LINEAR ALGEBRA MAT3105

1 • 6 credits

• Textbooks:
Linear Algebra , R. Kaye & R. Wilson (Oxford Science Publications)
Linear Algebra with Applications , R. Nicholson (McGraw Hill) ISBN 0070880468
Linear Algebra and Matrix Theory, E.D. Nering (Wiley)

• Bibliography:
Topics in Algebra, I. N. Herstein (Wiley)

Linear Algebra, S. Lipschutz (Schaum)

Elementary Linear Algebra, Anton (Wiley)

Linear Algebra, Larry Smith (Springer)

Modern Algebra, Gardiner, (Wiley)

********* • Aim:
1. To study Vector Spaces and their wide application in other areas of mathematics and other sciences.

2. Use the software package Mathematica to work out algorithms involving matrices.

3. Emphasise the power of matrices.

4. Include computational and geometrical aspects.

2 LINEAR TRANSFORMATIONS

Definition: $V(F)$ is said to be
a vector space over a field F
if V is an Abelian group under $+$,
and under scalar multiplication V is
1. closed: $\alpha v \in V$,
2. distributive: $\alpha(v+w) = \alpha v + \alpha w$ and $(\alpha+\beta)v = \alpha v + \beta v$,
3. associative: $(\alpha\beta)v = (\alpha\beta)v$ and
4. $1v = v$.

Definition: $W \subset V(F)$ is said to be
a subspace over the field F
if $\forall w_1, w_2 \in W$ and $\forall \alpha, \beta \in F$
then $\alpha w_1 + \beta w_2 \in W$.

Lemma: Let U and W be subspaces of V. Then $U \cap W$ is a subspace of V.

Definition: $T : V \longmapsto W$ is said to be
a linear transformation
if $\forall u, v \in V$ and $\forall \alpha, \beta \in F$
then $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$.

Theorem:
$$\begin{aligned}\dim V &= \dim(\mathrm{Ker}(T)) + \dim\,(\mathrm{Im}(T)) \\ &= \text{nullity}+\text{rank}\end{aligned}$$

Definition: $\mathrm{Hom}(V, W)$ is the set of linear transformations from V to W.

Theorem: If $T : V \longrightarrow W$ with $\mathrm{Ker}(T) = K$
then $T(V) \simeq \frac{V}{K}$

Theorem: dimension $\frac{V}{W} = \dim(V)$ - $\dim\,(W)$

Theorem: If $S \subset V$, then
$\mathcal{L}(S)$ is a subspace of V.

$\mathcal{L}(\mathcal{L}(S)) = \mathcal{L}(S)$

$\mathcal{L}(S) + \mathcal{L}(T) = \mathcal{L}(S \cup T)$

Definition: $V(F)$ is said to be
finite dimensional if $\exists S$, finite, such that $\mathcal{L}(S) = V$.

Definition: $v_1, v_2, \ldots, v_n$ are said to be linearly dependent in V if $\exists$ scalars
$\alpha_1, \alpha_2, \ldots, \alpha_n$ in F (not all zero) s.t.
$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$.

- depends on the vectors and on F.

- eg $1, i$ in R and in C.

- A set of vectors containing the zero vector is linearly dependent.

Definition: A Basis for V is s.t.b. a linearly independent spanning set.
Dim(V) = number of elements in a basis for V.

Theorem: $v_1, v_2, \ldots, v_k$ are linearly dependent in $V \longrightarrow$ some v_i is a linear combination of predecessors.

Theorem: $v_1, v_2, \ldots, v_m$ are linearly independent in $V \longrightarrow m \leq n$

Theorem: Any two bases have same number of elements.

Theorem: $F^{(m)} \simeq F^{(n)} \Longleftrightarrow m = n$.
$V(F)$ of dimension $n \ \simeq F^{(n)}$

Theorem: A set of linearly independent vectors can be extended to a basis.

Theorem:
dimension subspace $W \leq$ dimension V

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3 Direct Sum

Definition: Let $V(F)$ be a vector space over F. Let $U,\ W$ be subspaces. The sum $U+W$ is $\{u+w : u \in U \text{ and } w \in W\}$.

Lemma: $U+W$ is a subspace of V.

Definition: Let V be a vector space over F. Let $U_1, \ldots, U_n$ be subspaces of V. V is said to be the internal direct sum of $U_1, U_2, \ldots U_n$ if $\forall v \in V,\ v = u_1 + u_2 + \ldots + u_n,\ u_i \in U_i$ uniquely.
Notation: $V = U_1 \bigoplus U_2 \bigoplus \ldots \bigoplus U_n$.

