On the Rank of Graphs[1]

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Abstract

The properties of singular graphs obtained in a previous paper "On the construction of graphs of nullity one", lead to the characterization of graphs of small rank. The minimal configurations that are contained in singular graphs were identified as "grown" from certain cores. A core of a singular graph G is a subgraph induced by the vertices corresponding to the non-zero components of an eigenvector in the nullspace of the adjacency matrix of G. In this paper it is shown that an arbitrary singular graph Z without isolated vertices has core-sizes corresponding to a minimal basis for the nullspace of $\mathbf{A}$ bounded below by 2 and above by $r(Z) + 1$, $r(Z)$ being the rank of Z. For $r(Z) \geq 6$, these bounds are sharp.

Keywords: *rank, nullspace, core, singularity, core-width, nut graphs.*

1 Introduction

All the graphs we consider are **simple.** The **adjacency matrix** $\mathbf{A}(G)$ or $\mathbf{A}$ of a graph G with vertex set $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\}$, is often represented by $(R_1, R_2, \ldots, R_n)^T$ where R_i is the ith row vector of $\mathbf{A}$ corresponding to vertex v_i.

A graph is said to be **singular** if its adjacency matrix $\mathbf{A}$ is a singular matrix; then $\{\mathbf{v_0} : \mathbf{A}\mathbf{v_0} = \mathbf{0}\}$ is the nullspace of $\mathbf{A}$ denoted by $\mathcal{E}_0(\mathbf{A})$. The **nullity** of G, denoted by $\eta(G)$, is the dimension of $\mathcal{E}_0(A)$, which is the multiplicity of the zero eigenvalue of $\mathbf{A}$, since $\mathbf{A}$ is symmetric. The **rank** of a graph G, denoted by $r(G)$, is the rank of its adjacency matrix $\mathbf{A}$ which is $n(G) - \eta(G)$ where $n(G)$ denotes the order of G.

[1] SCIRIHA, I. On the Rank of Graphs, in ed. Y.Alavi, D.R.Lick and A.Schwenk, Combinatorics, Graph Theory and Algorithms, (2), Michigan, (1999) 769-778.

2 Core Space

Definition: *Let* $\mathbf{u} \in \mathbb{R}^n$. *Then the* **weight** *of* $\mathbf{u}$ *denoted by* $wt(\mathbf{u})$ *is the number of non-zero entries of* $\mathbf{u}$.

Definition: *A* **kernel eigenvector** $\mathbf{v_0}$ *of a singular graph with adjacency matrix* $\mathbf{A}$, *is a non-zero vector in the nullspace* $\mathcal{E}_0(A)$.

Definition: *If* G *is a singular graph, with adjacency matrix* $\mathbf{A}$ *and* $\mathbf{v_0} \neq \mathbf{0}$ *is a vector in* $\mathcal{E}_0(A)$, *then the subgraph of* G *induced by the vertices corresponding to the non-zero components of* $\mathbf{v_0}$, *denoted by* $\chi_{\mathbf{v_0}}$, *is said to be the* **core** *(w.r.t* $\mathbf{v_0}$*). The number of vertices of the core is called its* **core-order**. *The set of vertices* $\mathcal{V}(G\backslash\chi)$ *is called the* **periphery** *of* G
(w.r.t $\mathbf{v_0}$*) and is denoted by* $\mathcal{P}$.

Remark: The set of cores corresponding to the set of vectors in $\mathcal{E}_0(A)$ is called the **core-space** $C_0(G)$.

In [1], M. Brown *et al* defined the **graph singularity** $\kappa(G) = \kappa$ of a singular graph G as the least core-order in the graph. In [5], the structure of singular graphs having one core of core-order up to 5 was investigated. In [6], the concept of a core-space and of a minimal basis, in which the core-order sequence is unique, was discussed. The first term of the sequence is κ and its last term is defined as the core-width τ. These terms will be defined formally in the next section. Graphs can be classified according to τ and then more finely according to the corresponding core.

