Vertex-deleted and Edge-deleted Subgraphs

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1 INTRODUCTION

Some classical problems in mathematics are of the following general type. If the structure S' is associated with each structure S, does S' determine S uniquely? In graph theory we ask what knowledge of the graph short of its full incidence relations is sufficient to determine the graph completely. The structure S is a graph and S could be its line graph, or chromatic polynomial, or spectrum, say.

The foremost problem in this area of graph theory is the Reconstruction Conjecture which states that a graph is reconstructible from its collection of vertex-deleted subgraphs. In spite of several attempts to prove the conjecture only very partial results have been obtained. Several of these results do however bring to light interesting structural relationships between a graph and its subgraphs. For example, can all the subgraphs of a nonplanar graph be planar? Some reconstruction results viewed in this light will be discussed here.

In the few special cases where the Reconstruction Conjecture has been proved it generally turns out that the full collection of subgraphs contains much more information than is required to determine the graph. This has motivated attempts to reconstruct graphs from only a restricted subcollection of all the vertex-deleted subgraphs. The newest idea in this direction is to determine what is called the reconstruction number of a graph which is the smallest number of vertex-deleted subgraphs of the graph which are sufficient to reconstruct it. This notion opens questions which might give a new lease of life to research on the Reconstruction Problem, and we also review here the main results obtained to date in this area.

While the Reconstruction Problem deals with the question of whether a pair of nonisomorphic graphs can have the same collection of vertex-deleted subgraphs, a seemingly related problem is to investigate whether two nonsimilar vertices in a graph can give rise to an isomorphic pair of vertex-deleted subgraphs. Such vertices are said to be pseudosimilar, and it is now part of the folklore of graph theory that this phenomenon was discovered as a flaw in a purported proof of the Reconstruction Problem. We discuss some results obtained in this area, focusing attention on the construction of large sets of pseudosimilar vertices and edges, and on the prospect of exploiting possible relationships between pseudosimilarity and the Reconstruction Problem.

We shall mostly follow the graph theoretic terminology of Harary (1969), the most notable exception being that here we use the terms vertex and edge instead of point and line respectively. All graphs considered are finite and simple.

2 THE RECONSTRUCTION PROBLEM

A *vertex-deleted subgraph* of a graph G is a subgraph $G-v$ obtained by deleting from G the vertex v and all the edges incident to it; similarly, an *edge-deleted subgraph* of G is a subgraph $G-e$ obtained by deleting the edge e. The *deck* of G, denoted by $\mathcal{D}G$, is the collection of all (unlabelled) vertex-deleted subgraphs of G; we note that $\mathcal{D}G$ is a multiset, in the sense that if two or more vertices of G give rise to isomorphic vertex-deleted subgraphs, then that particular subgraph appears in $\mathcal{D}G$ with multiplicity equal to the number of vertices to which it corresponds. The *edge-deck* of G, denoted by $\mathcal{ED}G$, is similarly defined to be the collection of all edge-deleted subgraphs of G. Figure 1.1 shows a graph and its deck.

Figure 2.1

A *reconstruction (edge-reconstruction)* of G is a graph H with $\mathcal{D}G = \mathcal{D}H$ ($\mathcal{ED}G = \mathcal{ED}H$). A graph G is *reconstructible (edge-reconstructible)* if every reconstruction (edge-reconstruction) of G is isomorphic to G. In other words, G is reconstructible (or edge-reconstructible) if it can be determined uniquely, up to isomorphism, from its deck (or edge-deck).

The graphs K_2 and $2K_1$ are nonisomorphic reconstructions of each other, therefore neither of these graphs is reconstructible. Also, the two graphs $2K_2$ and $K_1+K_{1,2}$ are edge-reconstructions of each other, as are the two graphs $K_{1,3}$ and K_1+K_3. Therefore these graphs, and any obtained from them by adding isolated vertices, are not edge-reconstructible. The two main conjectures in graph reconstruction assert that these are the only non-reconstructible graphs.

The Reconstruction Conjecture (Kelly, 1942; Ulam 1960). *All graphs on at least three vertices are reconstructible.*

The Edge-Reconstruction Conjecture (Harary, 1964). *All graphs on at least four* *edges are edge-reconstructible.*

Two important sources of information on these two conjectures are the survey articles by Bondy and Hemminger (1977) and Nash-Williams (1978). Reviews on some of the work done since these two surveys were published can be found in a number of more recent expository articles (Lauri, 1987; Ellingham, 1988; Manvel, 1988; Stockmeyer, 1988; Bondy, 1991).

It is an easy exercise to see that the number of edges and the degree sequence of G are reconstructible from $\mathcal{D}G$ (that is, any graph having the same deck as G has the same number of edges and the same degree sequence as G). This then easily gives that regular graphs are reconstructible — because from the degree sequence one obtains that G is regular of degree r, say, and therefore the only way to reconstruct from any given vertex-deleted subgraph in the deck is to add a new vertex and join it the r vertices of degree $r-1$.

A less trivial, but still not too difficult, task is to show that disconnected graphs are reconstructible. Probably the shortest and simplest proof of this is given by Manvel (1976).

In spite of much effort by a number of researchers, relatively few classes of graphs, apart from regular and disconnected graphs, have been shown to be reconstructible. The very first such result obtained was the reconstruction of trees (Kelly, 1957), a result which has been extended in various directions by several people (Harary and Palmer, 1966a; Bondy, 1969; Manvel (1970); Lauri, 1983; Krasikov, 1987; Harary and Lauri, 1987; Myrvold, 1990). We shall be looking at some aspects of the reconstruction of trees in Sections 4 and 8. Maximal planar graphs form one other important class of graphs which has been shown to be reconstructible (Fiorini and Lauri, 1981; Lauri, 1981). We shall be looking at maximal planar graphs in the next section.

Intuition suggests that edge-reconstruction is easier than reconstruction — more of the graph to be reconstructed is left in an edge-deleted subgraph than in a vertex-deleted subgraph. This is, in fact, true. Greenwell (1971) has shown that, if G is a graph without isolated vertices, then $\mathcal{D}G$ is reconstructible from $\mathcal{ED}G$, that is, the information given by the deck of G is all contained in the edge-deck. Therefore if G is reconstructible then it is also edge-reconstructible. Hence, regular graphs, disconnected graphs, trees and maximal planar graphs are all edge-reconstructible. Moreover, other classes of graphs have been shown to be edge-reconstructible when it has not yet been determined whether or not they are reconstructible. These include planar graphs with minimum degree 5 (Lauri, 1979), 4-connected planar graphs (Fiorini and Lauri, 1982), graphs which triangulate the real projective plane and graphs with connectivity 3 which triangulate some surface (Fiorini and Lauri, 1982), bidegreed graphs (Myrvold, Ellingham and Hoffman, 1987); claw-free graphs (Ellingham, Pyber and Yu, 1988) and hamiltonian graphs of sufficiently large order (Pyber, 1990).

