# Archimedes

Joseph Muscat

last revised 15 Sep 2019

## 1 The Sand Reckoner

"There are some, king Gelon, who think that the number of the sand is infinite in multitude; or that, if finite, no number has been named which is great enough to exceed its multitude."

### 1.1 Accepted Magnitudes

[Let ☾ represent the Moon's diameter, $\oplus$ the Earth's, $\odot$ the Sun's, ♈ the Universe's; let also M represent a *myriad* which is a hundred hundred.]

Most astronomers accept that the Sun's diameter is less than ten times the Moon's, though some take it to be twenty times; let us accept that it is less than thirty times to avoid dispute. All accept that the Moon's diameter is less than the Earth's, so

$$\odot < 30☾ < 30\oplus$$

It is also accepted that the Sun's angle is $\frac{1}{2}^{\circ}$, as can be easily verified by placing a disk to cover the Sun at sunrise and taking note of its distance from the eye. However, this is the angle subtended at the Earth's surface; at the centre of the Universe it would be slightly larger, but not by much, and certainly less than a thousandth part of the whole zodiac. Thus

$$3♈ < \text{zodiac} < 1000\odot < 3\text{M}\oplus$$

$$♈ < \text{M}\oplus$$

It is accepted by geometers that Earth's circumference is at most 300 myriad stadia, so

$$\oplus < 100\text{Mstadia}$$

Finally, let us agree that a poppy seed contains at most a myriad sand grains, a finger width is not more than forty poppy seeds, and that a stadium is less than a myriad fingers.

## 1.2 New Numbers

Traditionally, numbers have been named up to a myriad myriad. Let us call these, numbers of the 1st order. Take the myriad myriad as a *unit*, so one can talk about, say, a thousand of these units; we will call these numbers of the 2nd order. Similarly, a myriad myriad of the 2nd order will be a unit of the 3rd order, and so on. One can build numbers of higher orders up to the myriad myriadth order. Even this can be considered a unit $\pi$ of the first period. Then comes the 2nd period, the 3rd, etc.up to the myriad myriadth period. Thus to specify a number one can talk about the number of units of a certain order of a certain period.

How to multiply these numbers? Consider the geometric series

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 10 | 100 | 1000 | M | 10M | 100M | 1000M | MM | 1 MM | 10 MM |

It is straightforward to find the position of a number of a certain order, and conversely. Every order ends at a position which is a multiple of 8; thus the 20th position is the 4th position in the 3rd order, that is, a thousand of the 3rd order.

To multiply two numbers in this series, add their positions and subtract one.

The reason is that, in a geometric series, the ratio of any two numbers that are equally distant is the same, e.g. the ratio of the myriad (5th position) to ten (2nd position) is the same as ten myriad myriad (11th position) to a hundred myriad (8th position). So the ratio of the number which occupies the position of their sum minus one, to one of them, is the same as the ratio of the other number to the first, namely 1. Hence this number is the product of them.

### 1.2.1 The size in the Universe

Whenever a solid is expanded by a ratio, its size is expanded by the ratio thrice over. So

$$\begin{aligned}
\text{sphere of finger width} &< 40 \times 40 \times 40 \text{ poppy seeds} \\
&< 64000 \text{ myriad sand grains} \\
&< 10 \text{ myriad myriad sand grains} \\
&= 10 \text{ of 2nd order} \\
&= 10\text{th position}
\end{aligned}$$

If the sphere is expanded a myriad times, the magnitude increases in the ratio of a myriad $\times$ myriad $\times$ myriad, which is a 5th times a 5th times a 5th; i.e., a 9th times a 5th, which is in the 13th position.

$$
\begin{array}{rcl}
\text{sphere of diameter a stadium} & < & \text{sphere of diameter a myriad fingers} \\
 & < & 13\text{th} \times 10\text{th position} = 22\text{nd position} \\
\text{sphere of diameter a myriad stadia} & < & 13\text{th} \times 22\text{nd position} = 34\text{th position} \\
\text{sphere of diameter a MM stadia} & < & 13\text{th} \times 34\text{th position} = 46\text{th position} \\
\text{sphere of universe} & < & \text{sphere of diameter a hundred MM stadia} \\
 & < & 100 \text{ myriad times } 46\text{th position} \\
 & = & 7\text{th} \times 46\text{th position} \\
 & = & 52\text{nd position} \\
 & = & 1000 \text{ of } 7\text{th order of sand grains.}
\end{array}
$$

Now, Aristarchos claimed that the Universe is much larger, namely that the stars are as far away compared to the Sun as what we call the universe is larger than the Earth,

$$* : \Upsilon = \Upsilon : \oplus < \text{myriad} : 1$$

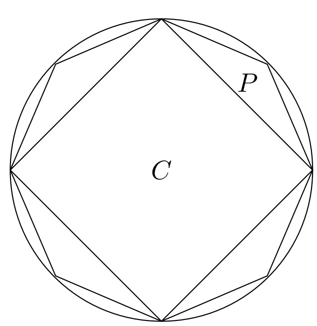
His universe's diameter is a myriad times our estimate of the universe, so the magnitude of his universe would be

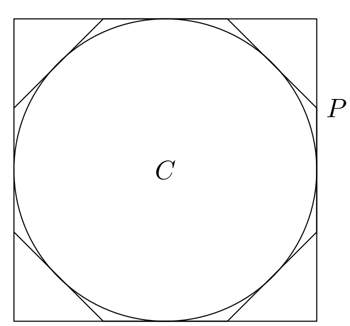
$$
\begin{array}{rcl}
\text{sphere of diameter a myriad } \Upsilon & < & 13\text{th} \times 52\text{nd position} \\
 & = & 64\text{th position} = 100 \text{ myriad of } 8\text{th order unit.}
\end{array}
$$

## 2 Measurement of a Circle

*Proposition* 2.1
**The area of a circle is that of a right triangle with sides equal to the radius and the circumference of the circle.**

PROOF: Let the area of the said triangle be $T$ and of the circle $C$. Suppose $C > T$. Inscribe the circle with a regular polygon (by bisection) until the its area $P$ is greater than $T$. The inner radius of the inscribed polygon is less than the circle's radius, and its perimeter less than the circumference (by axiom). Hence its area which is a sum of triangles is less than $T$, which is less than $P$, a contradiction.