In this paper we determine bounds on the core orders for an arbitrary singular graph Z without isolated vertices. We show that, for $r(Z) \geq 6$, these bounds are sharp. Moreover, we classify graphs according to their rank and extend the list of minimal configurations given in [5] to cover all those of rank up to 6.

3 Minimal Basis for the core-space

Definition: *Let* $B = (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u}_\eta)$ *be a basis for* $\mathcal{E}_0(A)$ *where* $\mathbf{A}$ *is* *the adjacency matrix of a singular graph G. The sequence of cores $B' = (\chi_{\mathbf{u_1}},\ \chi_{\mathbf{u_2}},\ \ldots, \chi_{\mathbf{u}_\eta})$ is called a* **core basis** *for G.*

The convention adopted will be to write **an ordered basis** such that the weights of its vectors are in **non-decreasing order.**

Definition: *Let B be an ordered basis $(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m})$ for a subspace of $\mathbb{R}^n$ of dimension m. The sequence of weights $t_1, t_2, \ldots, t_m$ of the vectors in B is said to be the* **weight-sequence** *of B.*

Definition: *Let G be a graph with adjacency matrix $\mathbf{A}$. If the weight-sequences of all bases for $\mathcal{E}_0(A)$ are in ascending lexicographic order then a basis $B = (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u}_\eta)$ with a minimal weight-sequence $S = (t_1, t_2, \ldots, t_\eta)$ is called a* **minimal basis** *for $\mathcal{E}_0(A)$. The corresponding minimal core basis for $C_0(G)$ is $B' = (\chi_{\mathbf{u_1}},\ \chi_{\mathbf{u_2}},\ \ldots, \chi_{\mathbf{u}_\eta})$. The weight-sequence S is called* **the core-order sequence** *of G. The first term of the sequence is defined as* **the singularity** *$\kappa(G)$, (the order of $\chi_{\mathbf{u_1}}$) and the last term as the* **core-width** *$\tau(G)$, (the order of $\chi_{\mathbf{u}_\eta}$).*

Definition: *Let G be a graph with core space $C_0(G)$. A core of largest order in a minimal basis for $C_0(G)$ is called* **a min-max core***.*

The following theorem is proved in [6].

Theorem 1: *Let G be a graph with adjacency matrix $\mathbf{A}$. Let $B_1 = (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u}_\eta)$ be a minimal basis for $\mathcal{E}_0(A)$ and $B_2 = (\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w}_\eta)$ be another ordered basis for $\mathcal{E}_0(A)$ with weight-sequences $t_1, t_2, \ldots, t_\eta$ and $s_1, s_2, \ldots, s_\eta$ respectively. Then $t_i \leq s_i, \forall i$.*

This theorem shows that the core-order sequence which depends on a minimal core basis is unique and thus it is well-defined.

The graph Y in Fig. 1 has order 8 and nullity 4. A minimal basis for the core-space $C_0(G)$ relative to the labelling in the diagram is given by $R_1 + R_2 = R_3 + R_4 = R_5 + R_6$; $R_2 = R_7$; $R_3 = R_8$. However $R_2 - R_7 + R_3 - R_8 = 0$ corresponds to a core of order $\tau(G)$ which if included in a basis would not give a minimal core-order sequence and is therefore not a min-max core.

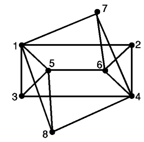

Figure 1: *The graph Y.*

4 Minimal Configuration

Definition: *A connected graph $\mathcal{T}$ is an* **extension** *of a graph G if G is an induced subgraph of $\mathcal{T}$ such that*
1. $n(G) < n(\mathcal{T})$
2. $< \mathcal{V}(T) - \mathcal{V}(G) >$ is null.
$\mathcal{T}$ is also said to be extended from G.