However, the list of classes of reconstructible or edge-reconstructible graphs falls far short of exhausting all possibilities. If only regarded as step-by-step efforts at obtaining an ultimate proof of the reconstructibility of all graphs, then the outlook is bleak — the cases solved are few and the techniques used to tackle one class of graphs do not generalise to other classes. It seems hardly likely that by working laboriously in this fashion at successive classes of graphs one can ultimately prove the Reconstruction Conjectures for all graphs. So why do graph theorists persist in nibbling away at this mighty and unyielding problem?

Although the ultimately wish of any reconstructor is to resolve the conjectures one way or the other, even when his investigations are far short of this elusive aim they still might unearth questions about important graph-theoretic concepts which are interesting in their own right. A good illustration of some of the very attractive general results which have been obtained is the recent survey by Bondy (1991) which focuses its attention mainly on powerful counting techniques in the spirit pioneered by Lovász (1972), Müller (1977) and Nash-Williams (1978). An example of the application of such techniques will be given in Section 4. In the next section, by considering maximal planar graphs as an example, we shall try to show how an interesting study of the interplay between more specific graph-theoretic properties can arise while investigating the Reconstruction Problem.

3 MAXIMAL PLANAR GRAPHS

An "obvious" way to reconstruct a maximal planar graph G from its deck would run as follows. Take any subgraph $G-v$ with $\deg(v) > 3$ and embed it in the plane such that all faces except one are bounded by triangles, with the exceptional face being bounded by a cycle of length $\deg(v)$ (such an embedding will be called a *deg*(v)*-embedding* of $G-v$). Then add an extra vertex and join it to all the vertices on the cycle bounding the exceptional face.

This procedure, however, raises a number of basic questions. First of all, how can we tell from $\mathcal{D}G$ that G is maximal planar? (It is sufficient to be able to determine that G is planar since the number of edges is known.) Knowing that every vertex-deleted subgraph of G is planar is not sufficient since a graph could easily be critically nonplanar (that is, G is nonplanar but every $G-v$ is planar). This question was studied by Fiorini (1978) and Fiorini and Manvel (1978) for graphs with minimum degree at least 4. The case for minimum degree 3 is, in general, not completely resolved, but by restricting themselves to maximal planar graphs, Fiorini and Lauri (1981) solved the problem by showing, essentially, that a graph G with $3|V(G)|-6$ edges cannot, in fact, be critically nonplanar.

Theorem 3.1 (Fiorini and Lauri, 1981). *If G has minimum degree 3 and either it has at least two vertices of degree 3, its order is at least 7, and $|E(G)| = 3|V(G)| - 6$, or else it has a unique vertex of degree 3 whose neighbours induce a cycle, then G cannot be critically nonplanar.*

So now we know that the graph to be reconstructed is maximal planar. Would not a $\deg(v)$-embedding of some $G-v$ with $\deg(v) > 3$ easily complete the reconstruction as described above? The snag now lies with the possible non-uniqueness of the embedding — if $G - v$ does not have a a unique plane embedding then this procedure is not guaranteed to give us a unique reconstruction. Connectivity can provide a helping hand here. A classical result of Whitney (1932) says that a 3-connected planar graph has a unique embedding in the plane. Also, by a theorem of Chartrand, Kaugars and Lick (1972), a 3-connected graph G with minimum degree at least 4 contains some vertex v such that $G - v$ is still 3-connected. Therefore a maximal planar (hence 3-connected) graph G with minimum degree 4 has a vertex-deleted subgraph $G-v$ which is uniquely embeddable in the plane, and by the above procedure we can conclude that such graphs are reconstructible (Fiorini, 1978).

This leaves us with minimum degree 3. Connectivity does not help here — we could get a maximal planar graph G all of whose vertex-deleted subgraphs $G - v$ with $\deg(v) > 3$ are not 3-connected and not uniquely embeddable in the plane. However, since we know that the graph to be reconstructed is maximal planar, we need only consider $\deg(v)$-embeddings of $G - v$. Could it still be possible that no $G - v$ with $\deg(v) > 3$ has a unique $\deg(v)$-embedding? This can very well happen, and a smallest possible example (Lauri, 1981) of a maximal planar graph which presents this problem is shown in Figure 3.1.

Figure 3.1

Reconstruction of maximal planar graphs therefore ultimately rests with the consideration of graphs such as those in Figure 3.1. By a detailed study of the structure of such graphs coupled with the fact that the degree sequence of G is known from $\mathcal{D}G$ Lauri (1981) showed that such graphs are also reconstructible, hence completing the reconstruction of maximal planar graphs.

Theorem 3.1 led Fiorini and Lauri to make the following conjecture regarding triangulations of surfaces other than the plane (Research Problem 17, *Discrete Math.* bf 40 (1982), 125-126). In this conjecture, by a *k-embedding*, $k \geq 4$, of a graph G on a surface $\mathcal{S}$ (not necessarily the plane) we mean a 2-cell embedding of G on $\mathcal{S}$ in which every face except one is bounded by a triangle, the exceptional face being bounded by a cycle of length k. Theorem 3.1 effectively proves the conjecture for the plane.

Conjecture 3.1. *If G is a graph such that, for every vertex v of degree greater than 3, $G - v$ has a deg(v)-embedding on the surface $\mathcal{S}$, then G triangulates $\mathcal{S}$.*

4 RECONSTRUCTION FROM SUBDECKS

While it is in general difficult to prove that a certain class of graphs is reconstructible, reconstruction seems, in one sense, to be "easy" — in most of the results obtained, only a few of the subgraphs in the deck have actually been required to determine G uniquely. This situation is particularly exemplified by trees.

Since Kelly (1957) first showed that trees are reconstructible, various authors have obtained this result but using only those vertex-deleted subgraphs corresponding to specified vertices of the tree, such as, vertices of degree 1 (Harary and Palmer, 1966b), vertices at a maximum distance from the centre (Bondy, 1969) and vertices of degree at least 2 (Lauri, 1983). In fact, in the next section we shall see that not more than three vertex-deleted subgraphs are required to reconstruct a tree. In that section we shall see one way of formalising the notion of reconstructing a graph from as few subgraphs as possible. Here we shall look at a question which is suggested by Harary and Palmer's result for trees, and we shall use this as an opportunity to illustrate the powerful counting techniques introduced by Lovász, Müller and Nash-Williams to which brief reference was made in Section 2.

An *endvertex* is a vertex of degree 1. The collection of endvertex-deleted subgraphs of G will be called the *endvertex-deck* of G and will be denoted by $\mathcal{D}_1 G$; G will be called *endvertex-reconstructible* if it is uniquely reconstructible from its endvertex-deck, that is, if $\mathcal{D}_1 G = \mathcal{D}_1 H$ implies that G and H are isomorphic. Harary and Palmer have therefore shown that trees are endvertex-reconstructible. Motivated by this, Bondy (1969) conjectured that all graphs with sufficiently many endvertices are endvertex-reconstructible. The problem was investigated by various authors (Bondy, 1969; Greenwell and Hemminger, 1969; Krishnamoorthy and Parthasarathy, 1976) but the final *coup de grâce* was delivered by Bryant (1971) who showed that, for any k, there exist nonisomorphic graphs G and H with k endvertices each and with $\mathcal{D}_1 G = \mathcal{D}_1 H$.