Suppose $C < T$. Circumscribe the circle with a regular polygon until its area $P$ is greater than $C$ but less than $T$. The radius of the circle is less than that of the polygon, and its circumference is less than the perimeter, hence $P > T$, a contradiction.

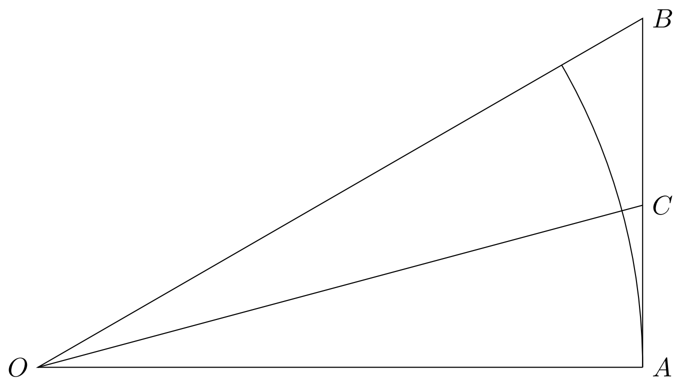
□

*Proposition* 2.2
**The ratio of the circumference to the diameter is between $223 : 71$ and $22 : 7$.**

Proof: Starting with a third of a right angle, continue bisecting. Then the ratio $OB : AB = 2 : 1$, and the ratio of the height to the radius $OA : AB$ is greater than $265 : 153$. For each bisection, $OB : OA = BC : AC$ (Euclid VI.3), so $OB + OA : OA = AB : AC$ and $OB + OA : AB = OA : AC$. Hence

$$OA : AC = OA + OB : AB > 571 : 153$$

Therefore $OC^2 : AC^2 > (571^2 + 153^2) : 153^2 = 349450 : 23409$, so $OC : AC > 591\frac{1}{8} : 153$. Continue like this for four bisections to get $OA : AG > 4673\frac{1}{2} : 153$; the angle is now the 48th of a revolution, so the ratio of the diameter to the perimeter is more than $4673\frac{1}{2} : 153 \times 48 \times 2 = 4673\frac{1}{2} : 14688 > 7 : 22$.



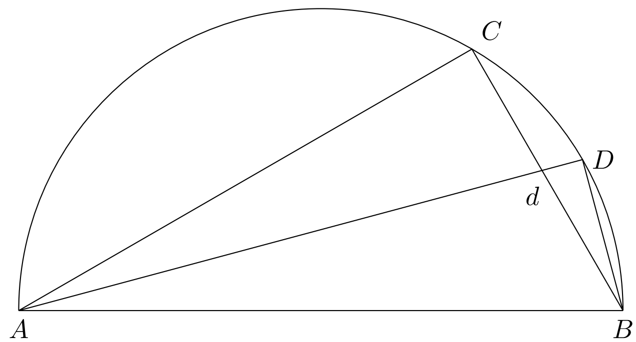

On a semicircle with diameter $AB$, start with a point $C$ at an angle of 30°. Then $AB : BC = 2 : 1$ and $AC : BC < 1351 : 780$. Repeatedly bisect the angle $BAC$. Then $AD : BD = BD : Dd = AB : Bd = AC : Cd$ by similar triangles, so equal to

$$AD : BD = AB + AC : Bd + Cd = AB + AC : BC,$$

i.e., $AD : BD = AB + AC : BC < 2911 : 780$. So

$$AB^2 : BD^2 = AD^2 + BD^2 : BD^2 < (2911^2 + 780^2) : 780^2$$

and $AB : BD < 3013\frac{3}{4} : 780$. Continue like this four times to get $AB : BG < 2017\frac{1}{4} : 66$. The angle $BAC$ is the 48th of a revolution, so the 24th at the centre; hence the ratio of the perimeter to the diameter is more than $6336 : 2017\frac{1}{4} > 223 : 71$.

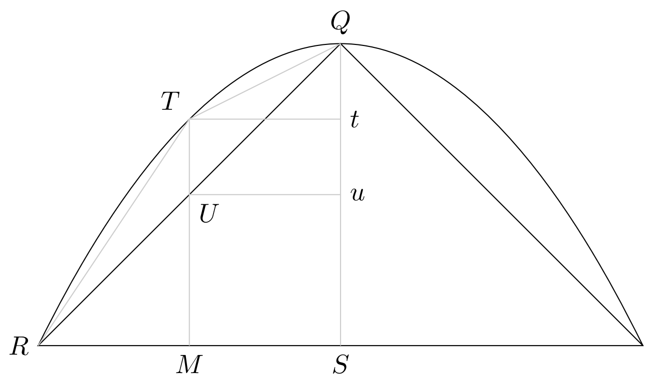



$\square$

# 3 Quadrature of the Parabola

"Though many have tried to square the circle or the ellipse, it is recognised by most people that the problem has not been solved. However I have discovered the solution of finding the area of a parabolic segment."

*Proposition* 3.1
**The area of a parabolic segment in relation to the triangle with same base and height is** $4:3$**.**




PROOF: Let $P$ be the parabolic segment and $A$ the first inscribed triangle. One can keep on inscribing triangles with parallel diameters and bisecting the base. Let $Q$ be the vertex, $Rr$ the base, $S$ the midpoint of the base, $M$ the midpoint of $RS$, $T$ the vertex of the second triangle, $U$ the midpoint of its base, $t, u$ points on the main diameter with the same heights as $T, U$.

The 'height' $TU$ of each smaller triangle is a fourth of that of the preceding one $QS$. For, $QS : Qt = SR^2 : SM^2 = 4 : 1$; but $Qu = uS$ hence $2Qu = QS = 4Qt$, so $Qt = tu$ and $2TU = UM$. Therefore each smaller triangle is eight times smaller than the preceding one, since $QRr = 2QRS = 4QRM = 8QTR$. Thus the addition of each new area is a fourth of the preceding area. That is, there is a geometric progression of reducing areas.