Definition: *A singular graph Γ of order $n \geq 3$, having a core χ and periphery $\mathcal{P} := \mathcal{V}(\Gamma) - \mathcal{V}(\chi)$ is a* **minimal configuration,** *of* **core-number** *$n(\chi)$, if the following conditions are satisfied:*
(i) $\eta(\Gamma) = 1$,
(ii) $\mathcal{P} = \varphi$ or $\mathcal{P}$ induces a null graph,
(iii) and in the case when $\mathcal{P} \neq \varphi$, the deletion of $v \in \mathcal{P}$ increases the nullity of Γ.

Definition: *A minimal configuration is called a* **nut graph** *if $\mathcal{P} = \varphi$.*

Since $\mathcal{P} = \varphi$ or $\mathcal{P}$ induces a null graph, it follows that a minimal configuration Γ is connected [7]. In a nut graph each entry of a kernel eigenvector is non-zero.

Lemma 1: *Let Γ be a minimal configuration of core-number p. Then $\kappa = \tau = p$.*

Proof: The core-order sequence S of the core-basis for Γ, is $\{p\}$. Since κ is the first term and τ the last term of S, the result follows.\\

In related work [3, 7], it is shown that for $r(G) \geq 6$, the upperbound $r(G)+1$ for $\tau(G)$ can be attained by nut graphs.

Lemma 2: *Let G be a nut graph of order n.*
Then $\kappa(G) = \tau(G) = r(G) + 1 = n$.

Proof: Since G is a minimal configuation, it has only one core χ. Since $\mathcal{P} = \phi$, then $\chi = G$. Thus $\kappa(G) = \tau(G) = n(\chi) = n(G) = n$.\\

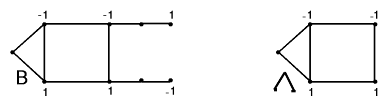

Figure 2: *Minimal Configurations*

Thus B and Λ, in Fig. 2, are two minimal configurations with cores C_4 and $C_4 \dot{+} \overline{K_2}$ respectively and kernel eigenvectors $(1, 1, -1, -1, 0)^t$ and $\mathbf{v_0} = (1, -1, -1, 1, 1, -1, 0, 0, 0)^t$ respectively. M. Ellingham defines a **basic subgraph** of a graph G as the subgraph induced by the vertices corresponding to a set of $r(G)$ linearly independent row vectors of $\mathbf{A}(G)$ [2]. It follows that a basic subgraph is non-singular. In [4] it is shown that a diagonal entry A_{ii} of the adjoint of the adjacency matrix of a graph of nullity one, corresponding to a vertex v_i of a core, is non-zero and that that corresponding to a vertex of the periphery is zero. Each of these diagonal entries corresponds to the determinant of the adjacency matrix of the subgraphs obtained when a vertex of the core is deleted. Thus the possible basic subgraphs of a minimal configuration are obtained by deleting each of the vertices of the core in turn.

5 Construction of a Core Basis

Henceforth a singular graph without isolated vertices will be denoted by Z.

Lemma 5.1 *Let χ be a core in a minimal basis for $C_0(Z)$. If χ is not itself a minimal configuration, then there is a minimal configuration,* *extended from χ, which is a (not necessarily vertex-induced) subgraph of Z.*

Lemma 5.2 *Let Z have core $\chi_{\mathbf{w}}$ and let H be a minimal configuration such that $H = \chi_{\mathbf{w}}$ or H is extended from core $\chi_{\mathbf{w}}$, and H is a subgraph of Z. Then $r(H) \leq r(Z)$.*

Theorem 5.3 *Let Z be of order n and rank r, with adjacency matrix $\mathbf{A}$. Let the first r row vectors of $\mathbf{A}$ be linearly independent vectors. Then there are $\eta(Z) = n - r$ cores in a basis for $C_0(Z)$. One such basis is given by the cores corresponding to the linear relations between each of the last $\eta(Z)$ row vectors and a subset of the first r row vectors of $\mathbf{A}$.*