Theorem:
Direct Sum $U_1 \bigoplus U_2 \bigoplus \ldots \bigoplus U_n \simeq$
Set of n-tuples or external direct sum $U_1 \times U_2 \times \ldots \times U_n$.

Remark:
internal direct sum and external direct sum are indistinguishable.

Theorem: Let $A,\ B$ be subspaces of V. Then
$\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B)$

Theorem: Let $A,\ B$ be subspaces of V. Then
$V = A \bigoplus B \Longleftrightarrow V = A + B$ and $A \cap B = \{0\}$.

Corollary:
$V = A + B$ and $\dim(V) = \dim(A) + \dim(B) \Longleftrightarrow V = A \bigoplus B$.

Third Isomorphism Theorem: Let A, B be subspaces of V. Then

$\frac{A+B}{B} \simeq \frac{A}{A\cap B} V = A + B$ and $A \cap B = \{0\}$.

Theorem: $V = U_1 \bigoplus U_2 \bigoplus \ldots \bigoplus U_n$.
Then Basis $B_V = B_{W_1} \bigcup B_{W_2} \bigcup \ldots \bigcup B_{W_n}$.

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4 MATRICES

Theorem: Let V, W be vector spaces of dimension n, m. There is an isomorphism between $\mathrm{Hom}(V, W)$ and the set of $m \times n$ matrices.

Theorem: An automorphism on V has rank $\dim(V)$

Theorem: If a linear transformation on V has rank $\dim(V)$ then it is an automorphism.

Theorem: $\mathrm{rank}(A)$ is the maximum number of linearly independent columns of A.
- rank (A)=columnrank(A)
- Linear transformation T is non-singular if $\ker T = \{0\}$.
- T is non-singular $\Longleftrightarrow$ T is invertible

$\Longleftrightarrow$ T is 1-1
$\Longleftrightarrow$ T is onto
$\Longleftrightarrow \mathrm{Det}[T] \neq 0$

Theorem: $\mathrm{rank}(AB) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$.

Theorem: If A is non-singular, then rank(A^{-1}) =rank(A).

Theorem: rank of $m \times n$ matrix remains unchanged by multiplying by a non-singular matrix.

Theorem: Let $AX = B$ be a system of linear equations.
The following three statements are equivalent.

i) $AX = B$ has a solution.
ii) B is a linear combination of columns of A.
iii) The coefficient matrix A and the augmented matrix $[A|B]$ have the same rank

Corollary: dim(Im(A)) =dim(column space(A))
=rank(A).

Theorem: Let $AX = 0$ be a homogeneous system of linear equations where $X \in F^n$ and rank$(A) = r$.
Then dim(solution space)=n-r.

Definition: {symm} denotes the subspace of symmetric matrices.

{antisymm} denotes the subspace of antisymmetric matrices.

Theorem: Let V be the vector space of $n \times n$ matrices over F. Then $V =$\{symm\}$\bigoplus$\{antisymm\}

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5 Change of Basis

Definition: Let X be a vector whose components are in terms of an " old" basis $\{e_i\}$

and let X' be in terms of a "new" basis $\{f_i\}$, for F^n.
If $X = PX'$, the $n \times n$ matrix $P =_e P_f$ is s.t.b. the transition matrix from the "old" basis to the "new" basis.

Theorem: Let $v_e \in V$ be the column vector X relative to $\{e_i\}$. If $v_f \in V$ is the same vector but with column vector X' relative to $\{f_i\}$, then $X = PX'$ where the columns of P are the column vectors of $\{f_i\}$ in terms of $\{e_i\}$.

Theorem: P is non-singular.

Theorem: Let $T : U \longrightarrow V$ be a linear transformation represented by matrix ${}_vA_e$, relative to $\{e_i\}$ in domain and relative to $\{f_i\}$ in co-domain.
i)Relative to $\{f_i\}$ in domain ${}_vA_e$ changes to ${}_vA_f.{}_vA_e$;

ii) Relative to $\{w_j\}$ in codomain ${}_vA_e$ changes to ${}_wA_e = {}_wP_{vv}A_e$;

iii) Relative to $\{f_i\}$ in domain and $\{w_j\}$ in codomain ${}_vA_e$ changes to ${}_wA_f = {}_wP_{vv}A_{ee}P_f$.

Theorem: Let A be an $m \times n$ matrix of rank r. There exists non-singular matrices $P,\ Q$ such that $A' = Q^{-1}AP = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$,

- called Associate Normal Form.