For such areas, $A + B + \cdots + Z + \frac{1}{3}Z = \frac{4}{3}A$. The reason is that $A = 4B$, $B = 4C$, so $B + \frac{1}{3}B = \frac{1}{3}A$, etc. Thus

$$(B + \tfrac{1}{3}B) + (C + \tfrac{1}{3}C) + \cdots + (Z + \tfrac{1}{3}Z) = \tfrac{1}{3}A + \tfrac{1}{3}B + \cdots + \tfrac{1}{3}Y$$

so cancelling the common terms gives $B + C + \cdots + Z + \frac{1}{3}Z = \frac{1}{3}A$.

Now suppose that the area $P > \frac{4}{3}A$. Then inscribe enough triangles $V$ until $\frac{4}{3}A < V < P$. By the above, $V + \frac{1}{3}Z = \frac{4}{3}A$ gives a contradiction.

Suppose that $P < \frac{4}{3}A$. Again inscribe enough triangles that the last triangle $Z < \frac{4}{3}A - P$. Then $V + \frac{1}{3}Z = \frac{4}{3}A > P + Z > V + Z$, another contradiction.

□

## 4 On the Equilibrium of Planes

Postulates:

1. Equal weights at equal distances are in equilibrium, otherwise the farther weight inclines down.

2. If a configuration of weights is in equilibrium, then adding any weight to one side makes it incline downwards; equivalently, removing any weight from one side inclines it upwards.

3. Congruent figures have congruent centres of gravity; similar figures have similar centres of gravity.

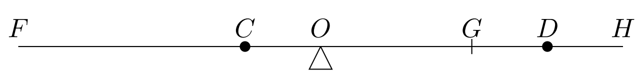
4. The centre of gravity of a convex figure is within the figure.

An immediate consequence is that unequal weights at the same distance are not in equilibrium, or equivalently, if they balance then the lighter weight is farther away (otherwise remove the difference).

Another consequence of Postulate 1 is that given an even number of equal weights separated by equal distances on a straight line, their centre of gravity is the midpoint of the two central weights. Otherwise, if the number is odd, the middle weight is the centre of gravity (remove the middle weight and a similar situation to an even number of weights is achieved).

*Proposition* 4.1
**Two weights balance when the ratio of their distances is inversely proportional to their weights.**

PROOF: Let commensurable weights $A > B$ be situated at points $C, D$ on a line, let $O$ be that point such that $OD : OC$ equals $A : B$. Let $E$ be a common measure of $OC, OD$. Let $F, G, H$ be points such that $GD = DH = OC$ and $CF = CG = OD$. Then $FG$ and $GH$ contain $E$ both an even number of times, say $n$ and $m$ respectively. Divide $A$ by $n$ and $B$ by $m$, to give smaller weights $I, J$ such that $A = nI$, $B = mJ$. As $A : B = FG : GH = n : m$, $I = J$, and so $A = nI$, $B = mI$, $FG = nE$, $GH = mE$. Distribute the weights of $A$ equally along $FG$ (one $I$ for each $E$), and those of $B$ along $GH$. Then their centres of gravity remains at $C, D$ respectively. But together they form a set of weights equally distributed along $FH$. Their centre of gravity is at their midpoint which is $O$ since $FO = FC + CO = OD + DH = OH$.

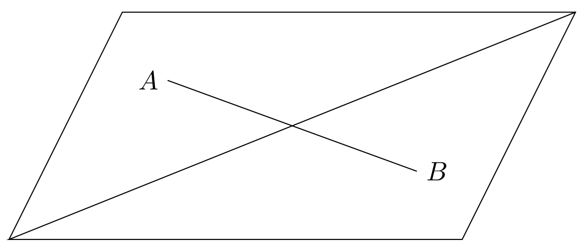



Suppose the weights $A', B$ are incommensurable and that $O$ is not the centre of gravity, say $A'$ is too great to balance $B$. Then it would be possible to remove a weight from $A'$ such that the remaining weight $A$ is commensurable with $B$ and is still too great to balance. Then since they are commensurable and don't balance, and $A : B$ is smaller than $A' : B = OD : OC$, then by the first part of the proof, $D$ will depress. But not enough weight has been removed for this to happen. Similarly, if $A'$ is too small to balance $B$.

□

*Corollary* 4.2
**If a figure has centre of gravity at $O$ and consists of two parts with weights $A, B$ with centres of gravity at $C, D$ respectively, then $OD : OC$ equals $A : B$.**

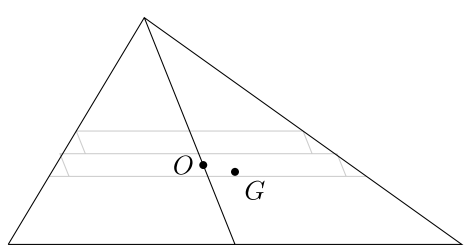
*Proposition* 4.3
**The centre of gravity of a parallelogram is the point of intersection of its diagonals.**




Proof: Cut the parallelogram by a diagonal. Since the triangles are congruent, their centres of gravity $A, B$ are on corresponding points. Join the line $AB$, so the centre of gravity of them together lies on their midpoint, which is the midpoint of the diagonal.

□

*Proposition* 4.4
**The centre of gravity of a triangle is the intersection point of its medians.**

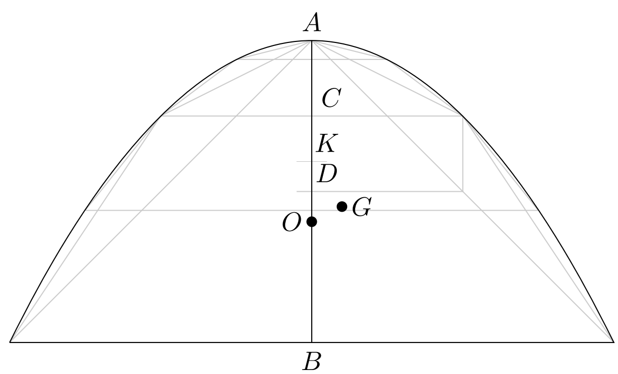



Proof: Suppose the centre of gravity, $G$, is not on its median, but is off by a distance $\epsilon$ of half the base. Divide the median into $n$ equal parts, draw parallels to the base through them, so that each part is less than $\epsilon$ of half the base. Thus $n : 1 > 1 : \epsilon$. Thus the triangle consists of a figure $A$ of parallelograms and the remaining part $B$ of small triangles. The centre of gravity of $A$ lies on the median since that of each parallelogram does so; call it $O$. Each triangle in $B$ is similar to a half of the main triangle, so the ratio of their areas is $1 : n^2$ and since there are $n$ of them, $n : n^2$. Thus $1 : n = B : T$. Now $T$ balances $A$ and $B$ together; if their centres of gravity are $G, O, H$ respectively, then $GH : GO = A : B = (n-1) : 1$. But this implies that $HO : GO = n : 1 > 1 : \epsilon$, so that $H$ is outside the triangle, contradicting the fourth postulate.

