The edges corresponding to nonzero entries of the (-2)-eigenvector are called *heavy edges*, while the others are the *light edges*. Let Θ be the subgraph of $\Phi + e$ consisting of all heavy edges of $\Phi + e$. Since Θ consists of heavy edges, then Θ is said to be the *heavy subgraph* of $\Phi+e$ (in [1], it is called the *core*). For each $e \in E(\Gamma \setminus \Phi)$ we get a different $\Phi + e$ (with a corresponding heavy subgraph Θ) from which we build a (-2)-eigenvector, which will be linearly independent from those similarly obtained. We have that Θ is either a balanced cycle, or a double-unbalanced infinite graph, or a double-unbalanced dumbbell (see again Fig. 3).

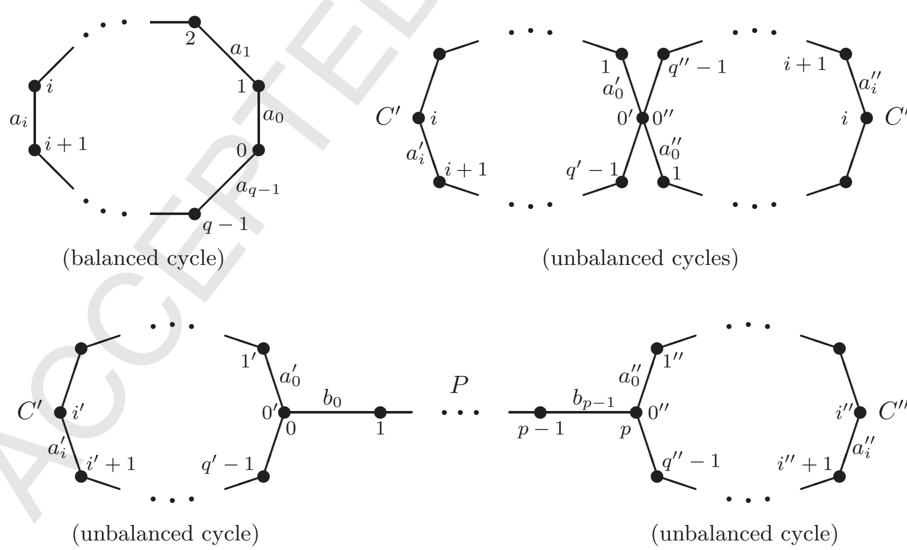

Fig. 3: Three types of heavy subgraphs.

The following theorems are proved in [1].

Theorem 3.1. *Let Θ be a balanced cycle and $\Theta_{\mathcal{L}}$ be its signed line graph. Then, under the above notation (see Fig. 3), the vector $\mathbf{a} = (a_0, a_1, \ldots, a_{q-1})^\top$, where*

$$a_i = (-1)^i \Big[\prod_{s=1}^{i} \nu(s) \Big] a_0 \quad (i = 0, 1, \ldots, q-1) \text{ and } \nu(s) = \sigma_L(e_{s-1}e_s) = \eta(s, e_{s-1})\eta(s, e_s),$$

is an eigenvector of $\Theta_{\mathcal{L}}$ for -2. Moreover, it can be extended to a (-2)-eigenvector of $\mathcal{L}(\Gamma)$ by putting zeros at all other entries.

Theorem 3.2. *Let Θ be a double-unbalanced infinite graph and $\Theta_{\mathcal{L}}$ be its signed line graph. Then, under the above notation (see Fig. 3), the vector $\mathbf{a}' \dot{+} \mathbf{a}''$, where $\mathbf{a}' = (a'_0, a'_1, \ldots, a'_{q'-1})^\top$, $\mathbf{a}'' = (a''_0, a''_1, \ldots, a''_{q''-1})^\top$, and*

$$\begin{aligned} a'_i &= (-1)^i \Big[\prod_{s=1}^{i} \nu(s) \Big] a'_0 \quad (i = 0, 1, \ldots, q' - 1); \\ a''_i &= (-1)^i \Big[\prod_{s=1}^{i} \nu(s) \Big] a''_0 \quad (i = 0, 1, \ldots, q'' - 1); \end{aligned}$$

is an eigenvector of $\Theta_{\mathcal{L}}$ for -2 provided $a'_0 \neq 0$ is arbitrary and $a''_0 = -\widehat{\nu}(0', 0'')a'_0$, where $\widehat{\nu}(0', 0'') = \eta(0', e'_0)\eta(0'', e''_0)$. Moreover, it can be extended to a (-2)-eigenvector of $\mathcal{L}(\Gamma)$ by putting zeros at all other entries.

Theorem 3.3. *Let Θ be a double-unbalanced dumbbell and $\Theta_{\mathcal{L}}$ be its signed line graph. Then, under the above notation (see Fig. 3), the vector $\mathbf{a}' \dot{+} \mathbf{b} \dot{+} \mathbf{a}''$, where $\mathbf{a}' = (a'_0, a'_1, \ldots, a'_{q'-1})^\top$, $\mathbf{b} = (b_0, b_1, \ldots, b_{p-1})^\top$, $\mathbf{a}'' = (a''_0, a''_1, \ldots, a''_{q''-1})^\top$, and*

$$\begin{aligned} a'_i &= (-1)^i \Big[\prod_{s=1}^{i} \nu(s) \Big] a'_0 \quad (i = 0, 1, \ldots, q' - 1), \\ b_i &= (-1)^i \Big[\prod_{s=1}^{i} \nu(s) \Big] b_0 \quad (i = 0, 1, \ldots, p - 1), \\ a''_i &= (-1)^i \Big[\prod_{s=1}^{i} \nu(s) \Big] a''_0 \quad (i = 0, 1, \ldots, q'' - 1), \end{aligned}$$

is an eigenvector of $\Theta_{\mathcal{L}}$ for -2 provided $b_0 \neq 0$ is arbitrary, $a'_0 = -\frac{1}{2}\widehat{\nu}(0, 0')b_0$ and $a''_0 = -\frac{1}{2}\widehat{\nu}(p, 0'')c_0$, where $\widehat{\nu}(0, 0') = \eta(0, e'_0)\eta(0, f_0)$ and $\widehat{\nu}(p, 0'') = \eta(p, f_{p-1})\eta(0'', e''_0)$. Moreover, it can be extended to a (-2)-eigenvector of $\mathcal{L}(\Gamma)$ by putting zeros at all other entries.