However, although a graph with an arbitrarily large number of endvertices need not be endvertex-reconstructible, in some cases it happens that if the proportion of endvertices in the graph is sufficiently large then it would be endvertex-reconstructible. For example, if G is a graph with minimum degree at least 2 and G' is obtained from G by attaching one endvertex to each vertex of G, then G' is endvertex-reconstructible. We now show that, in this situation, if the number of endvertices added to G is greater than $\left|V(G)\right|/2$, then the resulting graph G' is endvertex-reconstructible. The proof involves the straightforward application of counting techniques which have now become standard in reconstruction.

Let G be a graph with minimum degree at least 2 and let $S \subseteq V(G)$. Then the graph obtained from G by attaching one endvertex to each vertex in S is denoted by $G[S]$. It is convenient to consider $G[S]$ as the graph G with labels on its vertices: the vertices in $V(G) - S$ are given the label 0 while each vertex in S is given the label 1. Considered this way, an endvertex-deleted subgraph of $G[S]$ is a labelled graph obtained from $G[S]$ by changing one of its positive labels to 0. Two labelled graphs are isomorphic if there is an automorphism of G which preserves labels.

In the sequel, S_1 and S_2 will always denote subsets of $V(G)$ with $|S_1| = |S_2|$. The set of all isomorphisms from $G[S_1]$ to $G[S_2]$ will be denoted by $(G[S_1] \longrightarrow G[S_2])$. For $X \subseteq S_1$, $(G[S_1] \xrightarrow{X} G[S_2])$ will denote the set of automorphisms α of G such that: (i) if $u \in S_1 - X$ then the label of $\alpha(u)$ in $G[S_2]$ equals the label of u in $G[S_1]$ and (ii) if $u \in X$ then the label of $\alpha(u)$ in $G[S_2]$ does not equal the label of u in $G[S_1]$. The orders of these sets are denoted by $\left|G[S_1] \longrightarrow G[S_2]\right|$ and $\left|G[S_1] \xrightarrow{X} G[S_2]\right|$ respectively.

The next lemma is the analogue of the well-known result commonly referred to as Kelly's Lemma, and Theorem 4.1 is the analogue of the Nash-Williams Lemma in edge-reconstruction. Theorem 4.2, a special case of a theorem given by Lauri (1992a), is the analogue of Lovász's and Müller's results in edge-reconstruction.

Lemma 4.1. *Let $S_3 \subset S_1 \subseteq V(G)$ and let $\mathcal{D}_1 G[S_1] = \mathcal{D}_1 G[S_2]$. Then*

$$s(G[S_3], G[S_1]) = s(G[S_3], G[S_2])$$

where $s(G[S_i], G[S_j])$ denotes the number of subgraphs of $G[S_j]$ isomorphic to $G[S_i]$.

Proof. Let r be the total number of graphs isomorphic to $G[S_3]$ which appear as subgraphs of the graphs in $\mathcal{D}_1 G[S_1]$ (and hence $\mathcal{D}_1 G[S_2]$). Let p be the sum of the labels in $G[S_1]$ of all the vertices in $S_1 - S_3$. Clearly $p > 0$ since $|S_3| < |S_1|$ and the labels are positive. Then

$$r = p \cdot s(S(G[S_3], G[S_1]) = p \cdot s(S(G[S_3], G[S_2])$$

and the result follows since $p > 0$. □

Corollary 4.1. *If $\mathcal{D}_1 G[S_1] = \mathcal{D}_1 G[S_2]$ and $S_3 \subset S_1$ then*

$$\big|G[S_3] \longrightarrow G[S_1]\big| = \big|G[S_3] \longrightarrow G[S_2]\big|.$$

Proof. This follows easily from Lemma 4.1 since,

$$\begin{aligned} \big|G[S_3] \longrightarrow G[S_1]\big| &= s(G[S_3], G[S_1]) \cdot \big|G[S_3] \longrightarrow G[S_3]\big| \\ &= s(G[S_3], G[S_2]) \cdot \big|G[S_3] \longrightarrow G[S_3]\big| \\ &= \big|G[S_3] \longrightarrow G[S_2]\big|. \end{aligned}$$

□

Lemma 4.2. *Let $X \subseteq S_1$. Then*

$$\big|G[S_1] \xrightarrow{X} G[S_2]\big| = \sum_{Y \subseteq X} (-1)^{|Y|} \big|G[(S_1 - X) \cup Y] \longrightarrow G[S_2]\big|.$$

Proof. For $u \in X$ let A_u denote the set $(G[(S_1 - X) \cup \{u\}] \longrightarrow G[S_2])$. Note that

$$\cap_{i=1}^{r} A_{u_i} = (G[(S_1 - X) \cup_{i=1}^{r} \{u_i\}] \longrightarrow G[S_2]).$$

Since $\big|G[S_1] \xrightarrow{X} G[S_2]\big| = \big|G[S_1 - X] \longrightarrow G[S_2]\big| - \big|\cup_{u \in X} A_u\big|$, the result follows by applying the inclusion-exclusion principle. □

Theorem 4.1. *Let $\mathcal{D}_1 G[S_1] = \mathcal{D}_1 G[S_2]$, and let $X \subseteq S_1$. Then*

$$\begin{aligned} &\big|G[S_1] \longrightarrow G[S_2]\big| = \\ &\quad \big|G[S_1] \longrightarrow G[S_1]\big| + (-1)^{|X|}\big(\big|G[S_1] \xrightarrow{X} G[S_2]\big| - \big|G[S_1] \xrightarrow{X} G[S_1]\big|\big). \end{aligned}$$

Proof. By Lemma 4.2,

$$\left|G[S_1] \xrightarrow{X} G[S_2]\right| = \sum_{Y \subseteq X} (-1)^{|Y|} \left|G[(S_1 - X) \cup Y] \longrightarrow G[S_2]\right|$$

and

$$\left|G[S_1] \xrightarrow{X} G[S_1]\right| = \sum_{Y \subseteq X} (-1)^{|Y|} \left|G[(S_1 - X) \cup Y] \longrightarrow G[S_1]\right|.$$

Subtracting these two equations, all terms on the right hand side cancel (by Corollary 4.1) except for $Y = X$, giving the required result. □

Corollary 4.2. *Let $\mathcal{D}_1 G[S_1] = \mathcal{D}_1 G[S_2]$ and suppose that $G[S_1]$ is not isomorphic to $G[S_2]$. Let $X \subseteq S_1$. Then (i) if $|X|$ is odd, then $\left|G[S_1] \xrightarrow{X} G[S_2]\right| > 0$, and (ii) if $|X|$ is even, then $\left|G[S_1] \xrightarrow{X} G[S_1]\right| > 0$.*

Proof. Since $G[S_1] \not\simeq G[S_2]$, $\left|G[S_1] \longrightarrow G[S_2]\right| = 0$. Therefore when $|X|$ is odd, $\left|G[S_1] \xrightarrow{X} G[S_2]\right| = \left|G[S_1] \longrightarrow G[S_1]\right| + \left|G[S_1] \xrightarrow{X} G[S_1]\right| > 0$, and, when $|X|$ is even, $\left|G[S_1] \xrightarrow{X} G[S_1]\right| = \left|G[S_1] \longrightarrow G[S_1]\right| + \left|G[S_1] \xrightarrow{X} G[S_2]\right| > 0$. □

Theorem 4.2 *If either $|S| > |V(G)|/2$ or $|S| > 1 + \log_2 |AutG|$, then $G[S]$ is endvertex-reconstructible.*

Proof. Suppose $G[S]$ is not endvertex-reconstructible and let $G[S']$ be an endvertex-reconstruction of $G[S]$, not isomorphic to $G[S]$. Then taking $X = S$ in Corollary 4.2 implies that there is a $T \subseteq V(G)$ disjoint from S such that $G[(T)] \simeq G[S]$, if $|S|$ is even, or $G[(T)] \simeq G[S']$, if $|S|$ is odd. But this is impossible if $|S| > |V(G)|/2$.