□

It follows that the centre of gravity of a trapezium can be found by subtracting a triangle from another. In particular it lies on the line joining the midpoints of the parallel sides.

*Proposition* 4.5
**The centre of gravity of a parabolic segment lies on its diameter and divides it in the ratio of $3:2$.**




Proof: Let $AB$ be the diameter of the parabolic segment $P$, and $G$ its centre of gravity.

Bisect and insert triangles as shown. The segment divides into two parts: $A$ consisting of the triangles, and $B$ the remainder. Part $A$ can also be thought of as consisting of trapezia, so their centres of gravity lie on the diameter. Moreover, the parallel lines cut the diameter in a ratio of $1 : 3 : 5 : 7 : \cdots$ since they are the differences of the 'vertical' lengths, which are as $1 : 4 : 9 : 16 : \cdots$ when compared to the 'horizontal' lengths.

Suppose $G$ is not on the diameter but is at a distance of $\epsilon$ of half the base away. Then perform the above bisections until the area of $B$ is less than $\epsilon$ of that of $P$. Let the centres of gravity of $P, A, B$ be $G, O, H$. Then $P$ balances $A, B$, so $GH : GO > (1-\epsilon) : \epsilon$, so $HO$ is greater than half the base, which means that it is outside $P$, a contradiction. Hence $G$ lies on the diameter.

Let $S$ be the part of the parabolic segment not including the first triangle $T$. It consists of two parabolic segments which have centres of gravity similar to that of $P$. Mark points $K, C, D$ on the main diameter such that $K$ is level with the centres of gravity of $S$, $C$ is level with the vertices of $S$, and $D$ level with the midpoints. We get the following conditions:

1. $O$ is at the intersection of the medians of $T$, so $AO = 2OB$.
2. The diameters of $S$ are in $1:4$ ratio of the diameter of $P$, so $AD = DB = 2AC$. These imply that $2OB = AO = AD + OD = 2OD + OB$, so $2OD = OB$.
3. The parabolic segments are similar, so $CD : KD = AB : GB$, hence $AC : KD = 4AC : GB$ and $4KD = GB = OG + OB$.
4. $T$ and $S$ balance $P$, so $GK : GO = T : S = 3 : 1$ (from quadrature).

Therefore, $4OG = OK = KD + DO = (OG + 2OB)/4$; simplifying gives $OB = 5OG$. Thus the line $AGOB$ has ratios $9 : 1 : 5$, hence $AG : GB = 9 : 6 = 3 : 2$.

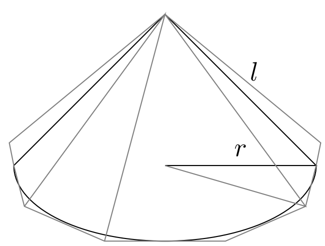
$\square$

# 5 On Spheres and Cylinders

*Proposition* 5.1

**The ratio of the area of a right cone to its base circle is the same as the ratio of the side to the base radius.**

PROOF: Let the cone have radius $r$, sloping side $l$, base area $C$, and sloping area $D$. Circumscribe the cone with a polygon and pyramid with areas $P, Q$ respectively. Then $P : Q = r : l$ (since triangles of same base); but $P : Q < P : D$ (cone inside pyramid) $< \lambda C : D$ (polygon can approximate circle).




Inscribe a polygon and pyramid. The area ratio $P : Q < r : l$ (since angle of pyramid triangle is more than cone's); and $P : Q > P : C > \lambda^{-1}C : D$. Since $\lambda$ is arbitrarily close to 1, $C : D = r : l$.

$\square$

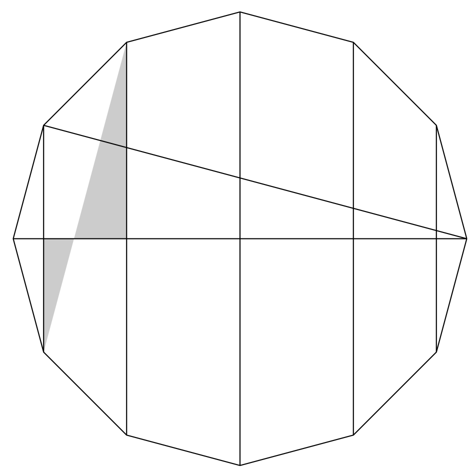
This is equivalent to saying that the area of $D$ is that of a circle with radius $\sqrt{rl}$, or to $lp/2$ where $p$ is the perimeter of the base circle.

*Corollary* 5.2
**The area of a slice of cone is the rectangle of the sloping side and the mean of the perimeters.**

PROOF: A slice is the difference of two cones. The ratio of their sides is the same as that of their perimeters. Hence one's side times the other's perimeter are the same. So subtracting the side times perimeter gives $l_2p_2 - l_1p_1 = l_2p_2 - l_1p_2 + l_2p_1 - l_1p_1 = (l_2 - l_1)(p_1 + p_2)$.

□

*Lemma* 5.3
**In a regular polygon with an even number of sides, the ratio of the sum of the parallels to the diameter is the same as that of the off-diagonal to a side.**



PROOF: Draw a second set of parallels, rotated by a side. These cut the diameter forming similar triangles, as in the shaded example. So the ratio of the perpendicular sides of these triangles is constant and equal to the ratio of the off-diagonal to a side (by similar triangles). Hence also their sum; but the sum of one set of sides forms a diameter.