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6 SIMILAR MATRICES

Definition: Let V be a vector space and let $T : V \longmapsto V$. Any two matrices representing the same linear transformation on V are s.t.b. similar.

Equivalently,

Definition: A and B are similar if
$\exists P$, P a non singular matrix,
such that $B = P^{-1}AP$.

Theorem: Similar matrices
i) have the same determinant
ii) have the same rank
iii) have the same eigenvalues and characteristic polynomial
iv) have the same minimal polynomial.
v) have the same trace.

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7 EIGENVALUES

Definition: Let A be a square matrix over a field F. If $\exists x \neq 0,\ x \in F^n$ and $\lambda \in F$, s.t.
$Ax = \lambda x$, then λ is an eigenvalue of A.

- For a particular λ,
$\{x : x \in F^n \& Ax = \lambda x\}$ is the eigenspace of λ.

- The eigenspace of λ is a subspace of V.

- A non-zero vector $x \in F^n$ in the eigenspace of λ is an eigenvector corresponding to λ.

Definition: v is a cyclic vector for T
if V is spanned by
$B = v, T(v), T^2(v), \ldots T^n(v)$.

Theorem: Every linear transformation has at least one eigenvalue over the closure of F.

Proof: $v, A(v), A^2(v), ..., A^n(v)$ are linearly dependent.
N.B. JNF

Lemma: If x is an eigenvector corresponding to λ,
then so is kx, $\forall k \in F, k \neq 0$.

- Eigensystem

If λ is an eigenvalue of A,
- then $f(\lambda)$ is an eigenvalue of $f(A)$,
- Also λ^r is an eigenvalue of A^r,
- and λ^{-1} is an eigenvalue of A^{-1}.

Definition: x is s.t.b. a normalized eigenvector in the $\lambda-$space of A if
i) $Ax = \lambda x$ for some $\lambda \in F$ and
ii) $||x|| = 1$.

Theorem: Let A be $n \times n$ matrix over F. $\lambda \in F$ is an eigenvalue of $A \iff A - \lambda I$ has a non-empty kernel.

Theorem: Let A be $n \times n$ matrix over F.
Eigenspace of λ is $\ker(A - \lambda I)$.

Lemma:
$\exists x,\ x \neq 0$, s.t. $Ax = \lambda x, \iff \mathrm{Det}(A - \lambda I_n) = 0$.

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8 Characteristic Polynomial

Definition: The characteristic polynomial of A
is $\mathrm{Det}(A - \lambda I_n)$.

Definition: $\mathrm{Det}(A - \lambda I_n) = 0$. is s.t.b. the characteristic equation $\phi(A, \lambda)$ of A.

- The characteristic equation of A
is of degree n.

Lemma: Let $\phi(A, \lambda) = \lambda^n + q_{n-1}\lambda^{n-1} + \ldots + q_1\lambda + q_0$ of A. The coefficients $(-1)^{n-i} q_i$ is the sum of those principal MINORS of A which have $n - i$ rows and columns.

Theorem: Let A be $n \times n$ matrix over F. $\phi(A, \lambda) = \lambda^n -$ (trace$A$)$\lambda^{n-1} + \ldots_(- 1)^n \mathrm{Det} A$

Corollary: The solutions of the characteristic equation are the eigenvalues of A.

Theorem: The characteristic equation is an invariant of A.

Equivalently: The characteristic polynomial of a linear transformation $T : V \longrightarrow V$ is independent of the matrix representing T; i.e. of the basis for V.

- The characteristic polynomial is well defined.

Corollary: A is singular $\Longleftrightarrow$ one of the eigenvalues of A is zero.

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9 Upper Triangular Matrices

Lemma: Let $f : V \longrightarrow V$ be a linear operator and nullity$(f) \geq 1$. Then there exists a basis B_V w.r.t. which matrix $[f]$ has the last row equal to zero.

Theorem: Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a linear operator. Then there exists an upper triangular matrix representing f w.r.t. some basis $B_{\mathbb{C}^n}$.

Theorem: The eigenvalues of an upper triangular matrix are the diagonal entries.

Lemma: Let $p,\ q$ be two polynomials. The $p(A)q(A) = q(A)p(A)$.

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10 The CAYLEY HAMILTON Theorem

Theorem If $A = (a_{ij})$ is an $n \times n$ upper triangular matrix and $\phi(A, \lambda)$
then $(A - a_{11})(A - a_{22}) \ldots (A - a_{nn})$ is the zero matrix.

Theorem There exists a polynomial that annihilates A.

Theorem If A is an $n \times n$ matrix and $\phi(A, \lambda)$ its characteristic polynomial Det$(\lambda I - A)$,
then $\phi(A, A)$ is the zero matrix.