Proof: Since the maximum number of linearly independent row vectors of $\mathbf{A}$ is r, each of the $\eta(Z)$ row vectors $R_j,\ j > r$, is linearly dependent on a subset of the first r row vectors of $\mathbf{A}$. Each linear relation corresponds to a kernel eigenvector in the nullspace of $\mathbf{A}$. Since each of these $\eta(Z)$ kernel eigenvectors corresponds to a core with a unique vertex v_j (described by row vector $R_j,\ j > r$), these $\eta(Z)$ kernel eigenvectors are linearly independent and so form a basis B for $\mathcal{E}_0(\mathbf{A})$. The $\eta(Z)$ cores corresponding to the kernel eigenvectors in B form a basis for the core-space $C_0(Z)$. □

For graph Y of Fig.1, $\kappa(G) = 2$ corresponding to cores of order 2 one of which is $< v_2, v_7 >$. We have shown that given a graph Z with a kernel eigenvector $\mathbf{x_0}$ there exists a subgraph of Z which is one of a set of particular minimal configurations given in [5], that depend on $\mathbf{x_0}$ and its corresponding core of Z. The four minimal configurations extended from the cores $\overline{K_2}$, $\overline{K_2}$, C_4, C_4 in a minimal basis for $C_0(Y)$ are P_3, P_3, Λ and Λ respectively, where Λ is shown in Figure 2.

Theorem 5.4

$$2 \leq \kappa(Z) \leq \tau(Z) \leq r(Z) + 1. \tag{1}$$

Proof: Let Z be labelled so that the first r row vectors of the adjacency matrix $\mathbf{A}$ are linearly independent. A kernel eigenvector corresponds to a relation between $R_j,\ j > r(Z)$ and a subset of the first r row vectors as in the proof of Theorem 5.3. A corresponding core will therefore have at most $r+1$ vertices.

These kernel eigenvectors correspond to a basis B for $\mathcal{E}_0(\mathbf{A})$ (not necessarily minimal). Let the weight-sequence of B be $(t_1, t_2, \ldots, t_\eta)$. Each term of a weight sequence of a minimal basis does not exceed the corresponding term in a weight sequence of another basis [6]. Since $\tau(Z)$ is the last term of a minimal weight-sequence, it follows that $\tau(Z) \leq t_\eta \leq r(Z) + 1$. Since the smallest possible core is $\overline{K_2}$ with 2 vertices [5], the result follows. $\square$

6 Graphs of Rank at most 3

Theorem 6.1 *A graph is of rank 0 iff it is an empty graph. The nullity of an empty graph is its order.*

Theorem 6.2 *There are no graphs of rank 1.*

Theorem 6.3 *The only graphs Z of order n and rank 2 are the complete bipartite graphs $K_{r,n-r},\ 1 \leq r \leq \lfloor \frac{n}{2} \rfloor$. In this case $\kappa(Z) = \tau(Z) = 2$. Only $K_2 = K_{1,1}$ is non-singular and of rank 2.*

Proof: Let G be a graph of order n and rank 2. Then $\mathbf{A}(G)$ has two linearly independent row vectors R_1, R_2. If $n(G) = 2$, then $G = K_2$, which is non-singular. If $n(G) > 2$ then each of the row vectors $R_j,\ j \geq 3$ is a linear combination of the first two and G is singular. Let $G = Z$. Then $(\alpha_1, \alpha_2, 0, \ldots, 0, \alpha_j, 0, , 0, \ldots, 0)^t$ is an eigenvector in the zero eigenspace of $\mathbf{A}$. From [5], we know that possible solutions are given by

i) $R_1 + R_2 = R_j \Longrightarrow n(G) \geq 5$ & rank ≥ 4. The minimal configuration in this case is P_5. Since we require $r(Z) = 2$, this kernel relation is not possible.

ii) For $j \geq 3$, $R_j = R_1$ or $R_j = R_2$. Thus $Z = K_{r,s}$ which has rank 2. $\square$