□

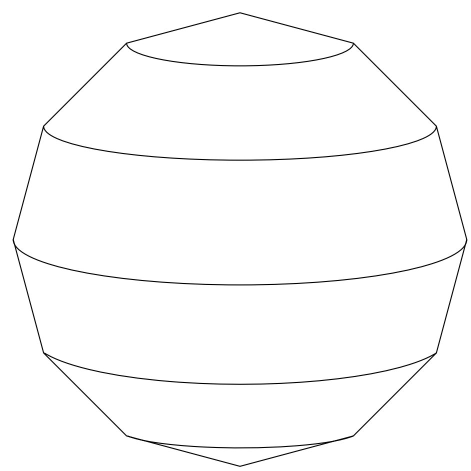
Note that if less parallels are added then we get that the ratio of the sum of the parallels with half the last parallel to the height of the segment equals the ratio of the off-diagonal to a side.

*Proposition* 5.4
**The surface [area] of a sphere is four times its base circle.**

PROOF: Circumscribe the circle and sphere by a regular even polygon and form its surface of revolution $S$. Then the area of $S$ is the sum of cone slices.

Each cone slice has area equal to a side times the mean perimeters; so all together the area is a side times the sum of the perimeters on the parallels. Now these perimeters bear the same ratio to their diameters [namely $\pi$], which are the parallels of the previous lemma. Since the sum of these parallels is equal to the diameter of $S$ times the ratio of the off-diagonal to a side, then the sum times the side is equal to the diameter of $S$ times the off-diagonal. Scaling to the circle, we get that the surface area is equal to the base circumference of $S$ times the off-diagonal. But, for a circumscribed polygon, the off-diagonal is the diameter of the sphere. Hence

$$\lambda\,\text{sphere} > \text{ area of surface} > \text{circle perimeter times diameter} = 4\,\text{base circle}$$



Inscribe the circle and sphere by a polygon and surface of revolution. Then, noting that the base perimeter of $S$ equals that of the sphere

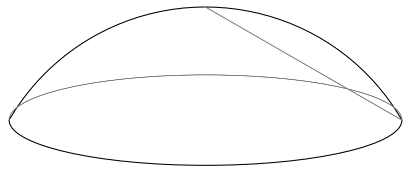
$$\begin{aligned}\lambda^{-1}\,\text{sphere} < \text{area of surface} &= \text{sum of cone slices}\\ &= \text{base perimeter times off-diagonal}\\ &< \text{perimeter times diameter}\\ &= 4\,\text{base circle}\end{aligned}$$

Since $\lambda$ is arbitrarily close to 1, the sphere's area is four times base circle.

$\square$

*Proposition* 5.5

**The surface of a spherical segment is the same as that of a circle with radius equal to the line from the segment vertex to a base point.**

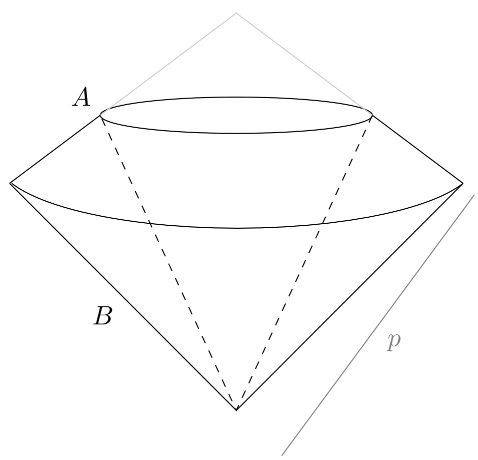


Proof: The proof is the same as above, except that a lesser sum of cone slices is taken. This sum is still the sum of the parallels times a side (times $\pi$), hence equal to the height of the segment times the off-diagonal. For an inscribed polygon, this is less than a diameter times the height, which equals the square of the said line. For a circumscribed polygon, the off-diagonal is the diameter but the height of the polygon-part is more than the height of the spherical segment; hence more than the diameter times the height.

□

*Lemma* 5.6

**If an even regular polygon is rotated about a diameter, then the volume is equal to that of a cone with base equal to the surface area of the rotated polygon and height equal to its radius.**



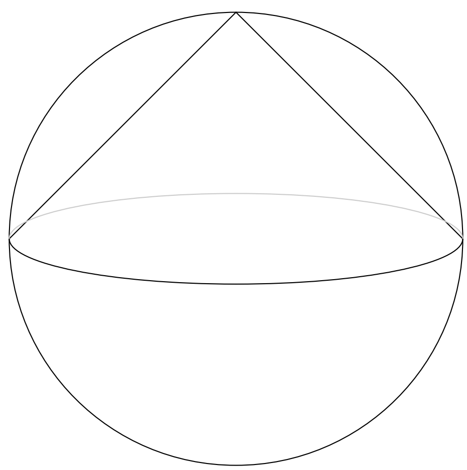

Proof: Join the centre to the vertices of the polygon. These radii partition the solid of revolution into parts, as shown, each of which is the difference of shapes [called 'rhombi' by Archimedes] that are made of two cones $A$, $B$, joined together at a common base. By Prop. 1, the ratio of the surface of $A$ to its base is that of its side to its radius; by similar triangles, it is the same as that of the total height to the perpendicular distance $p$ from the vertex of $B$ to the side of $A$. Now the 'rhombus' has volume equal to that of a cone with the same total height and the same base, and so by these ratios $[\frac{A}{\text{base}} = \frac{H}{p}]$, the same volume as that of a cone with base area equal to the surface of $A$ and height equal to $p$ [this is the Greek way of saying $V = \frac{1}{3}Ap$].

The part shown is the difference of two rhombi, each having the same perpendicular height $p$, namely the distance from the centre to the sides of the polygon. Thus its volume is equal to that of a cone with height equal to $p$ and base area equal to their difference in surface area, namely the outer area of the cone slice. Hence the sum of these volumes will be that of a cone with height $p$ and base area equal to the total area of the solid of revolution.

□

*Proposition* 5.7

**The sphere is four times the cone with base equal to a great circle and height equal to the radius.**



Proof: Circumscribe and inscribe the sphere with revolved polygons that can be made arbitrarily close to each other.

The volume of the inner solid is equal to a cone with base area equal to its surface area and height equal to $p$. By Prop. 4, this area is less than four great circles, and $p$ is less than the radius $r$. Hence the inner solid is less than four cones with base a great circle and radius $r$.