Also, if $G[S]$ is not endvertex-reconstructible then, by Corollary 2(ii), for every even subset X of S, $\left|G[S] \xrightarrow{X} G[S]\right| \geq 1$. There are $2^{|S|-1}$ even subsets of S and, since the sets $(G[S] \xrightarrow{X} G[S])$ are disjoint for different X, it follows that $|\mathrm{Aut}G| \geq 2^{|S|-1}$. Therefore if $|S| > 1 + \log_2 |\mathrm{Aut}G|$, then $G[S]$ is endvertex-reconstructible. □

(**Note:** The second condition of Theorem 4.2 can also be obtained as a special case of Corollary 2.4 of Alon et al., (1989).)

Bryant's counterexamples to Bondy's conjecture are, in fact, graphs with endvertices no two of which have a common neighbour and none adjacent to a vertex of degree 2. In view of this and of Theorem 4.2, the natural question to ask is: If G' is obtained from a graph G of minimum degree at least two by attaching one endvertex to each of k vertices of G, what is the largest value of $|k| \,/\, |V(G)|$ which can give a graph G' which is not endvertex-reconstructible?

The survey by Bondy (1991) gives a comprehensive treatment of the counting techniques associated with the Nash-Williams Lemma.

5 RECONSTRUCTION NUMBERS

When research on the Reconstruction Problem seemed to be slowing down, Harary and Plantholt (1985) came up with the idea of reconstruction numbers. Although their concept actually makes reconstruction more difficult, it has managed to bring to light several questions which, although still retaining sufficient complexity to make them worthy of investigation, seem more tractable than the apparently unscalable heights of the Reconstruction Conjectures.

The *reconstruction number* $rn(G)$ of G is defined to be the minimum number of vertex-deleted subgraphs in $\mathcal{D}(G)$ which can determine G uniquely, that is, $rn(G)$ is the size of the smallest subcollection of $\mathcal{D}(G)$ which is not contained in any other $\mathcal{D}(H), H \not\simeq G$.

The simplest observation one can make about reconstruction numbers is that $rn(G) > 2$ for any graph G. This can easily be seen as follows. Suppose $G - u$ and $G - v$ are two saubgraphs from $\mathcal{D}(G)$. Construct H as follows: if $uv \in G$ then $H = G - uv$ otherwise $H = G + uv$. Then $H \not\simeq G$ but $G - u, G - v \in D(H)$. Therefore $G - u$ and $G - v$ cannot alone distinguish between G and H.

By means of a computer search amongst all graphs of order at most 7, Harary and Plantholt (ibid.) found that only five of these graphs, together with their complements, had reconstruction number greater than the minimum possible value of 3. This lead them to make, amongst a number of other conjectures, the conjecture that almost every graph has reconstruction number equal to 3. (We say that almost every graph has a certain property if the proportion of graphs on n vertices which have the property tends to 1 as n tends to ∞.)

Myrvold demonstrated a simple proof of Harary and Plantholt's conjecture by exploiting an earlier result of Müller. Bollobás also proved the conjecture in essentially the same way but obtaining an independent (and simpler) proof of Müller's result.

Theorem 5.1 (Myrvold, 1988; Bollobás, 1990). *Almost every graph has reconstruction number 3.*

We defer presenting the ideas involved in the proof of this theorem to a later stage when we use essentially the same methods to obtain an analogous result for edge-reconstruction numbers (Theorem 5.3 below).

Although the reconstruction of regular graphs is trivial and that of disconnected graphs quite straightforward, the determination of their reconstruction numbers is not so easy. Work on this has been done by Myrvold. We here summarise some of her results and point to some of the immediate outstanding questions which emerge.

Myrvold (1989) showed that the reconstruction number of disconnected graphs equals 3, except in the case when the connected components are all isomorphic. In this case she showed that $rn(G) \leq c+2$, where $c \geq 3$ is the order of a connected component of G. This upper bound is attained when G consists of k copies of the complete graph K_c, since any $c+1$ vertex-deleted subgraphs of G are also vertex-deleted subgraphs of the graph which consists of $k-2$ copies of K_c and one copy of each of K_{c-1} and K_{c+1}. Some questions still remain here, mainly: (i) are these the only disconnecetd graphs for which the reconstruction number equals $c+2$? (ii) can one characterise those other disconnected graphs for which the reconstruction number is greater than 3?

Noticing that, when G consists of just two copies of K_c, $rn(G) = 2 + n(G)/2$, brings to mind an early conjecture made by Harary and Plantholt.

Conjecture 5.1 (Harary and Plantholt, 1985). *For any graph G,*

$$rn(G) \leq \frac{n(G)}{2} + 2$$

and equality holds iff G is a path on four vertices, or two copies of K_c or the complete bipartite graph $K_{c,c}$.

Of course, a proof of this conjecture would be quite remarkable since it includes the Reconstruction Conjecture as a special case.

For r-regular graphs G of order n Myrvold (1988) showed that

$$rn(G) \leq \min\{r+3, n-r-2\} \leq \lfloor n/2 \rfloor + 2.$$

Again, the maximum is attained by $K_{c,c}$ or $2K_c$.

In order to make the study of reconstruction numbers more manageable the following weakening of the problem has been considered by various authors. Let C be a class of graphs and let $G \in C$. Then the *class reconstruction number* $Crn(G)$ is defined to be the smallest number of cards of G which can determine G given that $G \in C$. That is, $Crn(G)$ is the size of the smallest subcollection of $D(G)$ which is not contained in any other $D(H), H \not\simeq G, H \in C$. As a trivial example we note that if C is the class of regular graphs then $Crn(G) = 1$ for any $G \in C$.

From the discussion in Section 3 it should be clear that, if C is the class of maximal planar graphs, then $Crn(G) = 1$ unless G is a graph such as that shown in Figure 3.1, that is, unless G the property that no $G - v$ with $\deg(v) > 3$ has a unique $\deg(v)$-embedding. By studying the degrees of the vertex-deleted subgraphs of such graphs, Harary and Lauri (1987) showed that their class reconstruction number is always equal to 2, thereby determining completely the class reconstruction number of maximal planar graphs.