Theorem 6.4 *The only graphs Z of rank 3 are the complete tripartite graphs $K_{a,b,c}$ with $\kappa(Z) = \tau(Z) = 2$. The only graph of rank 3 which is non-singular is $K_{1,1,1} = K_3$.*

Proof: Let G be a graph of rank 3 with adjacency matrix $\mathbf{A}$. Let R_1, R_2, R_3 be linearly independent row vectors of $\mathbf{A}$. If $n(G) = 3$ then $G = K_3$. If $n(G) > 3$, then $\forall j \geq 4,\ \exists(\alpha_1, \alpha_2, \alpha_3, 0, .., \alpha_j, 0, ..0)^t \in \mathcal{E}_0(\mathbf{A})$. Let $G = Z$. If $\tau(Z) = 4$ then from [5], we know that the possible cores are C_4 and $\overline{K_4}$ when the corresponding minimal configurations have ranks 4 and 6 respectively and so by Lemma 5.2, $r(Z) \geq 4$ and $r(Z) \geq 6$ for the respective cores.

Thus for $r(G) = 3$, $\tau(Z) \leq 3$. Let $\tau(Z) = 3$. Then from [5], we know that there exists a labelling such that $R_1 + R_2 = R_3$ and hence $r(Z) \geq 4$, again too large. Thus $\tau(Z) \leq 2$. Hence R_1, R_2, R_3 are linearly independent and $\forall J \geq 4,\ R_J = R_I,\ I \in \{1, 2, 3\}$. Hence Z is $K_{a,b,c}$. $\square$

It is convenient to classify singular graphs according to a min-max core from which they can be "grown". For a graph G of nullity one, $\kappa(G) = \tau(G)$ and G is necessarily in one class. In [5], we show that larger graphs called intermediate configurations and maximum configurations can be "grown" from a minimal configuration with core χ as extensions of χ. These are included in the class determined by a min-max core.

The following table shows the results obtained above.

Graphs of rank at most 3.

Rank $r(G)$	Minimum order	Core	Kernel eigenvector	Graph	Singularity $\kappa(G)$	Core-width $\tau(G)$
0	1		$(1, 1, ...1)^t$	$\overline{K_n}$		
1	–	–	–	–		
2	2	–	–	K_2		
2	3	$\overline{K_2}$	$(-1, 1, 0, ..)^t$	$K_{r,n-r}$	2	2
3	3	–	–	K_3		
3	4	$\overline{K_2}$	$(-1, 1, 0, ..)^t$	$K_{a,b,c}$	2	2

7 Graphs of Rank 4, 5

Graphs of rank 2 and of rank 3 are the complete multipartite graphs and their cores are empty graphs. Graphs of larger rank can have cores other than the empty graph.

Theorem 7.1 *If a graph Z is of rank 4, then Z is one of the graphs in List #1 with kernel eigenvector $\mathbf{x_0}$, and the corresponding minimal configuration Γ as a subgraph.*

List #1: *Minimal configurations Γ which are subgraphs of Z.*

$\mathbf{x_0}$	core	Core-width of Z	Γ
$(1,-1,1,-1,0,..)^t$	C_4	4	Λ of Fig. 2
$(1,1,-1,0,..)^t$	$\overline{K_3}$	3	P_5
$(-1,1,0,..)^t$	$\overline{K_2}$	2	P_3