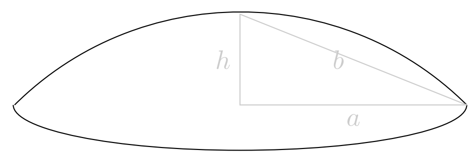
The outer solid has area greater than four great circles, hence its volume is greater than four cones with base a great circle and radius $r$.

Thus the ratio of the outer to the inner solids is greater than that of four cones to a sphere; at the same time it is greater than that of the sphere to four cones. As this ratio can be made arbitrarily close to unity, this shows that the sphere is at least and at most as large as four cones.

□

*Proposition* 5.8
**A spherical segment is equal to a cone whose base is the same as the segment's and whose height has ratio with the segment's height equal to the ratio of the sum of the radius and the complementary height to the complementary height.**

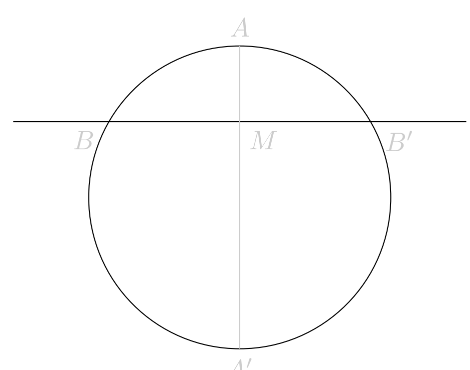
[The complementary height is the difference between the diameter and the height. The formula is $V = \frac{1}{3}Ah\frac{3r-h}{2r-h}$.]




Proof: By a proof similar to the above, the volume of a spherical sector is equal to a cone with base equal to the segment's spherical area and height equal to the radius. Now, the ratio of the cone's base area to the segment's base area is the square of the ratio $b : a$, hence is the ratio of the diameter to the complementary height [$\frac{b^2}{a^2} = \frac{b^2}{b^2-h^2} = \frac{dh}{dh-h^2}$]. Thus the volume is the cone with the segment's base and the height specified in the proposition [$\frac{dh}{d-h}$]. But the cone is equal to the rhombus with the same base but with the sphere's center as a second vertex. Removing the common cone leaves the volume of the segment, equal to the remaining cone as described.

□

*Proposition* 5.9
**To cut a given sphere by a plane so that the segment surfaces have a given ratio.**




Proof: Let $AA'$ be a diameter and $BB'$ a perpendicular plane cutting the diameter at $M$. Then the surfaces have areas equal to circles with radii $AB$ and

$A'B$, so are in the ratio of the squares on them, which is the same as the ratio of $AM$ to $A'M$. Hence let the plane cut the diameter in the given ratio.

□

*Proposition* 5.10

**To cut a given sphere by a plane so that the segment volumes have a given ratio.**

PROOF: The ratio of the segments' volumes is equal to the ratio of the cones mentioned in Proposition 8 above, hence to their heights, since they have the same base. Hence there are three 'equations' in three unknowns $M$, $H$, $H'$, namely

$$OA' + A'M : A'M = HM : MA$$

$$OA + AM : AM = H'M : MA'$$

$$HM : H'M = r \text{ (given ratio)}$$

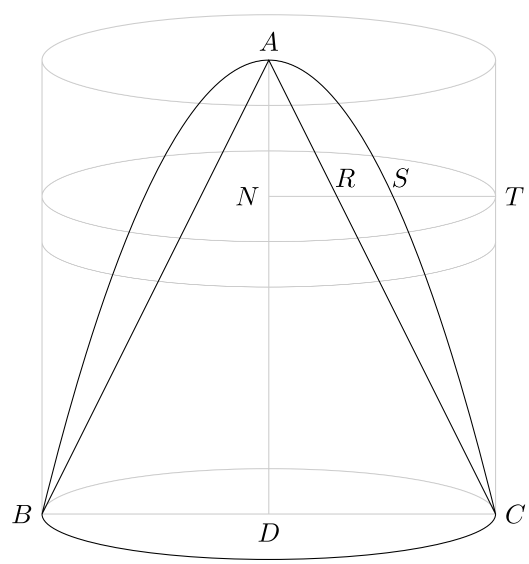
This leads to a convoluted argument to show that there is a solution.

□

# 6 On Conoids and Spheroids

*Proposition* 6.1

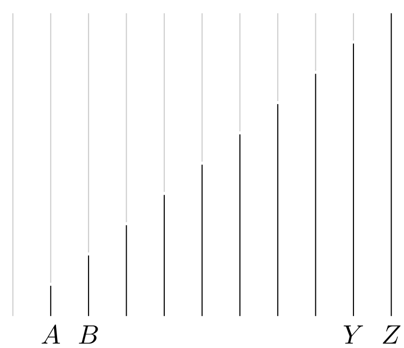
**A segment of a conoid [paraboloid] cut off by a plane is one and a half times as large as the cone with the same base and height.**



Proof: Let $A$ be the vertex, $AD$ the axis of revolution, $BC$ a diameter of the base. Slice the segment by parallel planes and form cylinders that circumscribe and inscribe the solid. Consider a slice $MN$ of inner cylinder with radius $NS$. Then the ratio of the cylinder slice to the inner cylinder slice is as $NT^2 : NS^2 = AD : AN = CD : NR$. Summing these ratios gives that the outer cylinder to the inscribed cylinders has ratio equal to the sum of $CD$s to the sum of $NR$s, which are in arithmetic progression.

Similarly, the ratio of a cylinder slice to an outer cylinder slice is the same but summed from 1 to $n$.

*Lemma.* For any arithmetic sum of $n$ terms $A$, $B$,…, $Y$, $Z$, a copy of them can be rotated to fill the remaining space, that is, $A + Y = Z$, $B + X = Z$, etc. So twice the sum to $Z$ is $nZ + Z$ which is greater than $nZ$. And twice the sum to $Y$ is $(n - 1)Z$ which is less than $Z$. Thus the ratio of $nZ$ to the sum up to $Z$ is less than $1 : 2$, while the ratio to the sum up to $Y$ is more than $1 : 2$.