Bange, Barkauskas and Host (1987) exploited properties of total graphs to show that their class reconstruction number is equal to 1. Harary and Lauri (1988) showed that the class reconstruction number of trees is at most 3, and this result was improved by Myrvold (1990) who showed that the reconstruction number of trees is 3. However, the following conjecture is still unresolved.

Conjecture 5.2 (Harary and Lauri, 1988). *Let C be the class of trees and let T be a tree. Then $Crn(T) \leq 2$.*

The *edge-reconstruction number* and the *class edge-reconstruction number* (denoted by $ern(G)$ and $Cern(G)$, respectively) of a graph G are defined in a manner analogous to the corresponding reconstruction numbers. But while the Edge-Reconstruction Conjecture is a weaker conjecture than the Reconstruction Conjecture, there does not seem to be any straightforward relationship between reconstruction and edge-reconstruction numbers. In fact, the latter often seem to be more difficult to determine.

Unlike the case for reconstruction numbers, $ern(G)$ could be equal to 1. This is so if and only if there is an edge $e \in E(G)$ such that the complement of $G - e$ is edge-transitive. Also, $ern(G)$ could equal 2, for example, when $G = K_{p,q}$. In fact, almost every graph has edge-reconstruction number equal to 2. We shall now show this using the same methods which have been employed to obtain Theorem 5.1. We first need a couple of definitions and a preliminary lemma.

A graph will be said to have *property P_k* if $G - X \not\simeq G - Y$ for any two distinct subsets X, Y of $V(G)$ with $|X| = |Y| = k$. A graph will be said to have *property EP_k* if $G - A \not\simeq G - B$ for any two distinct subsets A, B of $E(G)$ with $|A| = |B| = k$. The set of neighbours of a vertex v in G is denoted by $N_G(v)$.

Lemma 5.2. *If G has property P_{13} then it has property EP_3.*

Proof. Suppose, for contradiction, that A and B are two distinct subsets of $E(G)$ with $|A| = |B| = 3$ and such that $H = G - A$ and $K = G - B$ are isomorphic, and let $\alpha : H \to K$ be an isomorphism. Let A' and B' be the sets of vertices incident in G to A and B respectively. Let $S' = A' \cup \alpha^{-1}(B')$ and $T' = \alpha(A') \cup B'$. Then $|S'| = |T'| \leq 12$, and $H - S' \simeq K - T'$. But $H - S' = G - A$ and $K - T' = G - B$, therefore if $S' \neq T'$ we are done. We may therefore assume that $S' = T'$; let us call this set R. Note that $\alpha(R) = R$.

Now suppose that α does not act trivially on $V(G) - R$, and let $\alpha(x) = y, x \neq y, x, y \in V(G) - R$. Then $H - R - x \simeq K - R - y$, that is, $G - R - x \simeq G - R - y$, giving us the required contradiction.

We may therefore assume that α acts trivially on $V(G) - R$. Note that α cannot also act trivially on R, since there are some pairs of vertices in R adjacent in H but not in K. Therefore let $x, y \in R, \alpha(x) = y$. Since α acts trivially on $V(G) - R$, it follows that $N_H(x) \cap (V(G) - R) = N_K(y) \cap (V(G) - R)$. Therefore $H - (R - x) \simeq K - (R - y)$. But $H - (R - x) = G - (R - x)$ and $K - (T' - y) = G - (T - y)$, again giving us the required contradiction. □

(**Note:** The number 13 in the above lemma is sufficient for our purposes but it is certainly a very crude estimate. Also, the same proof applies for $|A| = |B| = k$ with property P_{13} replaced by P_{4k+1}.)

The next result was first obtained by Korshunov (1971), and then independently by Müller (1976) who used it to show that almost every graph is reconstructible. Bollobás (1990) subsequently gave another independent proof.

Theorem 5.2 (Korshunov, 1971; Müller, 1976; Bollobás, 1990). *For any fixed k, almost every graph has property P_k.*

Lemma 5.3. *Let G have property EP_3. Then G is reconstructible from any two edge-deleted subgraphs from its edge-deck.*

Proof. Let $G - a, G - b$ be two edge-deleted subgraphs of G. Find an edge x of $G - a$ and an edge y of $G - b$ such that $(G - a) - x \simeq (G - b) - y$. Then $x = b$ and $y = a$. Now let $\alpha : (G - a) - b \to (G - b) - a$ be an isomorphism. This isomorphism is unique because otherwise there would be, in $G - a - b$, edges e, f, $e \neq f$, such that $G - a - b - e \simeq G - a - b - f$, contradicting the property EP_3. We can therefore use α to label uniquely all the vertices in $(G - a) - b$ and $(G - b) - a$. We can therefore reconstruct G uniquely by adding to $G - a$ a new edge joining the two vertices which are joined by a in $G - b$. □

Corollary 5.1. *Almost every graph has edge-reconstruction number 2.*

However, it could happen that $ern(G) > rn(G)$. For example, if G consists of two copies of the star $K_{1,p}$, then $ern(G) = p + 2$ and $rn(G) = 3$. The situation is not made any clearer by considering class edge-reconstruction numbers. Harary and Lauri (1988), for example, have pointed out that the six trees shown in Figure 5.1 have class edge-reconstruction numbers equal to 3, although they conjecture that the class reconstruction number of any tree is at most 2 (and of course, the six trees shown here have class reconstruction number equal to 2); it is not known whether these six trees are the only ones with class edge-reconstruction number greater than 2.

Figure 5.1

It seems therefore that little is known about edge-reconstruction numbers. We would suggest, as the most interesting problems to investigate first, the study of the edge-reconstruction numbers of disconnected and regular graphs and of trees, and the class edge-reconstruction numbers of trees and maximal planar graphs.

We close this section with a problem which has been posed by Myrvold (1988). The reconstruction number can be considered as a game between players **A** and **B**. **A** is shown a graph G and she has to find the smallest number of vertex-deleted subgraphs which, when given to **B** will enable her to reconstruct G uniquely. This smallest number is the reconstruction number. In this game **A** and **B** are allies. We can consider an analogous game in which **A** and **B** are adversaries, where now **A** is required to find the largest number of subgraphs no proper subcollection of which will enable **B** to reconstruct G uniquely. In this vein Myrvold has defined the *adversary reconstruction number* $Arn(G)$ of a graph G to be one more than the maximum number of vertex-deleted subgraphs that G has in common with any graph not isomorphic to it. Therefore the basic question here is: What is the maximum number of vertex-deleted subgraphs that two nonisomorphic graphs on n vertices can have?

The following family of pairs of graphs, found by Myrvold, gives the nonisomorphic pairs which have the largest known number of common vertex-deleted subgraphs.

$$G = (p+1)K_p \cup (p-1)K_{p+1} \quad \text{and} \quad H = K_{p-1} \cup (p-1)K_p \cup pK_{p+1}.$$

The graphs G and H have $(p+1)(2p-1)$ vertices each and $p(p+1)$ vertex-deleted subgraphs in common, that is, n vertices and $n/2+\sqrt{(n+1.125)/8}+0.375$ vertex-deleted subgraphs in common. Myrvold therefore asks if there is a family of graph pairs on n vertices having more than this number of vertex-deleted subgraphs in common. She also makes the following two conjectures. As for Conjecture 5.1 above, a proof for any of them would be quite remarkable since each implies the Reconstruction Conjecture. However, for those who prefer disproving rather than proving conjectures, these are of course more promising candidates to tackle than the original Reconstruction Conjecture.