Proof: Let $\mathbf{A}(Z) = \mathbf{A}$ and let R_1, R_2, R_3, R_4 be linearly independent row vectors of $\mathbf{A}$. Then $\forall j \geq 5$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, .., 0, \alpha_j, 0, .., 0)^t \in \mathcal{E}_0(\mathbf{A})$, $\alpha_j \neq 0$. Suppose $\alpha_i \neq 0, \forall i \in \{1,2,3,4\}$. Let χ be the corresponding core on 5 vertices. Then $\mu = \eta(\chi) \geq 3$. If χ is the core of a minimum configuration, Γ, which is a subgraph of Z, then $n(\Gamma) = 5 + \mu - 1 \geq 7$ and so $r(Z) \geq 6$. Thus for $r(Z) = 4$, $\tau(Z) \leq 4$ when $n(\Gamma) = 7,\ 5,$ or 3 and $\chi = \overline{K_4},\ C_4,\ \overline{K_3},\ \overline{K_2}$. If Γ is a subgraph of Z, then $n(\Gamma) - 1 = r(\Gamma) \leq r(Z)$. So from [5], we deduce that only the graphs in List # 1 are possible. $\square$

Theorem 7.2 *A graph Z is of rank 5 iff Z is one of the graphs in List #1 with the corresponding Γ as a subgraph.*

Proof: Since the nullity μ of cores of order 6 is at least 2, for Z with core-width 6, $r(Z) = 5 + \mu - 1 \geq 6$. Thus for $r(Z) = 5$, $\tau(Z) \leq 5$. From [5], we know that if $\tau(Z) = 5$, then $\mu \geq 3$ and $r(Z) \geq 6$. It follows that $\tau(Z) \leq 4$ and so the same cores as for rank 4 are admissible. $\square$

8 Graphs of Rank 6

The nullity μ of the cores of the minimal configurations of core order 6, ranges from 2 to 6. The rank of a minimal configuration Γ is given by $r(\Gamma) = \tau + \mu - 2$. Since $\mu \geq 1$, for $r(\Gamma) = 6$, it follows that $\tau \leq 7$. More precisely, $(\tau, \mu) = (7, 1), (6, 2), (5, 3)$ or $(4, 4)$. In [5], we have described all the minimal configurations with core orders at most 5. Here we determine the cores of order 6 and choose the ones with nullity 2 to obtain the minimal configurations of rank 6. The cores of order 7 and nullity one are precisely the three nut graphs of order 7.

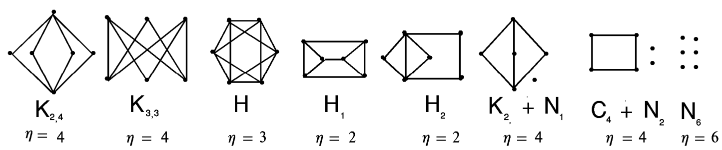

Figure 3: *Cores of order* 6.

Theorem 8.1 *Cores χ_6 on 6 vertices are the graphs in Fig. 3.*

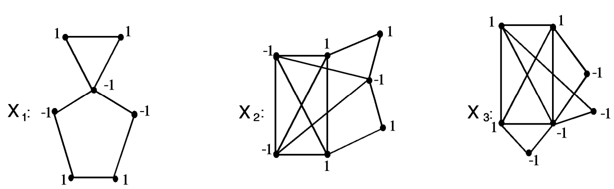

Figure 4: *Nut graphs of order* 7.

Theorem 8.2 *There are no nut graphs of order 6 or less, and three nut graphs of order 7 shown in Fig. 4.* ▭

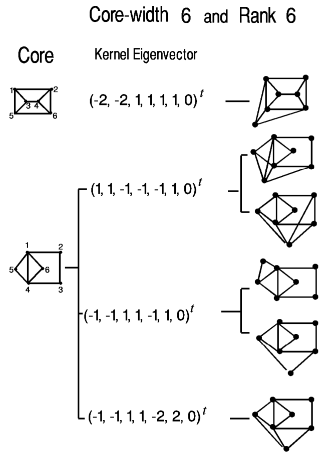

Figure 5: *Graphs with core-width* 6 *and rank* 6.

