Quadratic Forms and Local Extrema

Let H be a symmetric matrix. Then the function $q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$q(\mathbf{x}) = \mathbf{x}^T H \mathbf{x}$$

is called a *quadratic form*. For example if $H = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ then

$$q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 2xy + y^2.$$

Another example is the third term in the Taylor expansion of a scalar-valued function on $\mathbb{R}^n$:

$$\begin{aligned} q(\mathbf{h}) = \frac{1}{2}\mathbf{h}^T H_f(\mathbf{a})\mathbf{h} &= \frac{1}{2}\mathbf{h}^T \begin{pmatrix} D_1D_1f(\mathbf{a}) & \cdots & D_nD_1f(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ D_1D_nf(\mathbf{a}) & \cdots & D_nD_nf(\mathbf{a}) \end{pmatrix} \mathbf{h} \\ &= \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j D_j D_i f(\mathbf{a}). \end{aligned}$$

Definition: Let $q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then q is said to be

(a) *positive definite* if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$;

(b) *negative definite* if $q(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbb{R}^n$;

(c) *indefinite* if $q(\mathbf{x}) > 0$ for some $\mathbf{x} \in \mathbb{R}^n$ and $q(\mathbf{x}) < 0$ for some $\mathbf{x} \in \mathbb{R}^n$.

The symmetric matrix H is also sometimes said to be positive definite if its corresponding quadratic form is positive definite etc.

We can now generalize the second derivative test for classifying local extrema.

Theorem (Second Derivative Test): Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be of class $C^3(A)$, where A is open. Let $\mathbf{a}$ be a critical point of f. Then

(i) if the Hessian of f is positive definite, then $\mathbf{a}$ is a local minimum of f;

(ii) if the Hessian of f is negative definite, then $\mathbf{a}$ is a local maximum of f;

(iii) if the Hessian of f is indefinite, then $\mathbf{a}$ is a saddle point.

We note that a matrix is positive definite if all its eigenvalues are positive, negative definite if all its eigenvalues are negative, and indefinite if it has both positive and negative eigenvalues. (Recall that all the eigenvalues of a symmetric matrix are real, and it is always possible to find an orthonormal basis of eigenvectors.)

The following criterion helps us classify quadratic forms that arise from 2×2 matrices.

Proposition: Let H be the 2×2 symmetric matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Then the quadratic form $q(\mathbf{x}) = \mathbf{x}^T H \mathbf{x}$ is

(i) *positive definite* if $a > 0$ and $\det(H) > 0$;

(ii) *negative definite* if $a < 0$ and $\det(H) > 0$;

(iii) *indefinite* if $\det(H) < 0$.

Combining this with the above theorem, we get the following result.

Theorem : Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ be of class $C^3(A)$, where A is open. Then $\mathbf{a}$ is a (strict) local maximum of f if

(i) $D_1 f(\mathbf{a}) = D_2 f(\mathbf{a}) = 0$ (i.e. $\mathbf{a}$ is a critical point);

(ii) $D_1 D_1 f(\mathbf{a}) < 0$;

(iii) $D = \begin{vmatrix} D_1 D_1 f(\mathbf{a}) & D_2 D_1 f(\mathbf{a}) \\ D_1 D_2 f(\mathbf{a}) & D_2 D_2 f(\mathbf{a}) \end{vmatrix} > 0.$

If $D_1 D_1 f(\mathbf{a}) > 0$, then $\mathbf{a}$ is a strict local minimum, while if $D < 0$, then $\mathbf{a}$ is a saddle point.

Note that if $D = 0$, the test fails.