Systems of Equations

1. Use Gauss-Jordan row reduction to solve the following systems of equations:

   (i) $2x + y = 7$
   $8x - y = 3$

   (ii) $4x - 3y - 2z = 4$
   $3x + y - z = 8$
   $2x - y + 3z = -1$

   (iii) $x + 4y - z = 7$
   $2y + z = 4$
   $x + 6y + z = 13$

   (iv) $2y - z = 1$
   $x + y + 2z = 5$
   $x + 3y + z = 6$

   (v) $x - y = 4$
   $x + y + z = 8$
   $x + z = 1$

2. Determine a unique solution for $\mathbf{Ax} = \mathbf{y}$ in terms of a, b and c, in each of the following examples:

   (a) $\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 3 \\ 2 & a & b \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 0 \\ c \end{pmatrix}$.

Please turn over.

(b) $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 0 & 2 \\ 6 & a & b \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 5 \\ 8 \\ c \end{pmatrix}$.

(c) $\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 3 \\ 2 & a & b \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 0 \\ c \end{pmatrix}$.

(d) $\mathbf{A}(b) = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & b & b+2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 5 \\ 7 \\ c \end{pmatrix}$.

3. Determine also the conditions on a, b and c for the above systems of equations $\mathbf{Ax} = \mathbf{y}$ to have:

   (i) a unique solution;

   (ii) infinitely many solutions;

   (iii) inconsistent solutions;

Prof I. Sciriha

Determinants

1. Evaluate the determinants of the following matrices:
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 3 & 2 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 3 & 2 & -5 \\ 1 & 4 & 8 \\ 5 & 10 & 11 \end{pmatrix}.$$

2. Let the rows of the square matrix A be R_1, R_2 and R_3.

   (i) Determine E if $B := EA = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$.

   (ii) Use $\det(XY) = \det X \det Y$ to show that $\det B = -\det A$.

   (iii) Why is $\det \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = 0$?

3. Van der Monde's determinant is $f(a, b, c) := \det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$.

   (i) Show that $f(a, a, c) = 0$.

   (ii) Hence or otherwise show that $f(a, b, c) = k(a - b)(b - c)(c - a)$ and determine k.

4. Show that $\det \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \\ \sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta \end{pmatrix} = 1$

5. Show that $\det \begin{pmatrix} x & x+y & x-y \\ y & y+z & y-z \\ z & z+x & z-x \end{pmatrix} = 0$.

Please turn over.

6. Show that $\begin{vmatrix} \cos 2x & \cos 2y & \cos 2z \\ \sin 2x & \sin 2y & \sin 2z \\ 1 & 1 & 1 \end{vmatrix} = k \sin(x-y)\sin(y-z)\sin(z-x)$.
   Determine k.

Prof I. Sciriha

Matrices - Multiplication

1. Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 0 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 0 & 1 \end{pmatrix}$

   Determine the rank of **5A, A + B, 3A - 2B, BA,** $\mathbf{AA}^t$, $\mathbf{A}^t\mathbf{A}$, where possible.

2. Let **A** be a 2×4 matrix and let the order of **B** be 4×3.

   Determine the number of routes of $\mathbf{AA}^t, \mathbf{A}^t\mathbf{A}, \mathbf{BB}^t, \mathbf{B}^t\mathbf{B}, \mathbf{AB}$.

3. Solve for matrix **X** if $2\mathbf{A} + 3\mathbf{B} + \mathbf{X} = \mathbf{0}$.

4. Work out $\begin{pmatrix} 4 & -3 & 8 \\ 2 & 1 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

   and $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 & 8 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

5. Show that $\mathbf{AA}^t$ is symmetric.

6. If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ work out the rank of $\mathbf{xx}^t$ and $\mathbf{x}^t\mathbf{x}$.

Inverse of Matrices

1. Find the inverse of the following matrices:

$$\begin{pmatrix} 4 & -2 \\ 4 & -8 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

2. Determine the inverse of the matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

3. Show that $(\mathbf{I}+\mathbf{X})^{-1} = \mathbf{I} - \mathbf{X} + \mathbf{X}^2 - \mathbf{X}^3 + \ldots$ for a $n \times n$ matrix $\mathbf{X}$.

   Determine $(\mathbf{I}+\mathbf{X})^{-1}$ if $\mathbf{X} = \begin{pmatrix} 0 & .01 & .01 \\ .01 & 0 & .01 \\ .01 & .01 & 0 \end{pmatrix}$.

4. If $\mathbf{AB} = \mathbf{I}$, show that $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order.

5. Solve the following equations

$$\begin{aligned} 2x + 3z &= 8 \\ 2x + 2y - z &= 8 \\ z + y &= 8, \end{aligned}$$

   using

   i) the Gauss-Jordan Row Reduction;

   ii) the inverse of a matrix;

   iii) Cramer's rule.

6. Show that if A is a square matrix, then $(A^t)^{-1} = (A^{-1})^t$.

Eigenvalues of Matrices

1. a. Determine the eigenvalues of

$$\mathbf{A} := \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \ \mathbf{B} := \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{C} := \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

   b. Which matrices are diagonalizable?

   c. Express $\mathbf{A}$ as $\mathbf{P}^{-1}\mathbf{D}\mathbf{P}$.

   d. If $(\mathbf{A}+\mathbf{I}) = \mathbf{P}^{-1}\mathbf{E}\mathbf{P}$ determine $\mathbf{E}$.

   e. If $\mathbf{A}^5 = \mathbf{P}^{-1}\mathbf{F}\mathbf{P}\mathbf{P}$, determine $\mathbf{F}$.

   f. If $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, determine the product $\mathbf{CP}$ and deduce the eigensystem of $\mathbf{C}$.

2. a. Determine the eigenvectors $\mathbf{y}_1$ and $\mathbf{y}_2$ of unit length of

$$\mathbf{M} := \begin{pmatrix} 25 & -7 \\ -7 & 25 \end{pmatrix}$$

   b. Show that if $\mathbf{Q} = (\mathbf{y}_1\mathbf{y}_2)$, then $\mathbf{Q}\mathbf{Q}^t = \mathbf{I}$.

   c. If $f(x,y) = 25x^2 + 25y^2 - 28xy$, show $(x\ y)\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = f(x,y)$.

   d. Express the ellipse $f(x,y) = 576$ as $g(U,V) = \frac{U^2}{A^2} + \frac{V^2}{B^2} = 1$.

   e. Sketch $g(U,V) = 1$.
