Functions

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1 Definition of a well defined function

An expression for y in terms of a variable x can be referred to as a 'function of x' if for each unique value of x in the set X (the **domain**, see section 2 below) there is one and only one corresponding value of y.

For example:

(a) $y = x^2 - 1$ can represent a function of x (see fig. 1a) over all of $\mathbb{R}$, and we can write:

$$f(x) = x^2 - 1 \qquad x \in \mathbb{R}$$

(read f of x is equal to ...)

or

$$f : x \to x^2 - 1 \qquad x \in \mathbb{R}$$

(read f maps x onto ...)

(b) $y = \sin(x)$ can represent a function of x (see fig. 1b) over all of $\mathbb{R}$ and we can write:

$$f(x) = \sin(x) \qquad x \in \mathbb{R}$$

or

$$f : x \to \sin(x) \qquad x \in \mathbb{R}$$

(c) The possible values of y for a given $x \geq -1$ such that $y^2 = x + 1$ does not represent a function of x (see fig. 1c) since:

$$y^2 = x + 1 \quad , \quad x \geq -1$$

$$\Rightarrow y = \pm\sqrt{x+1} \quad , \quad x \geq -1$$

i.e. each value of $x > 1$, gives two possible values of y.

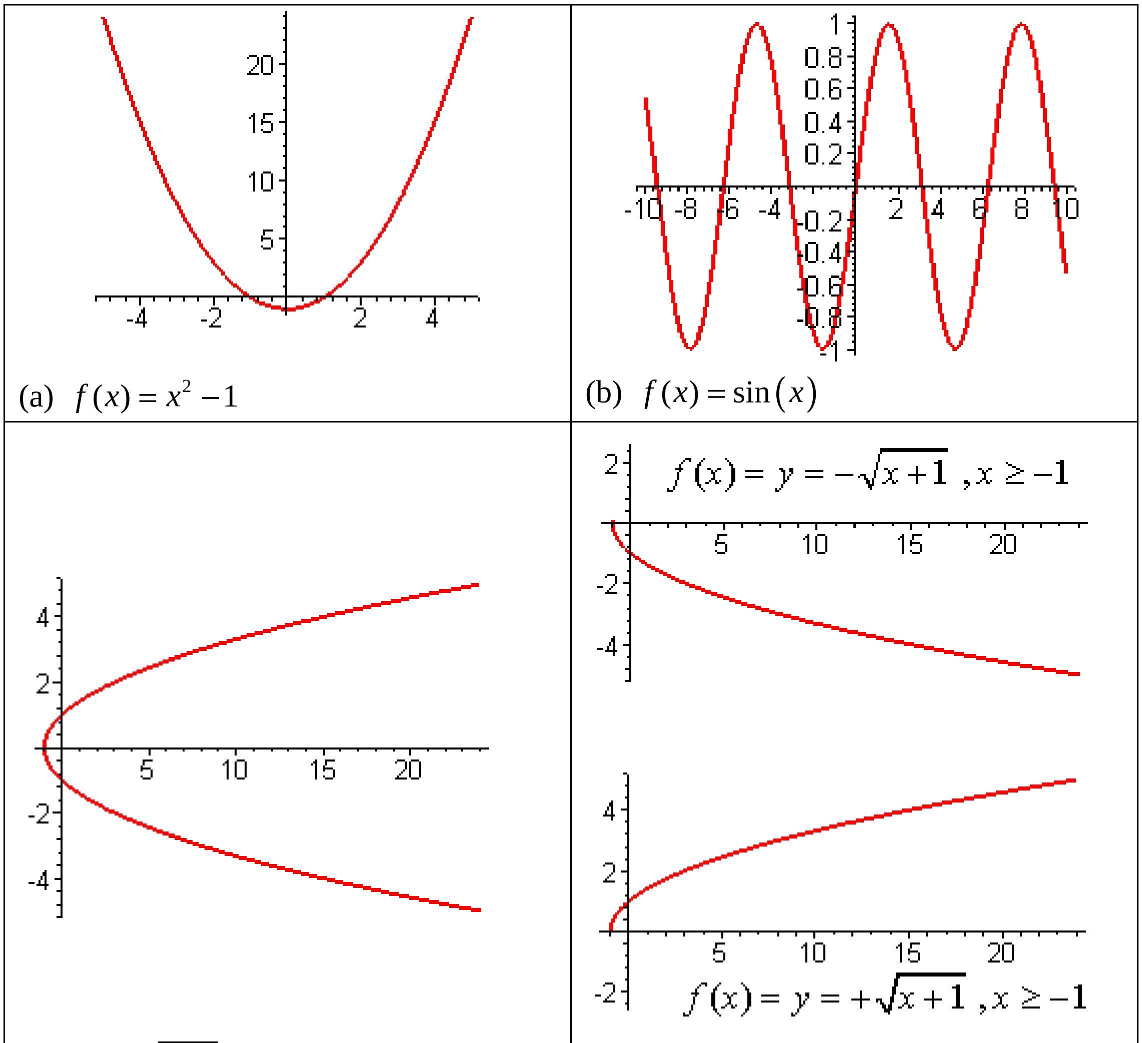
(d) $y = \pm\sqrt{x+1}$, $x \geq -1$ needs to be modified to be turned into a well defined function, by choosing to ignore the negative (or the positive) parts of the solution. Thus, whilst $y = \pm\sqrt{x+1}$, $x \geq -1$ does not fulfil the requirements of a well defined function, both:

