

## Some open problems

This is a set of open problems, mostly from graph theory. Please email me if you wish to have some more background information on any of these problems.

1. *With S. Fiorini.* Let  $S$  be a closed surface (orientable or non-orientable) and let  $H$  be a graph. For  $k \geq 4$  a  $k$ -embedding of  $H$  on  $S$  is a 2-cell embedding of  $H$  on  $S$  in which every face except one is bounded by a 3-cycle, the exceptional face being bounded by a  $k$ -cycle. Clearly, if  $G$  triangulates  $S$  then, for any vertex  $v$  of  $G$  with  $\deg(v) \geq 4$ ,  $G - v$  has a  $\deg(v)$ -embedding on  $S$ . Is the converse true? That is, *if  $G$  has the property that, for any vertex  $v$  with degree at least 4,  $G - v$  has a  $\deg(v)$ -embedding, then does  $G$  triangulate  $S$ ?* Fiorini and Lauri have proved that this is true when  $S$  is the sphere (that is,  $G$  is maximal planar). This proof makes essential use of Kuratowski's Theorem.
2. *With K. Asciak.* The *reconstruction number*  $rn(G)$  of a graph  $G$  is the minimum number of vertex-deleted subgraphs which are sufficient to reconstruct  $G$  uniquely up to isomorphism. Myrvold and Molina have proved that if  $G$  is disconnected and not all its components are isomorphic, then  $rn(G) = 3$ . So we assume here that  $G$  is made up of a disjoint union of copies of a graph  $H$  and we assume also that  $H$  has  $c$  vertices. Myrvold has observed that if  $H = K_c$ , the complete graph on  $c$  vertices, then  $rn(G) = c + 2$ . Asciak has proved the converse, and in fact that if  $rn(G) \geq c + 1$  then  $H = K_c$ . This means that there is no disconnected graph with all components isomorphic and having  $c$  vertices and such that  $rn(G) = c + 1$ . The problem therefore is this: *Let  $G = \cup H$  with  $|V(H)| = c$ . Determine the number  $g = g(c)$  such that if  $rn(G) \geq g$  then  $G$  is a union of copies of  $K_c$  but there is such a  $G$ , with  $H \neq K_c$ , such that  $rn(G) = g - 1$ .* The above result shows that  $g \leq c$ .
3. *With K. Asciak.* Asciak has also shown that if  $G$  is an  $r$ -regular graph (that is, all vertices have degree  $r$ ) and  $rn(G) = r + 3$ , then  $G$  is a union of copies of the complete graph  $K_{r+1}$ . Problem: *Determine the number  $f = f(r)$  such that, given an  $r$ -regular graph  $G$ , if  $rn(G) \geq f$  then  $G$  is a union of copies of  $K_{r+1}$ , but there exists a different  $r$ -regular graph  $G$  with  $rn(G) = f - 1$ .* Asciak's result shows that  $f \leq r + 2$ . Is the problem made easier if posed for vertex-transitive graphs?
4. *Asciak.* The *edge-reconstruction number*  $ern(G)$  of a graph is analogously defined. Asciak has shown that if  $G$  is an  $r$ -regular graph

then  $ern(G) \leq r + 2$ , but he conjectures: *If  $G$  is a regular graph then  $ern(G) \leq 2$ .* Is the problem made easier if we assume that  $G$  is vertex-transitive, or edge-transitive, or arc-transitive?