Theorem:
i) If λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$.
ii) $\phi(A, A)(x) = 0,\ \forall x_i,$ where $\{x_i\}$ is a basis of eigenvectors of A.
iii) $\phi(A, A) = 0,$ the zero matrix.

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11 The MINIMUM POLYNOMIAL

Definition: The monic polynomial of lowest degree which A satisfies is s.t.b. the minimum polynomial of A.

Definition: Let $T \in \text{Hom}(V, V)$. A subspace U of V is s.t.b. T-invariant if $\forall u \in U,\ T(u) \in U$.

- T_U, the restriction of T to U is a linear transformation from U to itself.

Lemma: The minimum polynomial of T exists and is unique. This justifies the well definition of $m_T(x)$.

Lemma: The minimum polynomial of T divides any other polynomial satisfied by T.

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12 Eigenvectors

Lemma: Let v be an eigenvector of T in $\mathcal{E}(\lambda_0)$ relative to basis B. Then for $\lambda_i \neq \lambda_0,\ v$ is not an eigenvector of T in $\mathcal{E}(\lambda_i)$.

Theorem: Let $v_1, v_2, \ldots, v_r$ be eigenvectors of $T : V \longrightarrow V$ each corresponding to different eigenvalues in $\lambda_1, \lambda_2, \ldots, \lambda_r$.
Then $v_1, v_2, \ldots, v_r$ are linearly independent.

Theorem: The eigenspaces are mutually disjoint. V is the direct sum of the distinct eigenspaces.

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Definition: Let A be $n \times n$ and
λ_0 be eigenvalue of $T : V \longrightarrow V$.
The highest power of $\lambda - \lambda_0$ is said to be the
ALGEBRAIC MULTIPLICITY of λ_0.
The dimension of $\mathcal{E}(\lambda_0)$ is said to be the
GEOMETRIC MULTIPLICITY of λ_0.

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13 Diagonalizable Matrices

Definition: Let $T \in \mathrm{Hom}(V, V)$ have a
diagonal matrix representation. Then T is s.t.b. diagonalisable.

Theorem: Let V be a vector space
of dimension n.
Let T be a linear operator $T : V \longrightarrow V$
with distinct eigenvalues.
There exists an invertible matrix P s.t. $P^{-1}[T]_B P = D$

CRITERION I: Theorem: Let T be a linear operator
$T : V \longrightarrow V$. T is diagonalizable $\Longleftrightarrow$ there exists
an ordered basis for V consisting entirely of eigenvectors of T.

Theorem: T is diagonalizable $\Longleftrightarrow$
$[T]_C$ is diagonalizable for any ordered basis C for V.

CRITERION II: Theorem: Let T be a linear operator
$T : V \longrightarrow V$. T is diagonalizable $\Longleftrightarrow$ $m_T(x)$ is the product of linear factors
only.

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Lemma: Let $T \in \text{Hom}(V,V)$ have minimal polynomial $g_1(x)g_2(x)$ where $g_1,\ g_2$ are
coprime polynomials. Then $V = V_1 \oplus V_2$ where V_i is a non-zero $T-$invariant subspace.
Moreover the minimum polynomial of T_{V_i} is $g_i(x)$.

Theorem: $\text{Im}(g_1(A)) = \text{Ker}(g_2(A)) =$

Primary Decomposition Theorem: If $m_T(x)$ is the minimum polynomial of T and
$m_T(x) = [\phi_1(x)]^{\alpha_1}[\phi_2(x)]^{\alpha_2}\ldots[\phi_s(x)]^{\alpha_s}$, where $\forall i,\ [\phi_i(x)]$ is irreducible distinct polynomials and $\phi_i(x) \neq [\phi_j(x)],\ i \neq j$, then $V = V_1 \oplus V_2 \oplus \ldots \oplus V_s$.
$V_i =$Ker$((T-\lambda_i)^{\alpha_i})$

Lemma: $T(v_0) = \lambda v_0 \iff q(T)v_0 = q(\lambda)v_0$

Theorem: $\lambda \in F$ is an eigenvalue of $T \in$Hom(V,V)
$\iff \lambda$ is a root of the minimal polynomial.

- $m_T(x)$ and $\phi(T,\lambda)$ have the same linear factors.
They differ only in the multiplicity of the factors.

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14 Primary Decomposition Theorem I

Theorem: Let T be a linear operator $T : V \longrightarrow V$ with eigenspaces
$\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m$.
Then $V = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \ldots \oplus \mathcal{E}_m$.
Basis B for V is $B_1 \cup B_2 \cup \ldots \cup B_m$.
Dim$V$=Dim $\mathcal{E}_1$ + Dim $\mathcal{E}_2$ + $\ldots$+Dim $\mathcal{E}_m$.