$$f(x) = y = +\sqrt{x+1} \ , \ x \geq -1$$

and

$$f(x) = y = -\sqrt{x+1} \ , \ x \geq -1$$

are well defined functions. (see Fig. 1d)



(a) $f(x) = x^2 - 1$

(b) $f(x) = \sin(x)$

(c) $y = \pm\sqrt{x+1}$ is not a well defined function of x: One value of x gives more than one value of y, i.e. $y \neq f(x)$

(d) $f(x) = y = +\sqrt{x+1}$, $x \geq -1$ and $f(x) = y = -\sqrt{x+1}$, $x \geq -1$ are both well defined functions.

Fig. 1

2 The domain and range of functions

2.1 The greatest possible domain

The **greatest possible domain** of a function *f(x)* is set of *x* over which the function is well defined. In particular, for real functions, it must exclude the following:

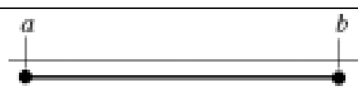
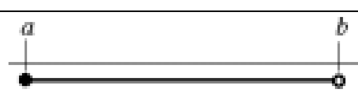
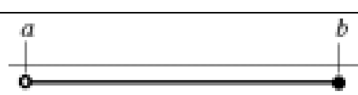
- Division by 0
- Square roots of negative numbers
- Logarithms of zero and negative numbers
- Tangents of $n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$

Thus, for example, as noneof the above are present for $f(x)=x^2-1$, then its greatest possible domain is $\mathbb{R}$.

2.2 The domain

The **domain** of a function *f(x)* is set of *x* over which the function is defined. It is a subset of the greatest possible domain and **has to be stated with the function**. *(Note: If not stated, it is appropriate to assume that the domain is the greatest possible domain.)*

Note that the domain can either be <u>discrete</u> values (e.g. *x*=1, 2 or 2.5) or <u>continues</u>, i.e. an interval. Intervals can be defined as follows:

Description	**Graphical representation**	**Representation using 'arrows'**	**Representation using brackets.**
Between *a* and *b*, both *a* and *b* included		$a \le x \le b$	$x \in [a,b]$
Between *a* and *b*, both *a* and *b* excluded		$a < x < b$	$x \in]a,b[$ **or** $x \in (a,b)$
Between *a* and *b*, *a* included and *b* excluded		$a \le x < b$	$x \in [a,b[$ **or** $x \in [a,b)$
Between *a* and *b*, *a* excluded and *b* included		$a < x \le b$	$x \in]a,b]$ **or** $x \in (a,b]$

Thus for example, although the greatest possible domain for $f(x)=x^2-1$ is $\mathbb{R}$, we may wish to restrict the values of x over which the function operates to x between 1 and 3, both values inclusive. This can be written as:

$$f(x) = x^2 - 1 \qquad , \qquad x \in [1,3]$$

2.3 The range

Given the domain X for a function $f(x)$, we may define the **range** of $f(x)$ as the set of output values of $f(x)$ which correspond to x in the domain.