Theorem 8.3 *If a singular graph Z is of rank* 6, *then Z is one of the graphs in List #1 above or one of the graphs in List #2 below with the corresponding* Γ *as a subgraph.*

Proof: By direct checking the condition can be proved sufficient. To prove that it is necessary, let Z be a graph of rank 6 with adjacency matrix $\mathbf{A}$. Let $R_1, R_2, R_3, R_4, R_5, R_6$ be a set of linearly independent row vectors of $\mathbf{A}$. Then $\forall j \geq 7$, $\mathbf{x_0} = (\alpha_1, \alpha_2, \ldots, \alpha_6, 0, ..\alpha_j, 0..)^t \in \mathcal{E}_0(\mathbf{A})$, $\alpha_j \neq 0$. Then core-width $\tau(Z) \leq 7$.

If there exists a minimal basis B for $\mathcal{E}_0(\mathbf{A})$ such that $\mathbf{x_0} \in B$, $\alpha_i \neq 0$, $\forall i$ and the corresponding core $\chi_{\mathbf{x_0}}$ has nullity μ then a minimal configuration Γ "grown" from it has order $7 + (\mu - 1)$ and is of rank $5 + \mu$. For $r(Z) = 6$, $(\tau, \mu) = (7, 1)$. Thus $n(\Gamma) = \tau(Z) = 7$. Thus Z is a nut graph of rank 6. The graph $\Gamma = \chi_{\mathbf{x_0}}$ is one of the three graphs shown in Fig. 4.

For $\tau(Z) = 6$ the possible graphs have one of the cores χ_6 given by Theorem 8.1. A minimal configuration Γ in Z is of order $6+(\mu-1)$ and rank $5+(\mu-1)$ where $\mu = \eta(\chi_6)$. Since $r(\Gamma) \leq r(Z) = 6$, $\mu \leq 2$. Thus $(\tau, \mu) = (6, 2)$. The minimal configurations for cores H_1 and H_2 whose nullity is 2 in each case are given in Fig. 5. Since the other cores of order 6 have nullity greater than 2, the rank of the minimal configurations grown from them is more than 6 and so they are not admissible.

bf List #2: *The minimal configurations* Γ *in a graph* Z *of rank at least* 6.

$\mathbf{x_0}$	core	$\tau(Z)$	Γ
$(-1,-1,-1,-1,1,1,1,0,\ldots,0)^t$	X	7	X [Fig. 4].
$(-1,-1,-1,-1,1,1,1,0,\ldots,0)^t$	X_2	7	X [Fig. 4].
$(-1,-1,-1,-1,1,1,1,0,\ldots,0)^t$	X_3	7	X [Fig. 4].
$(-2,-2,1,1,1,1,0,\ldots,0)^t$	H_1	6	[Fig. 5].
$(1,1,-1,-1,-1,1,0,\ldots,0)^t$	H_2	6	[Fig. 5].
$(-1,-1,1,1,-1,1,0,\ldots,0)^t$	H_2	6	[Fig. 5].
$(-1,-1,1,1,-2,2,0,\ldots,0)^t$	H_2	6	[Fig. 5].
$(1,1,-1,-1,1,0,..)^t$	$C_4\dot{+}K_1$	5	[Fig. 6].
$(2,1,-2,-1,1,0,..)^t$	$C_4\dot{+}K_1$	5	[Fig. 6].
$(1,-1,-1,1,,-2,0,..)^t$	$C_4\dot{+}K_1$	5	[Fig. 6].
$(1,1,-2,1,-1,0,..)^t$	$K_{2,3}$	5	[Fig. 6].
$(1,-1,1,-1,0,..)^t$	$\overline{K_4}$	4	P_7 and Γ_1 [Fig. 7].
$(1,1,-1,1,0..)^t$	$\overline{K_4}$	4	$S(K_{1,3})$ [Fig. 7].
$(1,1,-2,1,0,..)^t$	$\overline{K_4}$	4	$H(K_4)$ [Fig. 7].
$(2,1,-1,-1,0,..)^t$	$\overline{K_4}$	4	${}^1C_6$ [Fig. 7].