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15 The JORDAN NORMAL Form

Definition: Given an equivalence relation defined on matrices, a normal form is a simple representative of an equivalence class of matrices.

Theorem: 1. Similarity is an equivalence relation
2. The Associate Normal form corresponding to a matrix is

$$\left(\begin{array}{c|c} \mathbf{I_r} & 0 \\ \hline 0 & 0 \end{array}\right)$$

The Hermite Normal form is $B = Q^{-1}A$

The Jordan Normal form is $P^{-1}AP$, where P's columns are eigenvectors of A or the basis vectors of $\mathbb{R}^n$ forming the transition matrix P,

is of the form
$$\left(\begin{array}{cccc|cccc|c}
\lambda_1 & 1 & & & & & & & \\
 & \lambda_1 & 1 & & & & & & \\
 & & \ddots & 1 & & & & & \\
 & & & \lambda_1 & & & & & \\ \hline
 & & & & \lambda_2 & 1 & & & \\
 & & & & & \lambda_2 & 1 & & \\
 & & & & & & \ddots & 1 & \\
 & & & & & & & \lambda_2 & \\ \hline
 & & & & & & & & \ddots
\end{array}\right)$$

Lemma: The minimum polynomial of Jordan Normal Block

$$\begin{pmatrix} \lambda_1 & \text{a} & 0 \\ 0 & \lambda_1 & \text{a} \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

is $(\lambda - \lambda_1)^3 = 0$

Definition: If W is a subspace of R^n, then $\mathrm{Proj}_{|W}$:$\mathrm{R}^n \longrightarrow W$ is defined by
$\mathrm{Proj}_{|W}(w) = w \; \forall w \in W$
$\mathrm{Proj}_{|W}(v) = 0 \; \forall v \in W^{\perp}$

Lemma: JNB $\Longleftrightarrow$
the minimum polynomial is characteristic polynomial.

Lemma: A with $\phi(A) = (x-c)^n$ diagonalizable $\Longleftrightarrow$
the minimum polynomial is $(x-c)$ $\Longleftrightarrow$
$A = cI_n$.

Definition: Let T be a linear operator
$T : V \longrightarrow V$.
Let v be an eigenvector of T in c-space. Then $v_1, v_2, \ldots, v_n$ are s.t.b.
GENERALIZED EIGENVECTORS if

$$\begin{aligned}
&T(v_1) = cv_1\\
&T(v_2) = cv_2 + v_1\\
&T(v_3) = cv_3 + v_2\\
&\vdots\\
&T(v_n) = cv_n + v_{n-1}.
\end{aligned}$$

Lemma: The generalized eigenvectors form a basis for V
and T has matrix as JNF iff $\phi(T)$ has linear factors only.

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16 LINEAR FUNCTIONALS

THE DUAL SPACE

Definition: Let $V(F)$ be a vector space over a field F.
1. An element of $\mathrm{Hom}(V, F)$
(a linear transformation from V to F)
is s.t.b. a linear functional.

2. $\mathrm{Hom}(V, F)$ is s.t.b. the dual space of V;
- denoted by $\widehat{V}$.

- A linear functional is represented by a $1 \times n$ matrix.

Theorem:
Let V, W have dimension m, n over F.
Then $\dim(\ \mathrm{Hom}(V, W)) = mn$.
- $\dim(\ \mathrm{Hom}(V, F)) = m$.

Corollary: dim $(\widehat{V})$ =dim(V).

Theorem:$\widehat{V} \simeq V$.

Aim: To develop invariant proofs:
- proofs which are basis-free.

Definition:
Let $v_1, v_2, \ldots, v_n$ be a basis for $V(F)$.
Then $\widehat{v_1}, \widehat{v_2}, \ldots, \widehat{v_n}$
where $\widehat{v_i}(v_j) = \delta_{ij}$(Kronecker's delta)
for $1 \leq i, j \leq n$,
is a basis for $\widehat{V}$ and is s.t.b. the dual basis.

Theorem:
Let $v_1, v_2, \ldots, v_n$ be a basis for $V(F)$.
If $a_1v_1 + a_2v_2 + \ldots + a_nv_n$ is
the unique representation of v, $(a_i \in F)$,
then $a_i = \widehat{v_i}(v)$.