From the above construction (using also linearity) it follows that $B\mathbf{y} = \mathbf{0}$ for any vector $\mathbf{y} \in \mathcal{E}_A(-2; \mathcal{L}(\Gamma))$, and therefore $\mathbf{y}$ has no partners.

Summarizing the above considerations from Cases 1 and 2 (see also [1]), we obtain the following results for connected signed graphs (easy to extended to disconnected signed graphs):

Theorem 3.4. *Let $B = B_\eta$ be the incidence matrix of a connected signed graph $\Gamma = \Gamma_\eta$. Then we have:*

1^o *$\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_s}\}$ is a L-eigenbasis of Γ_η for $\mu \neq 0$ if and only if $\{B^\top \mathbf{x_1}, B^\top \mathbf{x_2}, \ldots, B^\top \mathbf{x_s}\}$ is an A-eigenbasis of $\mathcal{L}(\Gamma)$ for $\mu - 2$;*

2^o *$\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_t}\}$ is an A-eigenbasis of $\mathcal{L}(\Gamma_\eta)$ for $\lambda \neq -2$ if and only if $\{B\mathbf{y_1}, B\mathbf{y_2}, \ldots, B\mathbf{y_t}\}$ is a L-eigenbasis of Γ_η for $\lambda + 2$.*

Theorem 3.5. *Let $\Gamma = \Gamma_\eta$ be a connected graph. Then we have:*

1^o *If $\mu = 0$, then $\Gamma_\eta = (V_1 \cup V_2, E)$ is balanced and the corresponding L-eigenspace of Γ_η is spanned by $(x_1, x_2, \ldots, x_n)^\top$, where $x_i = -1$ if $v_i \in V_1$, and $x_i = +1$ if $v_i \in V_2$.*

2^o *If $\lambda = -2$, then the corresponding A-eigenspace of $\mathcal{L}(\Gamma)$ is spanned by the vectors constructed by the procedure given in Theorems 3.1, 3.2 and 3.3.*

Related to the Theorem 3.5(ii) it is worth mentioning:

Remark 3.6. *Note that, if Γ is balanced, we obtain $m-n+1$ independent balanced cycles. Otherwise, if Γ is unbalanced, we first construct all possible cycles ($m-n+1$ in total), keep those which are balanced, fix one unbalanced and adjoin to it in turn other unbalanced cycles to form either infinite graphs or dumbbells. In this way we obtain $m-n$ independent heavy subgraphs (note, one unbalanced cycle is not counted). Clearly, these heavy subgraphs, in both cases (balanced or unbalanced ones), give rise to the independent eigenvectors, which give the desired eigenbasis.*

Finally, we add that we have now (in view of Theorems 3.4 and 3.5) resolved the situations which were not covered for non-regular graphs related to Sachs's theorem mentioned in Section 1.

3.2 Relations among $\mathcal{E}_A(\lambda;\mathcal{S}(\Gamma))$, $\mathcal{E}_L(\mu;\Gamma)$ and $\mathcal{E}_A(\lambda;\mathcal{L}(\Gamma))$

Assume, as in the previous subsection, that $\mu_1,\mu_2,\dots,\mu_n$ are the L-eigenvalues of Γ, while $\lambda_1,\lambda_2,\dots,\lambda_m$ are the A-eigenvalues of $\mathcal{L}(\Gamma)$. Then the A-eigenvalues of $\mathcal{S}(\Gamma)$ are $\pm\sqrt{\mu_i}$ ($i=1,2,\dots,s$), for some $s\le n$; all other eigenvalues are equal to 0 (cf. Theorem 2.2). Note also that all non-zero L-eigenvalues of Γ and the corresponding A-eigenvalues $\mathcal{S}(\Gamma)$ have the same multiplicities. Next, if both μ_i and λ_i exist for some i, then $\mu_i=\lambda_i+2$.

Case 1: the A-eigenspaces of $\mathcal{S}(\Gamma)$ are known.

Now, our goal is to deduce, from the A-eigenspaces of $\mathcal{S}(\Gamma)$, the L-eigenspaces of Γ and the A-eigenspaces of $\mathcal{L}(\Gamma)$. To do this, we need some additional notation. Given a vector $\mathbf{v}\in\mathbb{R}^n$ whose entries are indexed by V, the vertex set of some signed graph, let $U\subset V$. Then $\mathbf{v}(U)$ denotes its restriction to U. If $|U|=t<n$, then $\mathbf{v}(U)\in\mathbb{R}^t$.