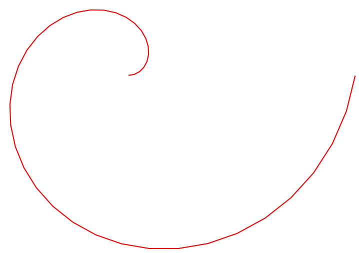



Hence for the above, the outer cylinder to the inscribed cylinder is more than 2; and the outer cylinder to the circumscribed cylinder is less than 2. Since the cylinders can be made arbitrarily close to the conoid, the ratio of the latter is exactly 2.

□

## 7 On Spirals

A spiral is that curve described by a point moving uniformly along a straight line which itself is uniformly rotating in a plane. The initial line is the starting postion of the line and the origin is the centre of rotation. After a complete revolution, the spiral encloses the first area and a circle with centre at the origin and passing through the endpoint is called the first circle. Similarly for the second area and second circle after the second revolution, etc.



*Proposition* 7.1

**Let $A, B, \ldots, Y, Z$ be $n$ magnitudes in arithmetic progression with common difference $A$. Then**

$$B^2 + \cdots + Z^2 : (n-1)Z^2 > A.Z + \frac{1}{3}Y^2 : Z^2$$

$$A^2 + \cdots + Y^2 : (n-1)Z^2 < A.Z + \frac{1}{3}Y^2 : Z^2$$

Proof: Claim: $A^2 + \cdots + Z^2 = \frac{(n+1)Z^2 + A(A+\cdots+Z)}{3}$. For

$$\begin{aligned} Z^2 &= (A+Y)^2 = A^2 + 2A.Y + Y^2 \\ Z^2 &= (B+X)^2 = B^2 + 2B.X + X^2 \\ &\cdots \\ Z^2 &= (Y+A)^2 = Y^2 + 2Y.A + A^2 \end{aligned}$$

$$\therefore\ (n-1)Z^2 = 2(A^2 + \cdots + Z^2) + 2(A.Y + B.X + \cdots + Y.A)$$

But from the sum of an arithmetic progression in a previous lemma

$$\begin{aligned}
B^2 &= 2A.B = A.2B = A.(B+2A)\\
C^2 &= 3A.C = A.(C+2C) = A.(C+2(A+B))\\
D^2 &= 4A.D = A.(D+3D) = A.(D+2(A+B+C))\\
&\cdots\\
Z^2 &= A.(Z+2(A+\cdots+Y))\\
\therefore\ A^2+B^2+\cdots+Z^2 &= A.(A+B+\cdots+Z)+A.((2n-1)A+\cdots+3Y+Z)\\
&= A.(A+\cdots+Z)+A.(2(n-1)A+\cdots+4X+2Y)\\
&= A.(A+\cdots+Z)+2(A.Y+\cdots+Y.A)\\
&= A.(A+\cdots+Z)+(n-1)Z^2-2(A^2+\cdots+Y^2)
\end{aligned}$$

Hence

$$A^2+\cdots+Z^2 = \frac{(n+1)Z^2+2(A.Y+\cdots+Y.A)}{3} > \frac{n}{3}Z^2$$

Also, $Z^2 = A(Z+2(A+\cdots+Y)) > A(A+\cdots+Z)$, hence from the previous identity

$$A^2+\cdots+Y^2 < \frac{n}{3}Z^2.$$

Similarly,

$$\begin{aligned}
B^2 &= (A+A)^2 = A^2+2A^2+A^2\\
C^2 &= (A+B)^2 = A^2+2A.B+B^2\\
&\cdots\\
Z^2 &= (A+Y)^2 = A^2+2A.Y+Y^2\\
\therefore\ B^2+C^2+\cdots+Z^2 &= (n-1)A^2+2A.(A+\cdots+Y)+(A^2+\cdots+Y^2)\\
&= (n-1)A^2+nA.Z+(A^2+\cdots+Y^2)\\
&> (n-1)A^2+nA.Z+\frac{n-1}{3}Y^2\\
&> (n-1)(A.Z+\frac{1}{3}Y^2)
\end{aligned}$$

Again

$$\begin{aligned}
A^2+\cdots+Y^2 &< (n-1)A^2+(n-2)A.Y+(A^2+\cdots+X^2)\\
&< (n-1)A^2+(n-1)A.Y+\frac{n-1}{3}Y^2\\
&= (n-1)(A^2+A.Y+\frac{1}{3}Y^2)\\
&= (n-1)(A.Z+\frac{1}{3}Y^2)
\end{aligned}$$

[Archimedes assumes this results remains true even if $A$ is not the common difference. In this case, the statement reads

$$B^2 + \cdots + Z^2 : (n-1)Z^2 > A.Z + \frac{1}{3}R^2 : Z^2$$

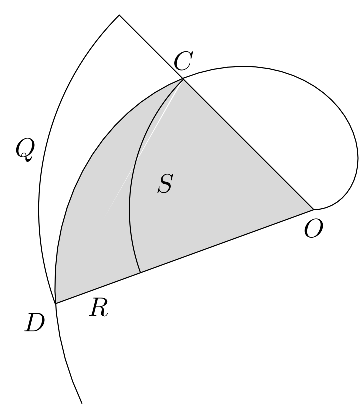
$$A^2 + \cdots + Y^2 : (n-1)Z^2 < A.Z + \frac{1}{3}R^2 : Z^2$$

where $R = Z - A$.]

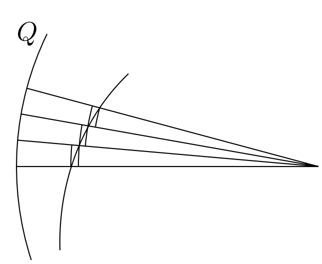
□

*Proposition* 7.2
**Let $CD$ be an arc of a spiral centred at $O$, less than or equal to a whole turn, and let $R$ be the difference between $OD$ and $OC$. Let $Q$ be that sector of a circle centred at $O$ with angle $COD$ extended. Then the ratio of the area of the spiral between $OC$ and $OD$ to the area of $Q$ is equal to**

$$OC.OD + \frac{1}{3}R^2 : OD^2$$




Proof: Consider the square of area $OC.OD + \frac{1}{3}R^2$, and let the sector $P$ have radius equal to the side of this square and angle the same as $Q$. Since the ratio $P : Q$ is $OC.OD + \frac{1}{3}R^2 : OD^2$, it is enough to show that the area $S$ of the spiral arc equals $P$.