Theorem:
Let $B = v_1, v_2, \ldots, v_n$ be a basis for $V(F)$ and let the dual basis be $\widehat{B} = \widehat{v_1}, \widehat{v_2}, \ldots, \widehat{v_n}$. If $B' = \{v'_1, v'_2, \ldots, v'_n\}$ is a new basis for V and P is the transition matrix from B to B'
then the matrix of transition from the old dual basis $\widehat{B}$ to the new basis $\widehat{B'}$ is $(P^{-1})^T$.

Theorem:
Let $w \in V$. If $\widetilde{w} : V \longmapsto \mathbb{R}$ is defined by $\widetilde{w}(\widehat{v}) = \widehat{v}(w)$,
then $\widetilde{w}$ is a linear functional on V.

Theorem: $\widetilde{V} \simeq \widehat{\widehat{V}}$

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17 ANNIHILATORS

Definition: Let $S \subseteq V$. The subset $S^{\perp}$ of $\widehat{V}$ defined by
$S^{\perp} = \{\widehat{v} \in \widehat{V} : \widehat{v}(v) = 0 \ \forall v \in S\}$
is s.t.b. the annihilator of S.

Definition: Let $w \in W$ and $\widehat{v} \in \widehat{V}$ s.t.
$\widehat{v}(w) = 0$.
Then $\widehat{v}$ and w are s.t.b. orthogonal.

- Not perpendicular.

Theorem: Let $S \subseteq V$.
Then $S^{\perp}$ is a subspace of $\widehat{V}$.

Theorem: Let $S \subseteq V$. Then $S^{\perp} =< S >^{\perp}$.

Theorem: Let S be a subspace of V.
Then dim(S) +dim($S^{\perp}$)=dim(V).

Theorem: Let S be a subspace of the finite dimensional V.
Then $(S^{\perp^{\perp}}) = S$.

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18 The Transpose

Definition: Let $T : V \longmapsto U$ be a linear transformation over F and let $\phi \in \widehat{V}$, $\phi : U \longrightarrow F$ be a linear functional.
The mapping $T^t : \widehat{U} \longrightarrow \widehat{V}$ defined by $T^t : (\phi) = \phi \circ T$ is s.t.b. the Transpose of T.

Theorem: T^t is a linear transformation.

Theorem: ker(T^t) is the annihilator of Im(T).

Corollary: rank T=rank T^t.

Definition: The transpose (T^t) of an $m \times n$ matrix T is the $n \times m$ matrix where the ith row of (T^t) is the ith column of T.

Corollary: Row Rank=Column Rank.

Theorem: $(A^{-1})^t = (A^t)^{-1}$

Theorem: If A is matrix representing
$T : V \longrightarrow U$
relative to bases B_U and B_V
then the matrix representing
$T^t : \hat{V} \longrightarrow \hat{U}$
relative to dual bases B_U^* and B_V^*
is A^t.

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19 INNER PRODUCT SPACES

Definition: The vector space $V(F)$ is s.t.b. an Inner Product Space if

$\forall u,\ v \in V$, the inner product $< u, v >$ is defined s.t.

i) $< u, v >= \overline{< v, u >}$

ii) $< u, u > \geq 0$ & $< u, u >= 0 \Longleftrightarrow u = 0$

iii) $\forall u,\ v,\ w \in V,\ \forall \alpha, \beta \in F$,
$< \alpha u + \beta v, w >= \alpha < u, w > + \beta < v, w >$

Definition: $f : V \times V \longrightarrow F$
defined by $f :< v, w > \mapsto < v, w >$
satisfying i) ii) & iii)
is s.t.b. an inner product.

Definition: In V, $||v||$ is s.t.b.
the norm of v, defined as $\sqrt{[v, v]}$.

Lemma: $< \alpha u + \beta v, \alpha u + \beta v >$
$= \alpha\overline{\alpha} < u, u > + \alpha\overline{\beta} < u, v > + \overline{\alpha}\beta < v, u > + \beta\overline{\beta} < v, v >$

Corollary: i) $||v|| \geq 0$ (non-negative)
ii) $||v|| = 0 \longleftrightarrow v = 0$ (zero)
iii) $||u + v|| \leq ||u|| + ||v||$ (triangle inequality)

Definition: The vectors $u,\ v$ are s.t.b. orthogonal to each other if $< u, v >= 0$.

Definition: If W is a subspace of V, the orthogonal complement $W^{\perp}$ of W is
$W^{\perp} = \{x \in V :< x, w >= 0\ \forall w \in W\}$.

Lemma: $W^{\perp}$ is a subspace of V.

Lemma: $W^{\perp} \cap W = \{0\}$.