Consider now $\Gamma=(G,\sigma)$ of order n and size m. Let $\hat\lambda\in\hat\sigma_A(\mathcal{S}(\Gamma))$, and let $\mathbf{z}\in\mathcal{E}_A(\hat\lambda;\mathcal{S}(\Gamma))$ be an A-eigenvector of $\mathcal{S}(\Gamma)$ for $\hat\lambda$. Recall that $V(\mathcal{S}(\Gamma))=V(\Gamma)\cup E(\Gamma)$. Denote by V_1 the set of vertices in $\mathcal{S}(\Gamma)$ originating from the vertices of Γ (say, the black vertices) and by V_2 the set of vertices originating from $E(\Gamma)$ (say, the white vertices). Let $\mathbf{x}=\mathbf{z}(V_1)$ and $\mathbf{y}=\mathbf{z}(V_2)$ be the restrictions of $\mathbf{z}$ to V_1 and V_2, respectively. Without loss of generality, we can assume that the first n entries of $\mathbf{z}$ correspond to the vertices, while the remaining m entries to the edges of Γ. So we can write $\mathbf{z}=\mathbf{x}\dot{+}\mathbf{y}$. Clearly, $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbb{R}^m$. Since $A(\mathcal{S}(\Gamma))\mathbf{z}=\hat\lambda\mathbf{z}$, then $A^2(\mathcal{S}(\Gamma))\mathbf{z}=\hat\lambda^2\mathbf{z}$. We also have that

$$A^2(\mathcal{S}(\Gamma))=\begin{pmatrix} BB^\top & O_n\\ O_m & B^\top B\end{pmatrix}=\begin{pmatrix} L(\Gamma) & O_n\\ O_m & A(\mathcal{L}(\Gamma))+2I_m,\end{pmatrix}=L(\Gamma)\dot{+}(A(\mathcal{L}(\Gamma))+2I_m).$$

where $B=B_\eta$. With the above decompositions of $A^2(\mathcal{S}(\Gamma))$ and of $\mathbf{z}$ in mind, we obtain

$$L(\Gamma)\mathbf{x}=\hat\lambda^2\mathbf{x}\ \text{ and }\ (A(\mathcal{L}(\Gamma))+2I_m)\mathbf{y}=\hat\lambda^2\mathbf{y}.$$

Since $\mu=\hat\lambda^2$ and $\lambda=\mu-2$ $(=\hat\lambda^2-2)$, we have

$$L(\Gamma)\mathbf{x}=\mu\mathbf{x}\ \text{ and }\ A(\mathcal{L}(\Gamma))\mathbf{y}=\lambda\mathbf{y}. \tag{8}$$

In what follows, recall that Γ is connected and, of course, $\mathcal{S}(\Gamma)$ is connected as well, and viceversa.

We next distinguish two subcases depending on $\hat\lambda$ and μ.

Subcase 1.1: $\hat\lambda\ne 0$.

We first claim that $\mathbf{x}$ and $\mathbf{y}$ are non-zero vectors. Otherwise, if $\mathbf{y}=\mathbf{0}$, applying the eigenvalue equations (at the black vertices in $\mathcal{S}(\Gamma)$), we obtain $\mathbf{x}=\mathbf{0}$, a contradiction (since $\mathbf{x}\ne\mathbf{0}$). Therefore $\mathbf{y}\ne\mathbf{0}$. Similarly, we obtain that $\mathbf{x}\ne\mathbf{0}$. So, if $\hat\lambda\in\sigma_A(\mathcal{S}(\Gamma))\setminus\{0\}$ and $\mathbf{z}\in\mathcal{E}_A(\hat\lambda;\mathcal{S}(\Gamma))\setminus\{\mathbf{0}\}$, then

$$\mu\in\sigma_L(\Gamma)\setminus\{0\},\ \mathbf{x}\in\mathcal{E}_L(\mu;\Gamma)\setminus\{\mathbf{0}\},$$

and

$$\lambda \in \sigma_A(\mathcal{L}(\Gamma)) \setminus \{-2\},\ \mathbf{y} \in \mathcal{E}_A(\lambda; \mathcal{L}(\Gamma)) \setminus \{\mathbf{0}\}.$$

Assume next that the eigenvalue $\hat{\lambda}$ in question is positive, and that its multiplicity is k. Let $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ be linearly independent A-eigenvectors associated with $\hat{\lambda}$. Without loss of generality, we can assume that they are mutually orthogonal. Consider next the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, the restrictions of the $\mathbf{z}_i$ $(1 \leq i \leq k)$ to V_1, and also $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$, the restrictions of the $\mathbf{z}_i$ $(1 \leq i \leq k)$ to V_2. Then, as already proven (see (8)), they are L-eigenvectors in Γ for $\mu \neq 0$ and A-eigenvectors in $\mathcal{L}(\Gamma)$ for $\lambda \neq -2$, respectively. We now prove, using orthogonality, that they are linearly independent. To this aim, consider two mutually orthogonal vectors $\mathbf{z}_i$ and $\mathbf{z}_j$ $(i \neq j)$. Let $\mathbf{z}_i = \mathbf{x}_i \dot{+} \mathbf{y}_i$ and $\mathbf{z}_j = \mathbf{x}_j \dot{+} \mathbf{y}_j$.
Then, by orthogonality, we have

$$\mathbf{z}_i \cdot \mathbf{z}_j = \mathbf{x}_i \cdot \mathbf{x}_j + \mathbf{y}_i \cdot \mathbf{y}_j = 0,$$

where $\cdot$ stands for the scalar product. On the other hand, since $\mathcal{S}(\Gamma)$ is bipartite, it is not difficult to see that if $\hat{\lambda} \neq 0$ is an A-eigenvalue for $\mathcal{S}(\Gamma)$ with corresponding eigenvector $\mathbf{z}_i = \mathbf{x}_i \dot{+} \mathbf{y}_i$, then $-\hat{\lambda}$ is an eigenvalue as well with corresponding eigenvector $\mathbf{z}'_i = \mathbf{x}_i \dot{+} \mathbf{y}'_i$, where $\mathbf{y}'_i = -\mathbf{y}_i$. Therefore, again by orthogonality, we have

$$\mathbf{x}_i \cdot \mathbf{x}_j - \mathbf{y}_i \cdot \mathbf{y}_j = 0.$$

From the latter two conditions we obtain $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ and $\mathbf{y}_i \cdot \mathbf{y}_j = 0$, and our claim follows. By simple counting, we conclude that we have found k linearly independent μ-eigenvectors for Γ. Also, we have found k linearly independent A-eigenvectors for $\mathcal{L}(\Gamma)$. In other words, if $\mu \neq 0$ we have found a basis for $\mathcal{E}_L(\mu; \Gamma)$, and if $\lambda \neq -2$ we have found a basis for $\mathcal{E}_L(\lambda; \mathcal{L}(G))$.