Conjecture 5.3 (Myrvold, 1988). *The number of vertex-deleted subgraphs that two nonisomorphic graphs on n vertices can have in common is at most $n/2 + O(\sqrt{n})$.*

Conjecture 5.4 (Myrvold, 1988). *Two nonisomorphic graphs on n vertices with the same degree sequence can have at most $(n+1)/2$ vertex-deleted subgraphs in common.*

6 PSEUDOSIMILARITY

Graph theorists seem to have stumbled on the concept of pseudosimilarity quite by accident. If two vertices u and v in a graph G are similar, that is, there is an automorphism of G which maps one into the other, then it is clear that $G - u$ and $G - v$ are isomorphic graphs. However the converse is not true, because $G - u$ and $G - v$ can be isomorphic without u and v being similar in G. An example of this is given by the graph in Figure 6.1. Nobody seems to have given this phenomenon any thought until, as reported by Harary and Palmer (1966), someone apparently "proved" the Reconstruction Conjecture using the "fact" that if $G - u$ and $G - v$ are isomorphic then u and v must be similar. To Harary and Palmer goes the credit of taking what could simply have remained a curious counter-example, and turning it into a graph theoretic concept worthy of investigation. Their 1965 and 1966 papers proved the first results and set the scene for further studies. In more than twenty-five years which have passed since the Harary and Palmer papers, a number of authors have found new results and discovered more problems.

Figure 6.1

As in Harary (1969), two vertices u and v of G are *similar* if there exists some automorphism α of G such that $\alpha(u) = v$; they are *removal-similar* if $G - u$ and $G - v$ are isomorphic, and they are *pseudosimilar* if they are removal-similar but not similar.

Although at first it might seem unexpected that pseudosimilar vertices do exist, upon further consideration one comes to realise that, in fact, such vertices should be a natural occurrence. The simplest way to create such vertices in a graph is possibly the following. Let G be a graph and let u and v be a pair of similar vertices in G. If u and v are adjacent in G, then let $H = G - uv$, whereas if they are not let $H = G + uv$. The vertices u and v are still removal-similar in H but the addition or deletion of the edge uv could have destroyed their similarity, making them pseudosimilar. One necessary condition for this to happen is that there is no automorphism α of G such that $\alpha(u) = v$ and $\alpha(v) = u$. This sets the scene for the investigation of why pseudosimilar vertices arise. We call two sets of vertices A and B in a graph G *interchange similar* if there is an automorphism α of G such that $\alpha(A) = B$ and $\alpha(B) = A$.

Theorem 6.1. *Let u, v be pseudosimilar vertices in a graph G, and let $A = N(u) \cap V(G - u - v)$, $B = N(v) \cap V(G - u - v)$. Then either A and B are similar but not interchange similar in $G - u - v$ or else $G - v$ contains a vertex pseudosimilar to u.*

Proof. Let $\alpha\colon G - v \to G - u$ be an isomorphism. If $\alpha(u) = v$ then restricting α to $V(G) - u - v$ gives an automorphism mapping A into B. Of course, A and B cannot be interchange similar in $G - u - v$ as otherwise u and v would be similar in G.

We therefore assume that $\alpha(u) \neq v$. Let $w = \alpha^{-1}(v)$; therefore $w \neq u$. It now follows that u and w are removal similar in $G - v$, for $(G - v) - w \simeq \alpha((G - v) - w) = (G - u) - v = (G - v) - u$.

Now suppose that u and w are similar in $G - v$. Let β be an automorphism of $G - v$ with $\beta(u) = w$. Then $\beta\alpha$ is an isomorphism from $G - v$ to $G - u$ with $\beta\alpha(u) = v$. We therefore again obtain, as above, that there is an automorphism of $G - u - v$ mapping A into B.

The only alternative left is that u and w are not similar in $G - v$, giving that $G - v$ contains a vertex pseudosimilar to u. □

This theorem easily gives, as a corollary, two results about trees, the first of which was one of the earliest results on pseudosimilarity. We first need to recall the following result on similar vertices in a tree. Recall also that an *endvertex* is a vertex of degree 1 and an *end-cutvertex* in a tree is a vertex having only one neighbour with degree greater than 1.

Theorem 6.2 (Prins 1957). *Any two similar vertices in a tree are interchange similar.*

Corollary 6.1. (Harary and Palmer, 1966) *(i) Any two removal-similar vertices in* *a tree are similar.* (Kirkpatrick, Klawe and Corneil) *(ii) Any two removal-similar endvertices in a tree are similar.*

Proof. We shall prove (i) by induction on the number of vertices of the tree T; the proof of (ii) is analogous. Suppose u, v are pseudosimilar endvertices of T. Let x, y be the neighbours of u, v respectively. Since, by the induction hypothesis, u cannot be pseudosimilar to any endvertex in $T-v$, it follows that x and y are similar but not interchange similar in $T-u-v$. But this contradicts Prins' Theorem. Therefore u and v cannot be pseudosimilar. □

So it seems that one way to try and obtain pseudosimilar vertices is to take a graph with two sets of similar but not interchange similar sets of vertices, and join two new vertices to them. Harary and Palmer (1966), by exploiting this idea, gave the first systematic way of constructing pairs of pseudosimilar vertices. Take any graph H, and let X and Y be two sets of vertices of H such that there is no automorphism α of H with $\alpha(X)=Y$ (for example, choose X and Y with $|X|\neq|Y|$). Then take three copies H_1,H_2,H_3 of H and form G by adding two new vertices u and v joining u to X in H_1 and to Y in H_2, and v to X in H_2 and Y in H_3. Then u and v are pseudosimilar in G. The graph in Figure 6.1 could have been constructed this way, with $H=K_2$, $X=V(H)$ and Y containing only one vertex. Godsil and Kocay (1982), by formalising these ideas, obtained a construction which explains in general how pairs of pseudosimilar vertices arise.

But the area where most of the unanswered questions about pseudosimilarity lie is the situation where a graph has several pseudosimilar vertices. We can ask for two ways in which this can happen. One way requires the graph to have several pairs of pseudosimilar vertices. Alternatively one can ask for the graph to have a large set of mutually pseudosimilar vertices (the vertices in a subset S of $V(G)$ are said to be mutually pseudosimilar if any two vertices of S are removal-similar but no two are similar). These two situations are discussed in the next section.

7 LARGE SETS OF PSEUDOSIMILAR VERTICES AND EDGES

We shall first give those definitions and results involving graphs and groups upon which most of the material in this sections depends.

Let G be a vertex-symmetric graph (that is, all of its vertices are similar) such that $\mathrm{Aut}(G)$ acts regularly on $V(G)$ (this means that the stabiliser of any vertex of G is the identity, or, equivalently, that $|\mathrm{Aut}(G)|=|V(G)|$). Then G is called a *graphical regular* *representation (GRR)* of Aut(G). Therefore GRR's are graphs which just manage to make it as vertex-symmetric graphs. We shall be requiring GRRs for groups of odd order, and it follows that such groups must be nonabelian (see Biggs (1974), for example). The following result assures us of the existence of GRR's.