Thus, for our example where:

$$f(x) = x^2 - 1 \qquad , \qquad x \in [1,3]$$

the range is between 0 and 8, i.e.:

$$f(x) = x^2 - 1 \qquad , \qquad x \in [1,3], \ f(x) \in [0,8]$$

whilst for:

$$f(x) = \sin(x) \qquad , \qquad x \in \mathbb{R}, \ f(x) \in [-1,+1]$$

3 Composite functions

The composite function $f \circ g(x)$, or simply $fg(x)$ is defined as:

$$f \circ g(x) = fg(x) = f(g(x))$$

For example:

Given: $f(x) = \sin(x), \ g(x) = x^2 - 1$

Then: (*i*) $f \circ g(x) = f(g(x)) = \sin(x^2 - 1)$

(*ii*) $g \circ f(x) = g(f(x)) = (\sin(x))^2 - 1$

Note that in general, $f \circ g(x) \neq g \circ f(x)$, as in this case where

$$f \circ g(x) = \sin(x^2 - 1) \neq g \circ f(x) = \sin^2(x) - 1 = \cos^2(x)$$

4 Inverse functions

4.1 Definition of the inverse function

Two functions f and g are said to be the **inverse** of each other if and only if:

$$f \circ g(x) = f(g(x)) = x$$

and:

$$g \circ f(x) = g(f(x)) = x$$

For example, $f(x) = 2x + 3$ and $g(x) = \frac{x-3}{2}$ are inverse of each other since:

$$f \circ g(x) = f(g(x)) = 2\left(\frac{x-3}{2}\right) + 3 = x$$

and:

$$g \circ f(x) = g(f(x)) = \frac{(2x+3)-3}{2} = x$$

Note that the inverse of $f(x)$ is usually denoted by $f^{-1}(x)$.

4.2 Finding the inverse function

The inverse of $f(x)$ is usually denoted by $f^{-1}(x)$ and is derived as follows:

Method:	**Example:**
Given: *f(x)*	$f\left(x\right)=2x+3$
Let *y* = *f(x)*	$y=2x+3$
Make *x* subject	$x=\frac{y-3}{2}$
'Interchange *x* and *y*'	$y=\frac{x-3}{2}$
To obtain inverse: Replace *y* by $f^{-1}\left(x\right)$	$f^{-1}\left(x\right)=\frac{x-3}{2}$

4.3 One-to-one functions

Not every function has an inverse function: A requirement for a function to have an inverse is that it must be **one-to-one (1-1)**, that is for every output value of *f(x)* in the range, there must be one and only one corresponding value of *x* in the domain.

Note that this goes further from the requirement that for any value of *x* in the domain of *f(x)*, the value of *f(x)* must be unique.

Thus for example, although:

$$f(x)=\sin\left(x\right) \quad , \quad x\in\mathbb{R},\ f(x)\in\left[-1,+1\right]$$

is a well defined function, it is not one-to-one, since several values of *x* give the same value of *f(x), e.g.*:

$$\sin\left(\frac{\pi}{6}\right)=\sin\left(\frac{5\pi}{6}\right)=\sin\left(2\pi+\frac{\pi}{6}\right)=\sin\left(2\pi+\frac{5\pi}{6}\right)=\ldots=\frac{1}{\sqrt{2}}$$

This means that $f(x)=\sin\left(x\right)$, $x\in\mathbb{R}$ does not have an inverse.

However, an inverse can be defined if we were to restrict the domain of *f(x)*=sin(*x*) to make f(*x*) one-to-one, for example, by restricting the domain to:

$$f(x)=\sin\left(x\right) \quad , \quad x\in\left[-\frac{\pi}{2},+\frac{\pi}{2}\right],\ f(x)\in\left[-1,+1\right]$$

The inverse for this 1-1 function exits and is what we refer to as $\sin^{-1}(x)$ or arcsin(*x*). This is illustrated in Fig. 2.

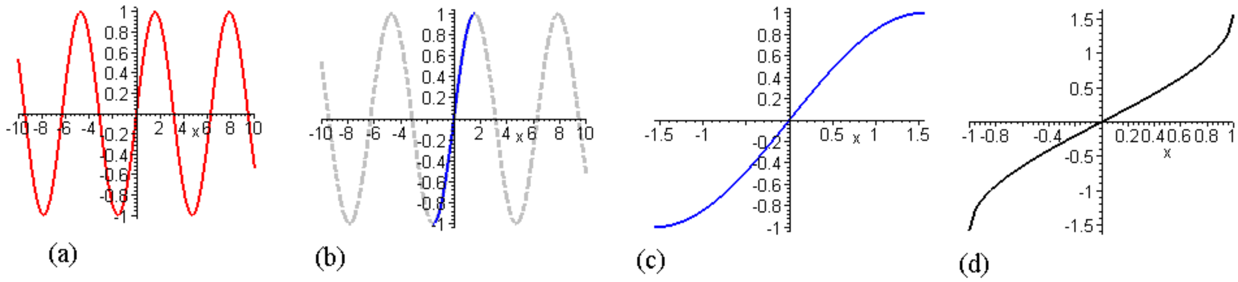


Fig. 2: (a) The plot of $y = f(x) = \sin(x), x \in [-10,+10]$,

(b) The plot of $y = f(x) = \sin(x), x \in [-10,+10]$ with $x \in [-\pi/2,+\pi/2]$ highlighted.