Subcase 1.2: $\hat{\lambda} = 0$.

Then $\mu = 0$ and $\lambda = -2$, and (8) is reduced to

$$L(\Gamma)\mathbf{x} = \mathbf{0} \quad \text{and} \quad A(\mathcal{L}(\Gamma))\mathbf{y} = -2\mathbf{y}.$$

Note now that $\mathbf{x}$ and $\mathbf{y}$ cannot be both 0-vectors (since $\mathbf{z} = \mathbf{x} \dot{+} \mathbf{y} \neq \mathbf{0}$). Also note that $\mathbf{x}$ and $\mathbf{y}$, as components of $\mathbf{z}$, are not related by $\mathbf{x} = B\mathbf{y}$ as in the case for $\hat{\lambda} \neq 0$. In oother words, if $\mathbf{x}$ is substituted with $\mathbf{x}'$ (for which $L(\Gamma)\mathbf{x}' = \mathbf{0}$), or if $\mathbf{y}$ is substituted with $\mathbf{y}'$ (for which $A(\mathcal{L}(\Gamma))\mathbf{y}' = -2\mathbf{y}'$), then any of the resulting vectors, say $\mathbf{z}' \neq \mathbf{0}$, is an A-eigenvector of $\mathcal{S}(\Gamma)$ for $\hat{\lambda} = 0$. The latter can be verified from the corresponding eigenvalue equations (for $\hat{\lambda} = 0$ in $\mathcal{S}(\Gamma)$). We can also say that $\mathbf{x}$ and $\mathbf{y}$ have no partners.

Recall that Γ is connected. By Theorem 2.2, the multiplicity of $\hat{\lambda} = 0$ in $\mathcal{S}(\Gamma)$ is $m-n+2$ if Γ is balanced, and $m-n$ otherwise. Assume first that Γ is balanced. As in the previous subcase, let $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ be mutually orthogonal A-eigenvectors associated with $\hat{\lambda} = 0$, while the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ and $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ are the restrictions of the $\mathbf{z}_i$ $(1 \leq i \leq k)$ to V_1 and V_2, respectively. According to (8), the $\mathbf{x}_i$ are in $\mathcal{E}_L(0; \Gamma)$ and the $\mathbf{y_i}$ are in $\mathcal{E}_A(-2; \mathcal{L}(\Gamma))$. If all the $\mathbf{x}_i$ $(1 \leq i \leq k)$ are 0-vectors, then the vectors $\mathbf{z}_i$ $(1 \leq i \leq k)$ cannot span the A-eigenspace of $\mathcal{S}(\Gamma)$ of dimension $m-n+2$ (since the vectors $\mathbf{y}_i$ $(1 \leq i \leq k)$ can span a subspace of dimension at most $m-n+1$). So, the vectors $\mathbf{x}_i$ should span just the $\mathcal{E}_L(0; \Gamma)$. On the other hand, if all the vectors $\mathbf{y}_i$ $(1 \leq i \leq k)$ span a space of dimension less than $m-n+1$ then the vectors $\mathbf{z}_i$ $(1 \leq i \leq k)$ cannot span the A-eigenspace of $\mathcal{S}(\Gamma)$ of dimension $m-n+2$ (since the vectors $\mathbf{x}_i$ span a subspace of dimension just 1). So the vectors $\mathbf{x}_i$ and the vectors $\mathbf{y}_i$ span the eigenspaces of dimensions 1 and $m-n+1$, respectively (as required). Finally, assume that Γ is unbalanced. Then all the vectors $\mathbf{x}_i$ are 0-vectors, and we deduce that the vectors $\mathbf{y}_i$ span the subspace of dimension $m-n$.

The above conclusions can be summarized as follows:

Theorem 3.7. *Let $\mathcal{S}(\Gamma) = (V_1 \cup V_2, E)$ be the signed subdivision graph of a connected signed graph Γ, where V_1 are the vertices originating from Γ, while V_2 are the inserted vertices. Let*

$$\{\mathbf{z_1} = \mathbf{x_1} \dot{+} \mathbf{y_1}, \mathbf{z_2} = \mathbf{x_2} \dot{+} \mathbf{y_2}, \ldots, \mathbf{z_k} = \mathbf{x_k} \dot{+} \mathbf{y_k}\}$$

be an A-eigenbasis of $\mathcal{S}(\Gamma)$ *for* $\hat{\lambda} \geq 0$*, where* $\mathbf{x}_i = \mathbf{z}_i(V_1)$ *and* $\mathbf{y}_i = \mathbf{z}_i(V_2)$ $(1 \leq i \leq k)$*.*

If $\hat{\lambda} = \sqrt{\mu} > 0$ *then*

1^o $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ *is an eigenbasis for* $\mathcal{E}_L(\mu; \Gamma)$*, and*

2^o $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}\}$ *is an eigenbasis for* $\mathcal{E}_A(\lambda; \mathcal{L}(\Gamma))$*, where* $\lambda = \mu - 2$*.*