Theorem 7.1 (Hetzel, 1976; Godsil, 1980) *Except for a finite number of known groups, all finite, nonabelian groups which are not generalised dicyclic groups have GRR's.*

Although given any group Γ there is a graph whose automorphism group is abstractly isomorphic to Γ, it is not true that every permutation group is equivalent to the group of a graph. However it is possible to obtain a graph with an automorphism group whose action on a subset of the vertices of the graph is equivalent to a given permutation group. This is what the next theorem says, a short proof of which can be found in Problem 12.21 of the book by Lovász (1979).

Theorem 7.2 (Bouwer, 1969). *Let Γ be a permutation group acting on a set X. Then there exists a graph G such that $X \subseteq V(G)$, X is invariant under the action of AutG and the restriction of AutG to X gives a permutation group equivalent to Γ.*

Kimble, Schwenk and Stockmeyer (1981) were the first to consider the problem of graphs with several pseudosimilar vertices. In their paper they gave a number of concrete examples of the following result.

Theorem 7.3 (Kimble, Schwenk and Stockmeyer, 1981). *There exist graphs in which every vertex has a pseudosimilar mate.*

Proof. Let Γ be a group of odd order and let H be a GRR of Γ (as we have noted above, Γ must be nonabelian and, by Theorem 7.1, such Γ and H do exist). We note that H is a regular graph and that the stabiliser of any vertex is just the identity element of Γ. Therefore, if r is any vertex of H, then $G = H - r$ has the identity automorphism group.

Now, let v be any vertex in G. There is an automorphism α of H mapping r to v. The vertices $\alpha^{-1}(r)$ and $v = \alpha(r)$ are distinct, because otherwise α would contain a cycle of length 2, which is impossible since Γ has odd order. Since α^{-1} maps $\{v, r\}$ onto $\{r, \alpha^{-1}(r)\}$, it follows that $G - v = H - r - v \simeq H - \alpha^{-1}(r) - r = G - \alpha^{-1}(r)$, that is, $v = \alpha(r)$ and $\alpha^{-1}(r)$ are removal-similar in G. But G has the identity automorphism group, therefore v and $\alpha^{-1}(r)$ are pseudosimilar. $\square$

Pseudosimilar edges can be defined in an analogous way to pseudosimilar vertices. Kimble (1981) tried to obtain a result like Theorem 7.3 for pseudosimilar edges, and he managed to construct a sequence of graphs in which the proportion of edges which have a pseudosimilar mate tends to 1. The following is a much simpler construction. Let C_1 and C_2 be two directed cycles, each on n vertices, n odd. Let G_n be a constructed as follows: Join every vertex of C_1 to every vertex of C_2 and replace each arc of C_1 by Gadget 1 and each arc of C_2 by Gadget 2, as shown in Figure 7.1; delete one of the edges joining a vertex of C_1 to a vertex of C_2.

Figure 7.1

The resulting graph G_n has the identity automorphism group, and each edge joining a pair of vertices from C_1 and C_2 has a corresponding edge to which it is removal-similar. Therefore G has $n^2 - 1$ edges which have pseudosimilar mates, and a total of $n^2 + 9n - 1$ edges. Therefore the proportion of edges which have a pseudosimilar mate tends to 1 as n tends to ∞.

But it is perhaps when investigating graphs with large sets of mutually pseudosimilar vertices that the most interesting open questions arise. This problem has been investigated by a number of authors (Krishnamoorthy and Parthasarthy, 1976; Kimble, Schwenk and Stockmeyer, 1981; Kimble, 1981; Kocay, 1982 and 1984; Godsil and Kocay, 1983; Lauri and Marino, 1991; Lauri, 1992a). It is clear that a graph G cannot have all its vertices mutually pseudosimilar. Otherwise G would be regular and a regular graph cannot have pseudosimilar vertices because if α is an isomorphism from $G - u$ to $G - v$, then α must map the neighbours of u into the neighbours of v, and therefore it can be extended to an automorphism of G mapping u into v. Therefore the main question which arises is to determine the largest size which a set of mutually pseudosimilar vertices in a graph of order n can have. This seems to be very difficult to settle, and below we shall consider a more restricted version of the question.

The first to construct graphs with large sets of mutually pseudosimilar vertices were, independently, Krishnamoorthy and Parthasarathy (1976) and Kimble, Schwenk and Stockmeyer (1981). The latter gave a method which constructs a sequence of graphs G_k having k mutually pseudosimilar vertices and a total of $O(k^2)$ vertices. Krishnamoorthy and Parthasarathy constructed a sequence G_k having 2^k mutually psuedosimilar endvertices and a total of $O(3^k)$ vertices. We shall now give a slightly more general construction as described by Lauri and Marino (1991) and Lauri (1991), which can be used to give a better "packing" of mutually pseudosimilar vertices in graph.

Let G' be a graph containing r endvertices, all of which are mutually pseudosimilar. Let G be the graph obtained from G' by removing all of its endvertices, and let R be the set of neighbours of the endvertices of G' (note that since no two endvertices are similar, no two can share a common neighbour, therefore $|R| = r$). Let X be the set of all those vertices of G which are similar to some vertex in R under the action of Aut$G$. We now construct a sequence of graphs G_t, $t = 1, 2, \ldots$, containing r^t mutually pseudosimilar endvertices. Let $G_1 = G'$ and let H_1 be G_1 less one of its endvertices. Having constructed G_t, let H_t be G_t less one of its pseudosimilar endvertices. Then, G_{t+1} is obtained by attaching a copy of G_t to each vertex in R and a copy of H_t to each of the other vertices in $X - R$. (By attaching a copy of G_t (or H_t) to a vertex v of G we mean joining v to every vertex of G_t (or H_t) which is not an endvertex.)

Each graph G_t so obtained has r^t mutually pseudosimilar endvertices and $O(|X|^t)$ vertices. Therefore if $k = r^t$ is the number of pseudosimilar endvertices, then the total number of vertices in G_t is $O(k^{\frac{\log|X|}{\log|R|}})$.

The crucial step in this construction is to find the starting graph G', that is, one with endvertices all of which are mutually pseudosimilar. Suppose Γ is a group of permutations acting on some set X such that, for some $R \subseteq X$, the following two conditions hold: (i) the setwise stabiliser $\Gamma_{\{R\}}$ of R is the identity and, (ii) for any two $(|R| - 1)$-subsets A, B of R, there is a permutation α in Γ such that $\alpha(A) = B$. Then, by employing Theorem 7.2, one can construct a graph G with minimum degree at least 2 and $X \subseteq V(G)$ and whose automorphism group has the same action as Γ on X. Therefore if we attach one endvertex to each vertex of $R \subset V(G)$ we obtain the starting graph G' all of whose endvertices are mutually pseudosimilar. Hence such starting graphs can be constructed if permutation groups satisfying conditions (i) and (ii) are found.