(c) The plot of $y = f(x) = \sin(x)$, domain = $[-\pi/2,+\pi/2]$, range = $[-1,1]$

(d) The plot of $y = f^{-1}(x) = \sin^{-1}(x)$, domain = $[-1,1]$, range = $[-\pi/2,+\pi/2]$

4.4 Plots of f(x) and their inverses.

The plots of $y = f(x)$ and $y = f^{-1}(x)$ are mirror images of each other about the line $y=x$. This is illustrated for a particular example in fig. 3.

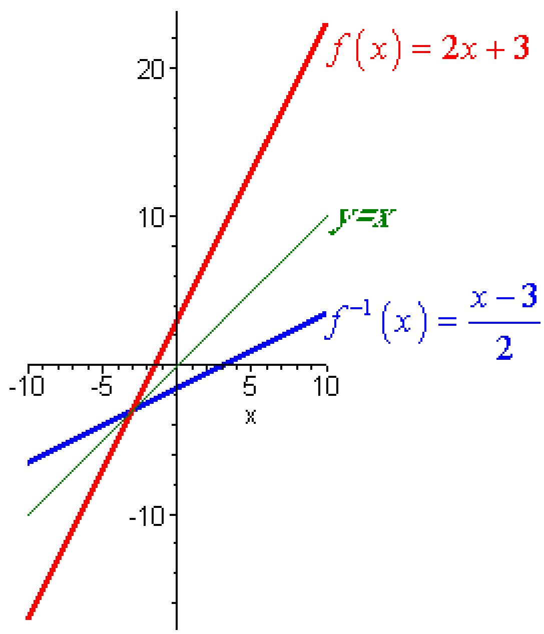


Fig. 3: An illustration showing that $f(x)$ and $f^{-1}(x)$ are mirror images of each other at about the line $y=x$.

One should also note that the domain and range of a function become the range and domain of the inverse function respectively, i.e.:

function	**domain**	**range**
$f(x)$	$[a,b]$	$[c,d]$
$f^{-1}(x)$	$[c,d]$	$[a,b]$

5 Special functions

5.1 The linear function

The equation of a straight line is given by:

$$y = mx + c$$

where *m* is the gradient and *c* is the y-intercept.

Another format of this equation is as:

$$y - y_o = m(x - x_o)$$

where *m* is the gradient and (x_o, y_o) is any point on the line.

Note that in this case, $f(x) = y = mx + c$ is always as well defined and one-to-one function

5.2 The quadratic function

The quadratic function has a general form of:

$$f(x) = ax^2 + bx + c$$

and has the well familiar $\cup$ or $\cap$ shape depending on the sign of *a* ($\cup$ if *a* is positive and $\cap$ if *a* is negative.) The solution of the equation $f(x) = ax^2 + bx + c = 0$ are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i.e. it will only have real roots (and maybe factorisable) if $b^2 - 4ac \geq 0$. The quadratic is also symmetric about the line $x = \frac{-b}{2a}$ (see fig. 4).

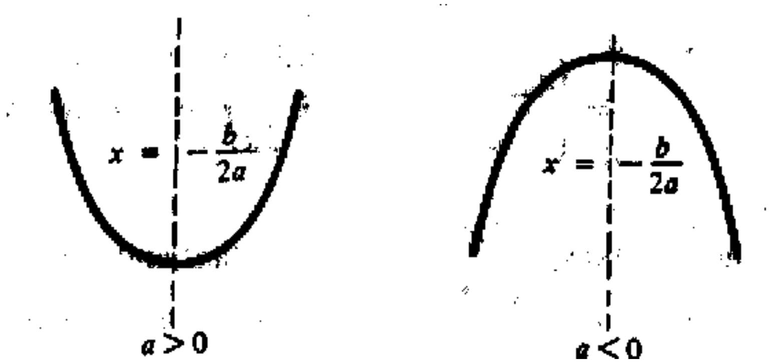


Fig. 4: Properties of $f(x) = ax^2 + bx + c$

5.3 The exponential and logarithmic functions

The exponential function is one were the variable appears as an exponent, e.g.: 2^x, 7^{-2x}

Exponential functions of the form a^{bx} where a > 1 and b > 0, have the following properties:

1. $f(x) > 0 \quad \forall x \in \mathbb{R}$
2. As x increases, $f(x)$ increases at a rapidly accelerating rate.
3. $f(0) = 1$
4. as $x \to -\infty$, $f(x) \to 0$, i.e. $\lim_{x\to-\infty}\left[f(x)\right] = 0$

One should at this point recall the rules of indices, i.e.:

$$A^a \times A^b = A^{a+b}$$
$$A^a \div A^b = A^{a-b}$$
$$A^{-a} = \frac{1}{A^a}$$
$$\left(A^a\right)^b = \left(A^b\right)^a = A^{ab}$$
$$A^{1/2} = \sqrt{A}$$
$$A^{1/a} = \sqrt[a]{A}$$
$$A^0 = 1$$

The most widely used exponential function (especially in chemistry) is that of e^x where

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

i.e.:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \approx 2.7183 \text{ (an irrational number)}$$

Logarithmic functions can be treated as the inverse of exponential function. In fact, one may define the log function as follows:

$$\log_a b = c \Leftrightarrow a^c = b$$

In theory, a (the base) can be any real number, but in practice, a is usually 10 ($\log_{10}$, or lg) or e ($\log_e$ or ln, read natural log). In these special cases:

$$\lg A = \log_{10} A = B \Leftrightarrow 10^B = A$$

$$\ln A = \log_e A = B \Leftrightarrow e^B = A$$

Logarithmic functions of the form $\log_a x$ have the following properties;

1. $f(x) = \log_a x$ does not exist for negative values of x.
2. For $x > 1$, $f(x) > 0$ and as $x \to \infty, f(x) \to \infty$
3. $f(0)$ is undefined but $\lim_{x\to 0}\left[\log_a(x)\right] = -\infty$
4. The sketch of $y = \log_a(x)$ is the mirror image of $y = a^x$ about the line $y=x$ (see fig. 5)

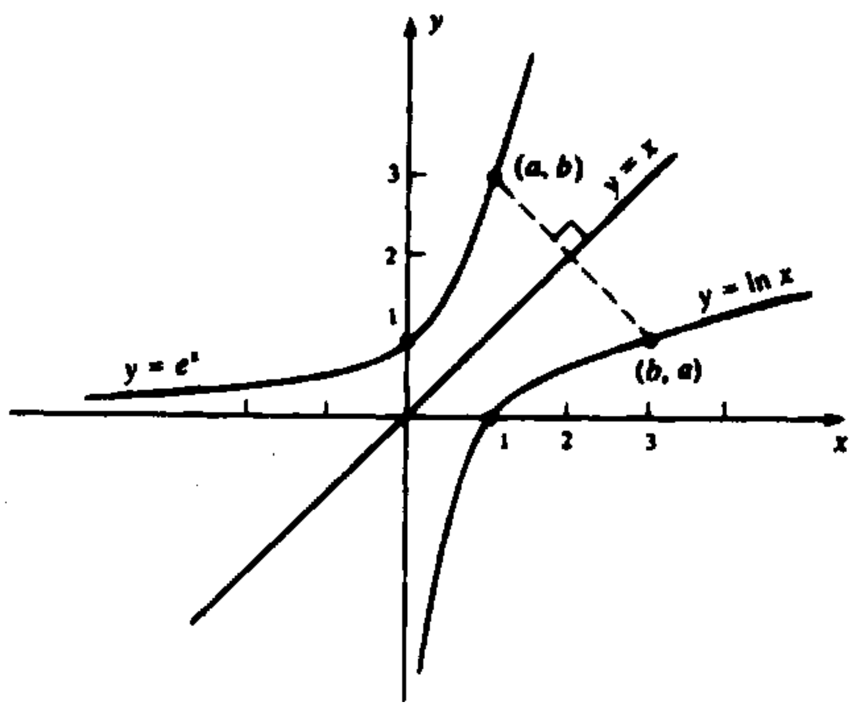


Fig. 5

Important things to remember:

$$\log_C A = B \Leftrightarrow C^B = A$$

$$\log_C C = 1$$

$$\log_C (AB) = \log_C (A) + \log_C (B)$$

$$\log_C (A \div B) = \log_C (A) - \log_C (B)$$

$$\log_C (A^B) = B \log_C (A)$$

$$\log_C (C^B) = B \log_C (C) = B$$

$$\log_C (A) = \frac{\log_B (A)}{\log_B (C)}$$

$$\mathrm{pH} = -\log_{10} \left[\mathrm{H}^+ \right]$$

5.4 The trigonometric functions