If $\hat{\lambda} = 0$ *then*

3^o $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ *spans* $\mathcal{E}_L(0; \Gamma)$*, and*

4^o $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}\}$ *spans* $\mathcal{E}_A(-2; \mathcal{L}(\Gamma))$*.*

Remark 3.8. *Note first that in Theorem 3.7, the statements* 1^o *and* 2^o *hold by taking* $\hat{\lambda} < 0$*, as well.*
The following facts deserve to be mentioned for the vectors $\mathbf{x}_i$ $(1 \leq i \leq k)$ *in* 3^o*, and for the vectors* $\mathbf{y}_i$ $(1 \leq i \leq k)$ *in* 4^o*, from the above theorem.*

- *If* Γ *is balanced then* $k = m - n + 2$*. In fact, the non-zero vectors* $\mathbf{x}_i$ $(1 \leq i \leq k)$ *are collinear with a vector described in Theorem 3.5* 1^o*. So the dimension of the L-eigenspace of* Γ *for* 0 *is* 1*. Next, either just one of the* $\mathbf{y}_i$ $(1 \leq i \leq k)$ *is a* 0*-vector, or none. In the former case we have only one choice for the A-eigenbasis of* $\mathcal{L}(\Gamma)$ *for* -2*, while in the latter* $m - n + 2$ *choices (each of the vectors* $\mathbf{y}_i$ *is a linear combination of the remaining ones). The dimension of the A-eigenspace of* $\mathcal{L}(\Gamma)$ *for* -2 *is* $m - n + 1$*. Therefore, we obtain* $k = m - n + 2$*.*
- *If* Γ *is unbalanced then* $k = m - n$*. In fact, each vector* $\mathbf{x}_i$ $(1 \leq i \leq k)$ *is a* 0*-vector, while* $\mathcal{E}_L(0; \Gamma)$ *is trivial. Now the vectors* $\mathbf{y}_i$ *form an eigenbasis for* $\mathcal{E}_A(-2; \mathcal{L}(\Gamma))$*. So,* $k = m - n$*.*

Case 2: either the L-eigenspaces of Γ, or the A-eigenspaces of $\mathcal{L}(\Gamma)$ are known.

Our goal now is to deduce, from the L-eigenspaces of Γ or A-eigenspaces of $\mathcal{L}(\Gamma)$, the A-eigenspaces of $\mathcal{S}(\Gamma)$. So, we actually consider the reverse problem with respect to Case 1.

We first prove the following (expected) result:

Lemma 3.9. *Let* $\mathcal{S}(\Gamma) = (V_1 \cup V_2, E)$ *be the signed subdivision graph of a connected signed graph* Γ*, where* V_1 *are the vertices originating from* Γ*, while* V_2 *are the inserted vertices. If* $\mathbf{z}$ *is an A-eigenvector of* $\mathcal{S}(\Gamma)$ *for* $\hat{\lambda} \neq 0$ *then we have:*

(i) $\mathbf{z} = \mathbf{z}(V_1) \dotplus B^\top \mathbf{z}(V_1)$*, or*

(ii) $\mathbf{z} = B\mathbf{z}(V_2) \dotplus \mathbf{z}(V_2)$*.*

Proof. If $\mathbf{z} = \mathbf{x} \dotplus \mathbf{y}$, where $\mathbf{x} = \mathbf{z}(V_1)$ and $\mathbf{y} = \mathbf{z}(V_2)$, then $L\mathbf{x} = \mu\mathbf{x}$ and $A_{\mathcal{L}}\mathbf{y} = \lambda\mathbf{y}$. On the other hand, using eigenvalue equations, we immediately get $\mathbf{y} = B^\top\mathbf{x}$ (when considering equations at vertices from V_2) and $\mathbf{x} = B\mathbf{y}$ (when considering equations at vertices from V_1), and the proof follows. □

Remark 3.10. *Note that, if we know the eigenspace* $\mathcal{E}_L(\mu; \Gamma)$ *for* $\mu \neq 0$*, or the eigenspace* $\mathcal{E}_A(\lambda; \mathcal{L}(\Gamma))$ *for* $\lambda \neq -2$*, then we can construct (using Lemma 3.9) the corresponding A-eigenspace of* $\mathcal{S}(\Gamma)$ *for* $\hat{\lambda} > 0$ *(recall,* $\hat{\lambda} = \sqrt{\mu} = \sqrt{\lambda + 2}$*). In addition, if* $\hat{\lambda} < 0$*, then* $\mathcal{E}_A(\hat{\lambda}; \mathcal{S}(\Gamma))$ *can be obtained from* $\mathcal{E}_A(-\hat{\lambda}; \mathcal{S}(\Gamma))$*. Observe, each eigenvector for* $-\hat{\lambda}$ *can be obtained by a reflection with respect to a hyperplane, determined by color classes, of an eigenvector for* $\hat{\lambda}$*.*

We have already observed that for $\hat{\lambda} = 0$, if $\mathbf{z} = \mathbf{x} \dotplus \mathbf{y}$ is an associated A-eigenvector for $\mathcal{S}(\Gamma)$, then $\mathbf{x}$ and $\mathbf{y}$ are not related. So $\mathbf{x}$ is either $\mathbf{0}$, or is constructed as in Theorem 3.5 1^o; similarly, $\mathbf{y}$ is either $\mathbf{0}$, or is is constructed as in Theorem 3.5 2^o. So, it follows that we can get the required number of A-eigenvectors (distinguishing balanced and unbalanced signed graph Γ) to form $\mathcal{E}_A(0; \mathcal{S}(\Gamma))$.