Definition: The vectors $v_1,\ v_2,\ \ldots\ v_n$ are s.t.b. orthonormal to each other if
i) $< v_i, v_j >= 0\ i \neq j$.
ii) Each v_i is of length 1; i.e. $< v_i, v_i >= 1\ \forall i$.

Theorem: The set of vectors $\{v_1,\ v_2,\ \ldots\ v_n\}$ is orthonormal $\Longrightarrow$ $\{v_1,\ v_2,\ \ldots\ v_n\}$ is linearly independent.

Corollary: If the set of vectors $\{v_1,\ v_2,\ \ldots\ v_n\}$ is orthonormal, then $\alpha_i =< w, v_i >$

GRAM SCHMIDT ORTHOGONALIZATION PROCESS

Lemma: Let $\{v_1,\ v_2,\ \ldots\ v_n\}$ be an orthonormal set in $V(F)$ and let $w \in V$ be l.i.of $\{v_i\}$.
Then $u = w- < w, v_1 > v_1- < w, v_2 > v_2 - \ldots - < w, v_n > v_n$ is orthogonal to each of $v_1,\ v_2,\ \ldots\ v_n$.

Theorem: Let V be a finite dimensional inner product space. Then V has an orthonormal basis.

Theorem: If W is a subspace of a finite dimensional inner product space, then
i) $V = W \oplus W^{\perp}$.
ii) $(W^{\perp})^{\perp} = W$.

Definition: If $\{e_1,\ e_2,\ \ldots\ e_n\}$ form an orthonormal basis and $v = \sum \alpha_i e_i$, then the Fourier Coefficients of v are $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\alpha_i =< v, e_i >$.

Definition: An orthogonal linear transformation T of $V_n(\mathrm{R})$ to itself is one for which $||Tx|| = x,\ \forall x \in V_n(\mathrm{R})$.

- Such a transformation preserves length.

Theorem: Let A be a matrix.
$||Ax|| = x,\ \forall x \in V_n(\mathrm{R}) \Longleftrightarrow A^{-1} = A^T$.

Definition: A square matrix A is orthogonal if $A^{-1} = A^T$.

Lemma: The only symmetric matrix S satisfying $x^T S x = x^T x$ is I.

Theorem: A is orthogonal
$\Longleftrightarrow < Ax, Ay >=< x, y >,\ \ \forall x, y \in V_n(\mathrm{R})$.

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20 Related Topics

PROJECT: Main Eigenvalues Orthogonal eigenvectors
Power of Matrices Walks in graphs

Definition: The Companion Matrix
of monic polynomial

$f(x) = x^n + a_{n-1}x^{n-1} + \dots a_0$

$$\text{is } Q = \begin{pmatrix} 0 & 0 & \dots & & & & & & -a_0 \\ 1 & 0 & & \dots & & & & & -a_1 \\ 0 & 1 & 0 & & & \dots & & & -a_2 \\ & & & \ddots & \ddots & \ddots & & & \vdots \\ & & & & & & 1 & 0 & -a_{n-2} \\ & & & & & & 0 & 1 & -a_{n-1} \end{pmatrix}$$

<u>Theorem:</u> The minimal polynomial of a
companion matrix= its characteristic polynomial

<u>Theorem:</u> The rank of Q is at least $n-1$.

<u>Theorem:</u> The JNF of Q has JN blocks of distinct eigenvalues along the diagonal.

<u>Lemma:</u>(i) B is an ordered basis for V.
(ii) $[T]_B$ is companion matrix for $\phi(T)$.
(iii) $m_T(x) = \phi(T, x)$.

<u>Theorem:</u> The minimal polynomial of a
diagonal partitioned matrix
is the l.c.m. of the minimal polynomials
of the diagonal blocks in JNF.

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21 THE ALGEBRA

of

LINEAR TRANSFORMATIONS

Definition: A ring R
- is an Abelian gp under $+$,
- is closed, (associative), and distributive over $+$.

Definition: An associative ring R is called an ALGEBRA over a field F
- if R is a vector space over F
- such that $\forall a, b \in R,\ \alpha \in F$,

$\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Theorem: Hom(V, V) is an algebra over F.

Analogue of Cayley's Theorem for groups
:$\forall G, \exists$ subgroups $\{S\}$ of S_n with $S \simeq G$.

Theorem: If A is an algebra with identity element I over F,
then A is isomorphic to a subalgebra of Hom (V, V) for some vector space V over F.

Definition: Let A be an algebra with identity element e over F. Then $a \in A$ is said to satisfy a polynomial $\alpha_0 + \alpha_1 x + \ldots \alpha_n x^n \in F[x]$ if $\alpha_0 e + \alpha_1 a + \ldots \alpha_n a^n = 0$.