For $\tau(Z)=5$, $(\tau,\mu)=(5,3)$. So at least one of the minimal configurations given in Fig. 6 (with a core $C_4\dot{+}K_1$ or $K_{2,3}$) is a subgraph of Z. For $\tau(Z)=$ 4,

$(\tau,\mu)=(4,4)$ or $(4,2)$. So at least one of the minimal configurations given in Fig. 7 (with a core $\overline{K_4}$) or Λ of Fig. 2 (with core C_4) is a subgraph.

For $\tau(Z)\leq 3$, the possible graphs of rank 6 are of the same structure as those of rank less than or equal to 5 given in List #1. $\square$

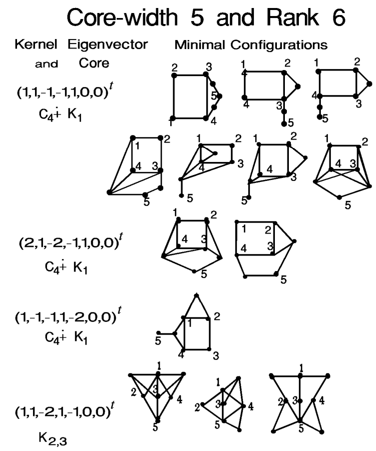

Figure 6: *Graphs with core-width* 5 *and rank* 6*.*

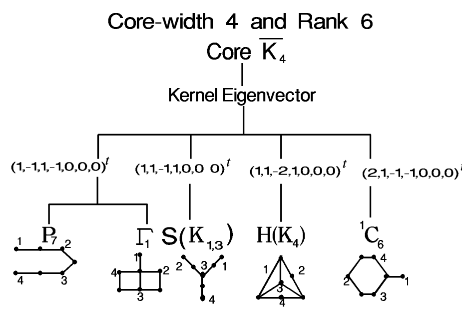

Figure 7: *Graphs with core-width* 4 *and rank* 6*.*

9 Bounds for the Core-Order-Sequence

The following thoerem has been proved.

Theorem 9.1 *If* $r(Z)$ *is* 2 *or* 3*, then* $\kappa(G) = \tau(G) = 2$*.*
If $r(Z)$ *is* 4 *or* 5*, then* $\kappa(G) \geq 2$ *and* $\tau(G) \leq 4$*.* □

Lemma 9.2 *For a given* $\tau(G) \leq 6$*, a lower bound for the rank of an* *arbitrary graph G is $\tau(G)$. The sharp bound is $\tau(G)$ if $\tau(G)$ is even and $\tau(G)+1$ if $\tau(G)$ is odd.* $\square$

Lemma 9.3 *For a given $\tau(G) \geq 7$, a lower bound for the rank of an arbitrary graph G is $\tau(G)-1$. This bound is sharp for each value of $\tau(G)$ in this range.*

Proof: A nut graph exists for each core order at least seven [3, 7]. So the graph G can have a min-max core χ which is a nut graph provided $\tau(G) \geq 7$ when $n(G) \geq n(\chi) = \tau(G)$ and $r(G) \geq r(\chi) = \tau(G) - 1$. Equality occurs for nut graphs. $\square$

For graphs of rank 6 or higher, the core-width has an upper bound that depends on the rank and that can be attained.

Theorem 9.4 *If $r(Z) \geq 6$, then $\kappa(Z) \geq 2$ and $\tau(Z) \leq r(Z) + 1$. Both bounds are attained.*

Proof: From [7], we know that a nut graph G exists for $n(G) \geq 7$, in which case $\tau(G) = r(G) + 1 = n(G)$. From Lemma 5.2, and Theorem 5.4, the result follows. $\square$

The rank of a singular graph Z imposes bounds on the core-width $\tau(Z)$ but not on the order $n(Z)$. For a given rank $r > 1$ of an arbitrary graph G, $n(G)$ is bounded below by r but is not bounded above since for example vertices that have the same set of neighbours can be added without changing the rank or core-width of the resulting graph.

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