The above conclusions can be summarized as follows:

Theorem 3.11. *Let Γ be a connected signed graph, and let $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ be a L-eigenbasis of Γ for $\mu \neq 0$. Then*

$$\{\mathbf{z_1} = \mathbf{x_1} \dot{+} (\pm B^\top \mathbf{x_1}),\ \mathbf{z_2} = \mathbf{x_2} \dot{+} (\pm B^\top \mathbf{x_2}), \ldots, \mathbf{z_k} = \mathbf{x_k} \dot{+} (\pm B^\top \mathbf{x_k})\}$$

is an A-eigenbasis of $\mathcal{S}(\Gamma)$ for $\hat{\lambda} = \pm\sqrt{\mu}$.

Analogously, we have:

Theorem 3.12. *Let Γ be a connected graph, and let $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}\}$ be an A-eigenbasis of $\mathcal{L}(\Gamma)$ for $\lambda \neq -2$. Then*

$$\{\mathbf{z_1} = \mathbf{y_1} \dot{+} (\pm B \mathbf{y_1}),\ \mathbf{z_2} = \mathbf{y_2} \dot{+} (\pm B \mathbf{y_2}), \ldots, \mathbf{z_k} = \mathbf{y_k} \dot{+} (\pm B \mathbf{y_k})\}$$

is an A-eigenbasis of $\mathcal{S}(\Gamma)$ for $\hat{\lambda} = \pm\sqrt{\lambda + 2}$.

Now, we assume that $\hat{\lambda} = 0$. Then we have:

Theorem 3.13. *Let Γ be a connected signed graph. If Γ is balanced, let $\mathbf{x_1}$ be a L-eigenvector of Γ for $\mu = 0$; otherwise, if Γ is unbalanced, let $\mathbf{x_1} = \mathbf{0}$. Let $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}\}$ be an A-eigenbasis of $\mathcal{L}(\Gamma)$ for $\lambda = -2$. Then*

$$\{\mathbf{x_1} \dot{+} \mathbf{0},\ \mathbf{0} \dot{+} \mathbf{y_1},\ \mathbf{0} \dot{+} \mathbf{y_2}, \ldots, \mathbf{0} \dot{+} \mathbf{y_k}\} \setminus \{\mathbf{0}\}$$

is an A-eigenbasis of $\mathcal{S}(\Gamma)$ for $\hat{\lambda} = 0$.

4 Switching and eigenvector components

In respect to signed graphs, as well-known, switching appears to be an important concept. Effects of switchings (on the eigenvectors, and so on the eigenspaces) within compound signed graphs have been already considered in [1] for the eigenvalue -2 of the signed line graphs. Here, we revisit the discussion on the the effects of switching and orientation on the eigenspaces of the compound (signed) graphs, and we include the signed subdivision graph in our considerations. In fact, we shall analyze what happens to the eigenspaces of the compound (signed) graphs when switching and/or orientation are applied to the root signed graph.

Let D be a diagonal matrix, whose i-th diagonal entry is d_i. It is well-known that in the product DA (or AD), all entries in the i-th row (resp. j-th column) are multiplied by d_i (resp. d_j). In particular, if $d_i = \pm 1$ for each i (or $d_j = \pm 1$ for each j), then all entries i-th row (resp. j-th column) change the sign whenever d_i (resp. d_j) is equal to -1.

We first discuss the effects of switching to a signed graph Γ. Let $\Gamma' = \Gamma^U$, where $U \subset V$ Now we have to consider both the effects of switching and orientation. As already known, switching produces an edge sign switching on the cut $[U, \Gamma \setminus U]$, and such a switching is represented by the matrix $S_U = D_1 = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, where $d_i = 1$ if $v_i \in U$ and $d_i = -1$ otherwise. It has been already observed that $D_1 A(\Gamma) D_1 = A(\Gamma')$ and $D_1 L(\Gamma) D_1 = L(\Gamma')$. Hence, we immediately deduce the following result.

Theorem 4.1. *Let $\mathbf{x}$ be a λ-eigenvector (μ-eigenvector) of $A(\Gamma)$ (resp., $L(\Gamma)$). If $\Gamma' = \Gamma^U$ is obtained from the switching D_1, then $D_1\mathbf{x}$ is a λ-eigenvector (μ-eigenvector) of $A(\Gamma')$ (resp., $L(\Gamma')$).*

Next, we consider the effects of switching and double-inversions on the incidence matrix $B = B_\eta$ of the bi-directed graph Γ_η. Let $B' = B(\Gamma'_{\eta'})$, for some orientation η' of $\Gamma' = \Gamma^U$. The matrix D_1 has an impact on the matrix B_η which can be seen as reversal orientation of arrows at switched vertices. In other words, the matrix $D_1 B$ results in a incidence matrix $\bar{B} = (\bar{\eta}(v, e))$, where $\bar{\eta}(v, e) = -\eta(v, e)$ if $v \in U$, otherwise $\bar{\eta}(v, e) = \eta(v, e)$. On the other hand, B' and $\bar{B}$ are two (possibly) different orientations of the same signed graph Γ'. Hence, they can differ only in double-inversion of edges in the sense of Fig. 1 (otherwise the orientations would not lead to the same signature of Γ'). Let $\bar{E} \subseteq E$ be the set of doubly-inverted edges, or equivalently the edges whose orientation in $\bar{\eta}$ is different from η', and let D_2 be the diagonal matrix such that $B' = \bar{B} D_2$ (now, if $e_j \in \bar{E}$ then $d_j = -1$, otherwise $d_j = +1$). If so, we have that $B' = D_1 B D_2$.

