Provisional Definition: A function is a ‘rule’ that assigns a unique element in a set $B$ to every element in a set $A$ (or equivalently, maps every element of $A$ onto a unique element of $B$).

Definition: A function $f$ from the set $A$ into the set $B$ is a relation from $A$ to $B$, such that for all $x \in A$, there exists a unique $y \in B$ s.t. $xRy$. In this case we write $f(x) = y$. The set $A$ is called the domain of $f$. The set $B$ is called the codomain of $f$.

Therefore, we say that a function $f : A \to B$ is well-defined if:

(i) for all $x \in A$, there exists a $y \in B$ s.t. $f(x) = y$;
(ii) if $y_1 = f(x)$ and $y_2 = f(x)$, then $y_1 = y_2$. 

A mapping which does not satisfy these two properties is not a function.

Notes: (i) This definition includes “one-to-one” and “many-to-one” mappings but not “one-to-many” mappings.
(ii) We must always specify the domain and codomain of a function.
(iii) The function must be defined for all elements of the domain, i.e. given any $x \in A$, there must be $f(x) \in B$. Furthermore, this $f(x)$ must be unique.

If $x \in A$, $y \in B$ and $f(x) = y$, then $y$ is s.t.b. the image of $x$ under $f$, or the value of $f$ at $x$.

The set $\{y \in B : \exists x \in A \text{ s.t. } f(x) = y\}$ is called the range of $f$, denoted $Rng(f)$.

The set $\{(x, y) : x \in A, y \in B, f(x) = y\}$ is called the graph of $f$.

If $C \subseteq A$, then the image of $C$ under $f$, $f(C)$, is the set $\{y \in B : \exists x \in C \text{ s.t. } f(x) = y\}$; i.e. it is the set of those elements of the codomain that are images of elements of $C$. (Note that the range of $f$ is the image of the domain of $f$.)

If $D \subseteq B$, then the inverse image of $D$ under $f$, $f^{-1}(D)$ is the set $\{x \in A : \exists y \in D \text{ s.t. } f(x) = y\}$; i.e. it is the set of those elements of the domain which the function maps onto elements of $D$.

Warning: The use of the symbol $f^{-1}(D)$ does not imply that the function $f$ has an inverse! In fact, the inverse image is well-defined even for functions that have no inverse.
**Definition**: A function $f : A \rightarrow B$ is s.t.b.
(i) **surjective** or onto if for all $y \in B$, there exists $x \in A$ s.t. $f(x) = y$;
(ii) **injective** or one to one if for all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$
(i.e. if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$;
(iii) **bijective** or a one-to-one correspondence if $f$ is both surjective and injective.

Note: (i) Surjective and injective are **not** opposites: a function may be surjective or injective or both or neither!
(ii) Surjective does **not** mean “many-to-one”.
(iii) A function is surjective if and only if its range and codomain are equal.

**Proposition**: Let $f : A \rightarrow B$. Then
(i) if $C \subseteq D \subseteq A$, then $f(C) \subseteq f(D)$;
(ii) if $C, D \subseteq A$, then $f(C \cap D) \subseteq f(C) \cap f(D)$;
(iii) if $C, D \subseteq A$, then $f(C \cup D) = f(C) \cup f(D)$;
(iv) if $C \subseteq D \subseteq B$, then $f^{-1}(C) \subseteq f^{-1}(D)$;
(v) if $C, D \subseteq B$, then $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$;
(vi) if $C, D \subseteq B$, then $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$;
(vii) if $C \subseteq A$, then $C \subseteq f^{-1}(f(C))$;
(viii) if $D \subseteq B$, then $f(f^{-1}(D)) \subseteq D$.

**Definition**: Let $f : A \rightarrow B$ and $g : A \rightarrow B$. Then the functions $f$ and $g$ are s.t.b. **equal** if for all $x \in A$, $f(x) = g(x)$.

**Definition**: Let $f : A \rightarrow B$ and $g : B \rightarrow C$. Then the **composition** of $f$ with $g$, denoted $g \circ f$, is the function:

\[
g \circ f : A \rightarrow C
\quad x \mapsto g(f(x)).
\]

Note that composition of functions is associative, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$.

**Proposition**: Let $f : A \rightarrow B$ and $g : B \rightarrow C$. Then
(i) if $f$ and $g$ are surjective, then $g \circ f$ is surjective;
(ii) if $f$ and $g$ are injective, then $g \circ f$ is injective.