1.1 Statements

A sentence is called a statement if we can decide whether it is true or false.

Examples: (1) The sun is shining.
(2) A dog has four legs.
(3) For all real numbers \(x\), \(x^2 > 0\).

The following are not statements: Come here!
What time is it?
This statement is false.

Statements are usually denoted by the letters \(p, q, r\) etc. A statement about \(x\), where \(x\) can take on several values, is sometimes denoted \(p(x)\).

1.2 New statements from old: Negation

Given a statement \(p\), we can form its negation, \(\neg p\), denoted \(\neg p\).

If \(p\) is the statement “The sun is shining” then \(\neg p\) is the statement “The sun is not shining”.

If \(p\) is false, then \(\neg p\) must be true and vice-versa.

What is the negation of this statement: “\(x > 5\)”?

A truth table tells us when a statement is true and when it is false. The truth table for \(\neg\) is:

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\neg p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

1.3 New statements from old: Compound Statements

Another way of forming new statements is by joining two or more statements together. For example, “The sun is shining” and “Rain is falling” may be joined together to form the following compound statements:
The sun is shining and rain is falling.
The sun is shining or rain is falling.
If the sun is shining, then rain is falling.
The sun is shining if and only if rain is falling.

The truth value of these compound statements depends on the truth values of the simpler statements from which they are composed.

1.3.1 And

The statement “p and q” is denoted \( p \land q \). It is true when both \( p \) and \( q \) are true, and is otherwise false.

\[
\begin{array}{c|c|c}
    p & q & p \land q \\
    \hline
    T & T & T \\
    T & F & F \\
    F & T & F \\
    F & F & F \\
\end{array}
\]

Example: “\( 2^2 = 4 \) and 2 is a negative number”.

1.3.2 Or

The statement “p or q” is denoted \( p \lor q \). It is false when both \( p \) and \( q \) are false, and is otherwise true.

\[
\begin{array}{c|c|c}
    p & q & p \lor q \\
    \hline
    T & T & T \\
    T & F & T \\
    F & T & T \\
    F & F & F \\
\end{array}
\]

Example: “\( 2^2 = 4 \) or 2 is a negative number”.

1.3.3 Implies

The statement “p implies q” or “if p, then q” is denoted \( p \Rightarrow q \). It is false only when \( p \) is true and \( q \) is false, and is otherwise true. (Careful!)
Note that a true statement can only imply a true statement i.e. if \( p \Rightarrow q \) and \( p \) is true, then \( q \) must be true. However, a false statement can imply both true and false statements i.e. if \( p \Rightarrow q \) and \( p \) is false, then \( q \) could be either true or false.

The negation of \( p \Rightarrow q \) is \( p \land \neg q \). (Prove this by constructing a truth table for \( p \land \neg q \).)

Exercise: Let \( p \) be “That animal is a dog”, and \( q \) be “That animal has four legs”. Which of the following statements are true?

- \( p \Rightarrow q \) is “If that animal is a dog, then it has four legs”.
- \( q \Rightarrow p \) is “If that animal has four legs, then it is a dog”.
- \( \neg p \Rightarrow \neg q \) is “If that animal is not a dog, then it does not have four legs”.
- \( \neg q \Rightarrow \neg p \) is “If that animal does not have four legs, then it is not a dog”.

1.3.4 Equivalence

The statement “\( p \) if and only if \( q \)” or “\( p \) is equivalent to \( q \)” is denoted \( p \iff q \).

It is true when \( p \) and \( q \) have the same truth value, and is otherwise false.

Example: “\( 2^2 = 4 \) if and only if 2 is a negative number”.

1.4 The truth value of a general statement

Truth tables help us to decide when a compound statement is true.

Example: When is the statement \( (p \lor q) \land \neg p \) true?

Solution: Fill in the following truth table:
1.5 Tautologies

A tautology is a statement which is always true. The following are some examples:

(i) \( p \lor \neg p \) (Law of the Excluded Middle)
(ii) \( \neg(p \land \neg p) \)
(iii) \( \neg \neg p \iff p \)
(iv) \( (p \Rightarrow q) \iff (\neg q \Rightarrow \neg p) \) (Law of Contrapositive)
(v) \( (p \land q) \Rightarrow p \)
\( p \Rightarrow (p \lor q) \)
(vi) \( (p \Rightarrow q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r) \) (Law of Syllogism)
(vii) \( \neg(p \land q) \iff (\neg p \lor \neg q) \) (De Morgan’s Laws)
\( \neg(p \lor q) \iff (\neg p \land \neg q) \)
(viii) \( p \lor (q \land r) \iff (p \lor q) \land (p \lor r) \) (Distributive Laws)
\( p \land (q \lor r) \iff (p \land q) \lor (p \land r) \)
(ix) \( \neg(p \Rightarrow q) \iff (p \land \neg q) \)

These tautologies may be verified using truth tables (do this as an exercise).

1.6 Quantifiers

Several mathematical statements contain phrases like “for all real numbers \( x \)” or “there exists an integer \( a \)”. Phrases such as ”for all”, ”for some” and ”there exists” are called quantifiers.

The symbol \( \forall \) denotes “for all”, e.g. \( \forall x \in \mathbb{R}, x^2 \geq 0 \).

The symbol \( \exists \) denotes “there exists” or “for some”, e.g. \( \exists x \in \mathbb{C} \) such that \( x^2 = -5 \) or \( \exists x \in \mathbb{R}, x^2 < x \).
There is a very simple rule which helps us to form the negation of a statement involving a quantifier. The negation of a statement of the type “$\forall x, p(x)$” is the statement “$\exists x, \neg p(x)$”.

Example: The negation of “$\forall x \in \mathbb{R}, x^2 > 0$” is “$\exists x \in \mathbb{R}, x^2 \leq 0$”.

Exercise: Which of the following statements is the negation of “All books have hard covers”?

(i) No books have hard covers.
(ii) Some books do not have hard covers.
(iii) All books have soft covers.
(iv) There exists at least one book which has a soft cover.

Important note: to prove that a statement of the type “$\forall x, p(x)$” is false, we need to find just one $x$ for which $p(x)$ does not hold (i.e. a counterexample).

The negation of a statement of the type “$\exists x, p(x)$” is the statement “$\forall x, \neg p(x)$”.

Example: The negation of “$\exists x \in \mathbb{R}, x^2 = -1$” is the statement “$\forall x \in \mathbb{R}, x^2 \neq -1$”.

Exercise: What is the negation of the following statement: ”For all real numbers $x$, there exists an integer $y$ such that $x < y$”?

Combining quantifiers: note that the statements “$\forall x \exists y p(x, y)$” and “$\exists y \forall x p(x, y)$” do not necessarily mean the same thing.