Example 4.2. Let $\Gamma = (G, \sigma)$ be the graph depicted in Fig. 4 and consider any corresponding bi-directed graph Γ_η. Consider next Γ', a signed graph switching equivalent to Γ and a corresponding bi-directed graph $\Gamma'_{\eta'}$. We will compute the matrices D_1 and D_2 which transform B_η in to $B'_{\eta'}$.

Assume that the signed graph Γ' is obtained by switching with respect to $U = \{v_2, v_3\}$. The state matrix corresponding to the switching is D_1. The switching on Γ_η produced the bi-directed graph $\Gamma'_{\bar{\eta}}$. The edges of $\Gamma'_{\bar{\eta}}$ which are doubly-inverted are $\bar{E} = \{e_2, e_3, e_6\}$ give rise to the matrix D_2. Hence, we have that $B'_{\eta'} = D_1 B_\eta D_2$.

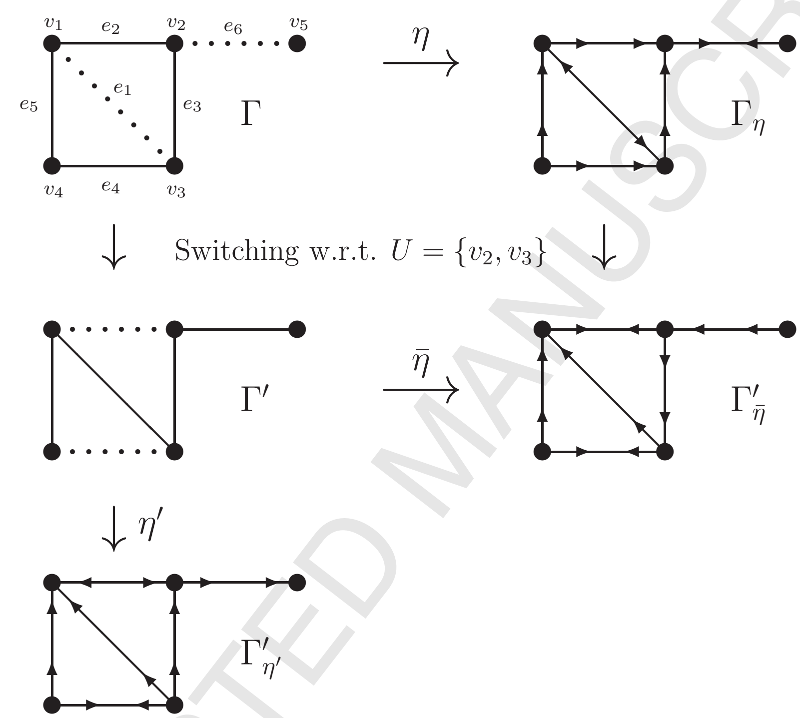

Fig. 4: Switching and double-inversion on equivalent bi-directed graphs.

From the above bi-directed graphs we obtain that

$$B_\eta = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B'_{\eta'} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By comparing $B(\Gamma'_{\eta'})$ and $B(\Gamma'_{\bar{\eta}})$, the doubly-inverted edges are $\bar{E} = \{e_2, e_3, e_6\}$, hence the "transition" matrices are

$$D_1 = \mathrm{diag}(1, -1, -1, 1, 1) \ \text{ and } \ D_2 = \mathrm{diag}(1, -1, -1, 1, 1, -1).$$

It is easy to verify that

$$B'_{\eta'} = D_1 B_\eta D_2.$$

□

Recall that two switching equivalent signed graphs do not necessarily produce signed line graphs or signed subdivision graphs with exactly the same signature. However, they produce switching equivalent compound signed graphs. The same holds for their induced (signed) subgraphs. We now survey the impact of switchings and double-inversions on the eigenspaces of the compound (signed) graphs.

Let us first discuss the effects of switching and double-inversion on the signed line graph. From the above discussion, let $\hat{B} = B_{\hat{\eta}}$ be the incidence matrix of a bi-directed graph $\hat{\Gamma} = \Gamma_{\hat{\eta}}$ arising from both of the above actions on $\Gamma = \Gamma_\eta$, i.e. from switching and double inversions. Then

$$\hat{B} = D_1 B D_2.$$

Recall that

$$B^\top B = 2I + A(\mathcal{L}(\Gamma)).$$

Therefore

$$B^\top B = D_2(D_2 B^\top D_1)(D_1 B D_2)D_2 = D_2 \hat{B}^\top \hat{B} D_2 = 2I + D_2 A(\mathcal{L}(\hat{\Gamma}))D_2,$$

and consequently

$$A(\mathcal{L}(\Gamma)) = D_2 A(\mathcal{L}(\hat{\Gamma}))D_2,$$

showing that the adjacency matrix of the line graph of a bi-directed graph is independent of the switching and only double inversion is relevant. Assume now that $\mathbf{x}$ is a λ-eigenvector for $A(\mathcal{L}(\Gamma))$. Then it immediately follows that

$$A(\mathcal{L}(\hat{\Gamma}))D_2\mathbf{x} = \lambda(D_2\mathbf{x}),$$

whence $\hat{\mathbf{x}} = D_2\mathbf{x}$ is a λ-eigenvector for $A(\mathcal{L}(\hat{\Gamma}))$ (clearly, it is a non-zero vector). We summarize the above fact in the following theorem.