Lauri (1992a) constructed such a permutation group with $|X| = 2|R| = 8$ as follows. Let Γ be the group of affine transformations of the field $GF(8)$. Although this group is not 3-transitive, it is 3-homogeneous (Livingstone and Wagner, 1965), that is, any two 3-sets are similar under the action of Γ on the set of 3-subsets of $GF(8)$. Therefore all we need is a 4-set R such that $\Gamma_{\{R\}}$ is the identity. If we represent $GF(8)$ as $Z_2[x]/p(x)$, where $p(x)$ is the primitive, irreducible (over Z_2) polynomial $x^3 + x + 1$, and if we let $R = \{0, 1, x, x^2\}$, then one can check that the only permutation in Γ which fixes R setwise is, in fact, the identity. Therefore the required permutation group has been obtained. The construction can then proceed as above, giving graphs G_t with $k = 4^t$ pseudosimilar endvertices and a total of $O(k^{3/2})$ vertices (Lauri, ibid.) and this therefore gives a denser "packing" of mutually pseudosimilar vertices in a graph than do the constructions of Krishnamoorthy and Parthasarthy or of Kimble, Schwenk and Stockmeyer.

These constructions suggest two problems. The first is to construct a sequence of graphs having k mutually pseudosimilar (end)vertices and a total of $O(k^{1+\epsilon})$ vertices, with ϵ as small as possible. The second problem is to construct permutation groups satisfying the above conditions (i) and (ii).

Lauri (1992a) presented the following construction of such permutation groups which is a slight simplification of one given by Cameron (1991). Let $X = F^{r-1}$, the vector space of dimension $r-1$ over the finite field F, and let Γ be the group of all linear automorphisms of X. Let $B = \{e_1, e_2, \ldots, e_{r-1}\}$ be a basis of X, $f = \sum a_i e_i$ an element of X and $R = B \cup \{f\}$. Suppose the following conditions on the a_i hold:

(1) $a_i \neq 0$, $1 \leq i \leq r-1$;
(2) $a_i \neq a_j, i \neq j$;
(3) $a_i a_j \neq 1$, $1 \leq i, j \leq r-1$;
(4) $a_i + a_j a_k \neq 0$, $1 \leq i, j, k \leq r-1$.

Then Γ and R have the required properties (i) and (ii). For, since none of the a_i is zero, any two $r-1$-subsets of R are bases of X, therefore similar under the action of Γ. Hence condition (ii) holds. Also, if, for some $\alpha \in \Gamma$ not equal to the identity, $\alpha(R) = R$ then, since the a_i are distinct, we cannot have $\alpha(B) = B$ and $\alpha(f) = f$; therefore, for some j, $\alpha(e_j) = f$, $\alpha(f) = e_{\pi(j)}$ and $\alpha(e_i) = e_{\pi(i)}$ for $i \neq j$, where π is a permutation of $\{1, 2, \ldots, r-1\}$. But then, since $f = \sum a_i e_i$, we have

$$e_{\pi(j)} = \sum_{i \neq j} a_i e_{\pi(i)} + a_j f.$$

This gives that $a_i + a_j a_{\pi(i)} = 0$ and $a_j a_{\pi(j)} = 1$, contradicting (3) and (4). Therefore the only permutation α with $\alpha(R) = R$ is the identity, and hence condition (i) also holds.

As discussed above, such a permutation group can be used to construct, via Theorem 7.2, a graph G all of whose endvertices are mutually pseudosimilar. Such a graph can be transformed, as follows, into one containing a set of mutually pseudosimilar vertices which are not endvertices. Let Δ be the maximum degree of G. Identify the endvertices of G with distinct vertices of the complete graph K_Δ. In the resulting graph, the vertices which were endvertices in G are still mutually pseudosimilar.

8 PSEUDOSIMILARITY AND RECONSTRUCTION

Is there any concrete relationship between pseudosimilarity and the Reconstruction Problem? Stockmeyer's (1977) construction of non-reconstructible digraphs depends on tournaments which have the property that every vertex has a pseudosimilar mate. However, there seems to be no way of adapting this construction to undirected graphs (Stockmeyer, 1988). On the other hand, trees are reconstructible from their endvertex-deleted subgraphs and also from their end-cutvertex-deleted subgraphs (Krasikov 1987, 1988) and, as we have seen, endvertices and end-cutvertices cannot be pseudosimilar in a tree. Is this a coincidence, or could one prove, as asked by Krasikov (1988), that a graph G with a "sufficiently large" set S of non-pseudosimilar vertices is reconstructible from its subgraph $G-v$, $v \in S$? A good place to start investigating this might be trees — it could be revealing if one could prove that trees are endvertex-reconstructible using mainly the similarity properties of endvertices in a tree. We shall briefly outline here a programme for doing this.

First we note that the Reconstruction Conjecture can be re-worded in a way that brings out better possible connections with pseudosimilarity. Let G be a non-reconstructible graph, and let H be a reconstruction of G, $H \not\simeq G$. Let u be a vertex of G. Then $G-u$ is isomorphic to a vertex-deleted subgraph of H. Therefore $H \simeq (G-u)+v$, where the vertex v is joined to vertices of $G-u$ which are, of course, not all neighbours of u. Let $K = G+v$, that is, K is obtained from $(G-u)+v$ by putting u back and joining it to the neighbours it had in G. Since $\mathcal{D}G = \mathcal{D}H$, for every vertex $x \in V(K) - \{u, v\}$ there is a vertex $x' \in V(K) - \{u, v\}$ (x' could be equal to x) such that $K-u-x \simeq K-v-x'$. Let us call the pair of vertices u, v which have this property in K *complementary vertices.* From these comments one easily sees that the Reconstruction Conjecture is equivalent to the following conjecture.

Conjecture 8.1. *If u and v are complementary vertices in a graph K, then $K-u \simeq K-v$.*

We now turn our attention more specifically to trees. In his investigation of the reconstruction of trees Krasikov (1991) found that the following theorem is very useful in proving reconstruction results for trees. First we need a definition. Let T be a tree $a, b \in V(T)$, and A, B two rooted trees. Then $T_{a,b}(A, B)$ denotes the tree obtained by identifying the roots of A and B with a and b respectively.

Theorem 8.1 (Krasikov, 1991). *If A and B are non-isomorphic rooted trees and $T_{a,b}(A, B) \simeq T_{a,b}(B, A)$, then a and b are similar in T.*

It is easily seen that this generalises Corollary 6.1 (let one of A or B be a single vertex). It turns out that, in order to tackle more fully the reconstruction of trees one needs the following theorem. (The definition of $T_{a,b,c}(A, B, C)$ is analogous to that of $T_{a,b}(A, B)$.)

Theorem 8.2 (Lauri, 1992b). *If A, B and C are mutually non-isomorphic rooted trees and $T_{a,b,c}(A,B,C) \simeq T_{a,b,c}(C,A,B)$, then a, b, c are similar in one of T, $T_{a,b,c}(A,A,A)$, $T_{a,b,c}(B,B,B)$ or $T_{a,b,c}(C,C,C)$.*

Combining Theorem 8.2 with the formulation of the reconstruction problem as in Conjecture 8.1 one can obtain the endvertex-reconstruction of trees (Lauri, 1992b).

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