5. *With C.Dandria.* A graph  $H$  is said to be a *topological minor* of  $G$  if  $G$  contains a subgraph which is a subdivision of  $H$  (that is, obtained by inserting vertices of degree 2 in the edges of  $H$ ).  $G$  is said to be  $H$ -critical if it contains a subdivision of  $H$  but, for all  $v \in V(G)$ ,  $G - v$  does not. Let  $\delta_T(H)$  be the largest possible minimum degree of a graph  $G$  which is  $H$ -critical. Certainly,  $\delta_T(H)$  exists for any  $H$  and (by a result of Szemerédi) is at most  $cm^2 \log m + 1$ , although the value of  $\delta_T(H)$  would seem to be much less than this in fact. It is easy to prove that  $\delta_T(K_4) = 3$ . Mitchem has shown that  $\delta_T(K_{2,3})$  also equals 3, and Fiorini has shown that  $\delta_T(K_{3,3}) = 4$ . Dandria has shown that  $\delta_T(K_{2,4}) = 4$ . She asks: *Are there constants  $c, t_0$  such that, for  $t > t_0$ ,  $\delta_T(K_{2,t}) \leq c$ .* She also asks: *What is  $\delta_T(K_5)$ ? In general, find the value of, or bounds for,  $\delta_T(K_n)$ .*
6. *M.A. Francalanza.* The *adversary reconstruction number* of a graph  $G$  is equal to one more than the number of vertex-deleted subgraphs which  $G$  can have in common with any other graph  $H$ . This number has been studied mostly by Wendy Myrvold. She considered the adversary reconstruction number of trees and she suggested the study of the maximum number of vertex-deleted subgraphs which a caterpillar can have in common with a sunshine graph (this is a unicyclic graph such that, removing all the vertices on the unique cycle results in a graph consisting only of isolated vertices). The edge adversary reconstruction number is analogously defined. Francalanza conjectures: *A caterpillar and a sunshine graph can have at most  $\frac{n}{3} + 1$  edge-deleted subgraphs in common.* She also gives an example showing that this bound is attained.
7. *With F. Harary.* Let  $\mathcal{C}$  be a class of graphs and let  $G \in \mathcal{C}$ . The *class reconstruction number*  $ern(\mathcal{C}; G)$  is defined to be the least number of vertex-deleted subgraphs which can determine  $G$  uniquely given the information that  $G$  is in  $\mathcal{C}$ . For example, the class reconstruction number for regular graphs is clearly equal to 1. The *edge class reconstruction number* is similarly defined in terms of edge-deleted subgraphs. Harary and Lauri have shown that if  $\mathcal{C}$  is the class of trees and  $T$  is any tree, then  $rn(\mathcal{C}; T) \leq 3$ . Myrvold strengthened this to  $rn(T) = 3$ , but in many of the cases which Harary and Lauri considered,  $rn(\mathcal{C}; T)$  turned

out to be 2. They, in fact, conjecture: *The class reconstruction number of trees is at most 2.*

8. Harary and Lauri also demonstrated six trees whose edge class reconstruction number is 3. They ask: *Are these the only trees with edge class reconstruction number greater than 2?*
9. Two vertices  $u, v$  in a graph  $G$  are said to be *pseudosimilar* if  $G - u$  and  $G - v$  are isomorphic but there is no isomorphism of  $G$  mapping  $u$  into  $v$ . A set  $S$  of vertices in  $G$  are said to be mutually pseudosimilar if any pair of vertices in  $S$  are pseudosimilar. *Is there a sequence of graphs  $\langle G_k \rangle$  such that each  $G_k$  has  $k$  mutually pseudosimilar vertices and  $|V(G_k)| = O(k)$ ? The best result obtained so far is a sequence with  $|V(G)| = O(k^{3/2})$ .*
10. This problem is motivated by a problem in computational molecular biology. A long string (which can be circular) of DNA or RNA is sometimes analysed as follows. It is treated with an enzyme which splits the string into smaller fragments at specified points. It is also treated with another enzyme which fragments the string at different points. It is then also treated with both enzymes which splits the string at both types of points. The problem is then to identify the original string from the fragments. This suggests the following idealised combinatorial problem. Let  $C$  be a circle and suppose there is a finite number of points identified on the circle. These points are of two types, Type A and Type B. Suppose that points of different type alternate around the circle (otherwise, the answer to the question below would be no). Suppose you are given the following information: (i) The lengths of the arcs between successive points of Type A, (ii) the lengths of the arcs between successive points of Type B, (iii) the length of all the arcs joining a point of Type A to a successive point of Type B (with the appropriate A to B orientation) and similarly (iv) the lengths of all arcs joining points to Type B to successive points of Type A. *Is the distribution of points round the circle reconstructible (up to orientation or reflection) from this information, or can one produce two essentially different distribution of points with the same set of information (i) to (iv)?* One can modify the problem to make it more difficult to reconstruct the original circle (or more easy to find two different circles with the same fragment information) by lumping together the information in (ii) and (iii) and not giving the orientation (A to B or B to A) of these fragments.

11. *With R. Scapellato.* Does there exist a  $p$ -group  $\Gamma$ ,  $p$  an odd prime, (or even a nilpotent group) having two subgroups  $\mathcal{H}$  and  $\mathcal{K}$  with the following properties (i)  $\mathcal{H} \cap \mathcal{K} = \{1\}$ ; (ii)  $\Gamma = \langle \mathcal{H} \cup \mathcal{K} \rangle$ ; and (iii) no nontrivial automorphism of  $\Gamma$  fixes  $\mathcal{H} \cup \mathcal{K}$  setwise?

**Note:** If  $\Gamma$  is not a  $p$ -group and not nilpotent either, then finding such subgroups is possible. For example, let  $\Gamma$  be the group

$$\langle a, w, c \mid a^5 = w^3 = c^{31} = 1, wa = awc, ca = ac^2, cw = wc^{25} \rangle.$$

Then it is easy to check that  $\mathcal{H} = \langle a \rangle$  and  $\mathcal{K} = \langle w \rangle$  have the required properties.