Theorem 4.3. *Let $\mathcal{L}(\Gamma_\eta)$ and $\mathcal{L}(\Gamma'_{\eta'})$ be two signed line graphs obtained from switching equivalent bi-directed graphs. Let B (B') be the incidence matrix of Γ_η (resp., $\Gamma'_{\eta'}$) with $B' = D_1 B D_2$. If $\mathbf{x}$ is a λ-eigenvector of $A(\mathcal{L}(\Gamma_\eta))$, then $D_2\mathbf{x}$ is a λ-eigenvector for $A(\mathcal{L}(\Gamma'_{\eta'}))$.*

Finally, for the subdivision graph $\mathcal{S}(\Gamma)$, in view of (8) we just need to combine the switching and double-inversion for both the L-eigenspaces of $L(\Gamma)$ and the A-eigenspaces of $A(\mathcal{L}(\Gamma))$, as already discussed in Theorems 4.1 and 4.3. In particular we get that if Γ_η and $\Gamma'_{\eta'}$ are two bi-directed graphs, leading to switching equivalent signed graphs Γ and Γ' with incidence matrices $B = B(\Gamma_\eta)$ and $B' = B(\Gamma'_{\eta'})$ such that $B' = D_1 B D_2$, then

$$(D_1 \dot{+} D_2)A(\mathcal{S}(\Gamma_\eta))(D_1 \dot{+} D_2) = A(\mathcal{S}(\Gamma'_{\eta'})). \tag{9}$$

Therefore, we get the following result.

Theorem 4.4. *Let $\mathcal{S}(\Gamma_\eta)$ and $\mathcal{S}(\Gamma'_{\eta'})$ be two signed subdivision graphs obtained from switching equivalent bi-directed graphs. Let B (B') be the incidence matrix of Γ_η (resp., $\Gamma'_{\eta'}$) with $B' = D_1 B D_2$. Let $\mathbf{x} = \mathbf{y} \dot{+} \mathbf{z}$ be a λ-eigenvector of $A(\mathcal{S}(\Gamma_\eta))$, where $\mathbf{y}$ and $\mathbf{z}$ are the components corresponding to $V(\Gamma)$ and $E(\Gamma)$, respectively. Then $\hat{\mathbf{x}} = D_1\mathbf{y} \dot{+} D_2\mathbf{z}$ is a λ-eigenvector for $A(\mathcal{L}(\Gamma'_{\eta'}))$.*

Proof. Observe that from (9) we get

$$\begin{pmatrix} O & D_1 B D_2 \\ (D_1 B D_2)^\top & O \end{pmatrix} = \begin{pmatrix} O & B' \\ B'^\top & O \end{pmatrix}.$$

Let $\mathbf{x} = \mathbf{y} \dot{+} \mathbf{z}$ be a λ-eigenvector of $A(\mathcal{S}(\Gamma_\eta))$. From $A(\mathcal{S}(\Gamma_\eta))\mathbf{x} = \lambda\mathbf{x}$ we get

$$A(\mathcal{S}(\Gamma_\eta))\mathbf{x} = \begin{pmatrix} O & B \\ B^\top & O \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} B\mathbf{z} \\ B^\top \mathbf{y} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix},$$

which implies

$$\begin{cases} B\mathbf{z} = \lambda\mathbf{y}; \\ B^\top\mathbf{y} = \lambda\mathbf{z}. \end{cases}$$

Consider now $\hat{\mathbf{x}} = D_1\mathbf{y}\dot{+}D_2\mathbf{z}$. We have

$$A(\mathcal{S}(\Gamma'_{\eta'}))\hat{\mathbf{x}} = \begin{pmatrix} O & D_1BD_2 \\ (D_1BD_2)^\top & O \end{pmatrix}\begin{pmatrix} D_1\mathbf{y} \\ D_2\mathbf{z} \end{pmatrix} = \begin{pmatrix} D_1B\mathbf{z} \\ D_2B^\top\mathbf{y} \end{pmatrix} = \begin{pmatrix} D_1\lambda\mathbf{y} \\ D_2\lambda\mathbf{z} \end{pmatrix} = \lambda\hat{\mathbf{x}}.$$

Hence $\hat{\mathbf{x}}$ is a λ-eigenvector for $A(\mathcal{L}(\Gamma'_{\eta'}))$. □

To conclude, it is worth to observe that switching and double-inversions preserve the moduli of eigenvector entries.

5 Conclusion

In this paper we have generalized the main results from [13], which featured for unsigned graphs, to signed graphs. Most of these results are formally the same, but now proved with more work in view of some technical difficulties when passing from unsigned graphs to signed graphs.

The following issues now also deserve to be mentioned:

(i) We have not paid any attention regarding the properties of being "main" or "non-main" for some eigenvalues (as is discussed in [13]) since this is not so important for signed graphs. However, these can be also considered here especially for signed line graphs (for the eigenvalue -2) and signed subdivision graphs (for the eigenvalue 0: this is left for readers interested in such topic).

(ii) We have also restricted our considerations to situations in which our "basic" signed graph has no multiple edges (loops are not allowed, as well). Then some results can be generalized on the lines of the papers [1, 2, 3], so that the so-called generalized line graphs (whose root graphs can be interpreted as signed multigraphs) and their signed counterparts are, in addition, included in our considerations.

Acknowledgment

This paper is based on [13], which appeared in issue of "Quaderni di Matematica" dedicated to the memory of Lucia Gionfriddo. We also dedicate this paper to her memory.
The research of the first author was supported by the Project "SGTACSMC" of the University of Naples "Federico II", and by the University of Primorska within the operation n. 3330-14-500033 and by the grant "UP and the World" co-financed by the European Social Fund. The research of the second author was supported by the University of Malta through the project MATRP01: Graph Spectra and Electrical Conductivity in Nano-Structures. The research of the third author was supported by the Serbian Ministry of Science (project 174033), by GNSAGA-INdAM (Italy) and by the PRIN 2012 "Strutture Geometriche, Combinatoria e loro Applicazioni".

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