Lemma: Let A be an algebra with identity element e over F and let $\dim(A) = m$.
Then $\forall a \in A$, a satisfies some polynomial $p(x) \in F[x]$ s.t. $\deg(p) \leq m$.

Corollary: If V is an n-dimensional vector space over F, then $\forall\ T \in$hom(V, V),

T satisfies a non-trivial polynomial
$q(x) \in F[x]$ such that $\deg(q)(x) \leq n^2$.

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22 System of Linear Equations

An Application

Theorem: The set of solutions of $AX = B$ is $\ker(A) + x_0$ where $Ax_0 = B$.
It is a coset of $\ker(A)$ not a subspace of R.

Definition: $[A|B]$ is the augmented matrix of A.

Theorem: $AX = B$ has a solution
$\Longleftrightarrow$ rank(A) =rank$([A|B])$.

Corollary: Let rank$(A) = r$ and A be $n \times n$.
$AX = B$ has a solution $\Longrightarrow$ all solutions $\{x_i\}$ can be expresed in terms of $n - r$ independent parameters.

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23 SELF ADJOINT OPERATOR

Definition: $T^H = T^*$ is $\overline{T^t}$.

Definition: T is said to be a
SELF ADJOINT OPERATOR
if $T^H = T$.
Over $\mathbb{C}$, T is s.t.b. HERMITIAN.
Over $\mathbb{R}$, $T^t = T$ is s.t.b. SYMMETRIC.

Lemma: Let T is self adjoint $\Longleftrightarrow$
$< T(v), w >=< v, T(w) >$.

Theorem: Let V be a vector space
of dimension n.
Let T be self adjoint operator $T : V \longrightarrow V$.
Then for distinct eigenvalues, eigenvectors are orthogonal.

Theorem: Let T be self adjoint linear operator. Then T has real eigenvalues.
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24 REAL SYMMETRIC MATRICES

Definition:
- A is a real symmetric matrix if $A = A^T$.
- A is a Hermitian matrix if $A = \overline{A^T}$ denoted by A^*.

Theorem: Let A be
a real symmetric (Hermitian) matrix. Then A has real eigenvalues.

Theorem: Let A be
a real symmetric (Hermitian) matrix
having distinct eigenvalues k_1, k_2.
Let x_1, x_2 be the corresponding eigenvectors. Then x_1, x_2 are orthogonal.

Theorem: Let A be a real symmetric (Hermitian) matrix. Then to every eigenvalue of multiplicity η there corresponds η linearly independent eigenvectors.

Theorem: Let A be a $(n \times n)$ real symmetric (Hermitian) matrix. Then A has n linearly independent eigenvectors. Thus A is
diagonalisable.

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25 SPECTRAL DECOMPOSITION THEOREM

Lemma: S is self adjoint $\Longrightarrow$
there exists a real orthogonal matrix U s.t.
Diag$=U^tSU$

Theorem: Let V be a vector space
of dimension n.
Let T be self adjoint operator $T : V \longrightarrow V$.
Then for distinct eigenvalues $\mu_1, \mu_2, \ldots \mu_s$,
$T = \mu_1 P_1 + \mu_2 P_2 + \ldots \mu_s P_s$
where $P_i = Proj_{W_i}$

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26 HERMITE NORMAL FORM

Definition: An $m \times n$ matrix of rank r is s.t.b. in Hermite Normal Form if

i) it is in echelon form and the first non-zero element (the pivot) in each of the first r rows is 1;

ii) the elements above and below the pivot 1_i are 0;

iii) Each entry of the rows $R_{r+1}, R_{r+2}, \ldots R_n$ is 0.

Remark: The H.N.F. is obtained by
applying elementary row operations,
i.e. by a change of codomain basis.

Theorem: Uniqueness of H.N.F.
If there exist non-singular matrices $Q_1,\ Q_2$ s.t. ${Q_1}^{-1}A$ and $Q_2^{-1}A$ are both in H.N.F., then ${Q_1}^{-1}A = Q_2^{-1}A$.

Theorem: Row Rank=Column Rank.
- =Row Rank of Transpose

Theorem: A non-singular matrix A is the product of elementary matrices.

Corollary: The H.N.F. of a non-singular matrix is I.

Theorem: $[A, I] \overset{row}{\equiv} [H.N.F., Q^{-1}],\ Q^{-1}A = A'$.

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Quadratic forms
Bilinear Forms
Definite Forms
Sylvester's Law of inertia