Divide the sector $Q$ into equal sectors whose individual areas are less than any given area $\epsilon$ (for example, by bisection). Limit the small sectors to an outer sector that just contains the spiral and an inner sector that is just inside the spiral. Each outer sector is identical to the following inner sector. Then, by cancellation, the difference in the sums of the outer sectors $F$ and the inner sectors $G$ is equal to the difference between the last outer sector and the first inner sector, and so smaller than $\epsilon$. Let the radii of the sectors be $A, B, \ldots, Y, Z$, in arithmetic progression by definition of a spiral, with $A = OC$, $Z = OD$; and $R = OD - OC$.

Suppose $S < P$ with $P - S = \epsilon$. Then by the previous proposition, the ratio of the sum $B^2 + \cdots + Z^2$ to $(n-1)Z^2$ is greater than $A.Z + \frac{1}{3}R^2 : Z^2$. Hence by converting to the tiny sectors' areas, the ratio of the outer area $F$ to the sector $Q$ is greater than $P$ is to $Q$; that is $F > P$, so $\epsilon > F - G > F - S > P - S = \epsilon$, a contradiction.

Suppose $S > P$, with $S - P = \epsilon$. Then $A^2 + \cdots + Y^2 : (n-1)Z^2$ is smaller than $A.Z + \frac{1}{3}R^2 : Z^2$. Again, by scaling, the ratio of the inner area $G$ to the sector $Q$ is smaller than $P$ is to $Q$; so $G < P$ and $\epsilon > F - G > S - G > S - P = \epsilon$.

□

*Corollary* 7.3
**The area under the first turn under the spiral is equal to one third of the first circle.**

Proof: Then $C = O$, $R = OD$, and the sector is the first circle, so the ratio is $1:3$.

□

*Corollary* 7.4
**The area added by the second turn is twice the first circle; the third turn adds twice this, the fourth thrice, and so on.**

Proof: Let $A, B, C, \ldots$ be the areas added at each turn in terms of the first circle. Then $A = \frac{1}{3}$. The second turn has area $2 + \frac{1}{3} : 4$ of second circle, which is $2 + \frac{1}{3}$ of the first, hence adds $2 = B$. The third turn has area $6 + \frac{1}{3} : 9$ of the third circle, so $6 + \frac{1}{3}$ of the first, so adds $4 = 2B$. The fourth turn has area $12 + \frac{1}{3}$ of the first, so adds $6 = 3B$. And so on.

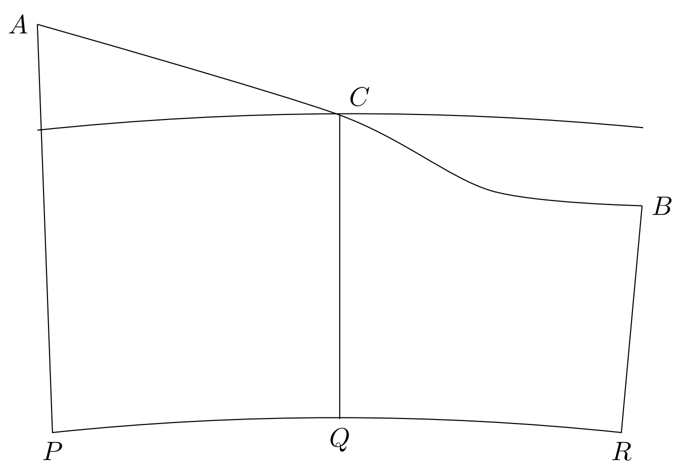
□

# 8 On Floating Bodies

Postulate: In a continuous fluid, that part which is thrust the less is driven along by that which is thrust the more, even if it has to move in a perpendicular direction.

*Proposition* 8.1
**The surface of a fluid at rest is part of a sphere with the same centre as Earth.**

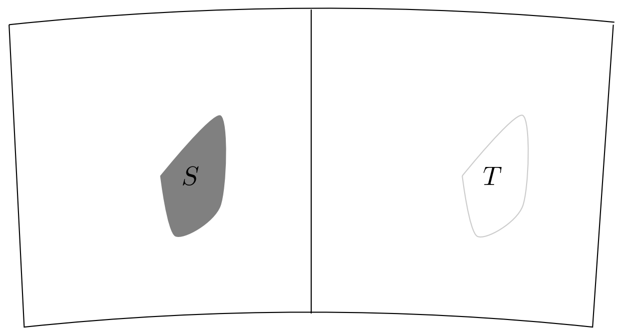
Proof: Suppose that it is not, then there are two points on the surface $A, B$ which do not have the same distance to the centre $O$. The plane through them cuts the surface in a curve that is not a circle with centre $O$. Take a point $C$ between them such that the curve $AC$ is higher than $C$ and the curve $CB$ is lower than $C$. Draw, further, a circle through $C$ centre $O$, and another smaller circle with radius less than $B$, cutting the radii at $P, Q, R$.




The segment $PQ$ beneath $AC$ is compressed by more fluid than the segment $QR$ beneath $CB$. Therefore there will not be rest.

□

*Proposition* 8.2
**If a solid is lighter than a fluid of the same volume, then it will float on the fluid, with part of it projecting above the surface. The part which is immersed has a weight equal to the weight of the fluid displaced.**




Proof: Suppose the solid $S$ at rest is completely submerged in the fluid. Consider two adjacent congruent pyramids with vertex $O$, bounded by spheres, with one of them containing the solid. Consider also a volume $T$ in the second pyramid congruent to $S$ but containing fluid. The part of the fluid on the lower boundaries of the pyramids are compressed by different amounts since they are identical to each other except for $S$ and $T$, of which the former is lighter. Hence they will not be at rest.

Suppose that part $W$ of the volume $V$ is immersed. As above, consider the two adjacent congruent pyramids, but let $T$ be congruent to $W$. Then, at rest, the two parts of the lower boundaries are compressed by the same amounts. Thus the weights of $T$ and $V$ are the same, as required.

□

*Proposition* 8.3
**If a solid is lighter than a fluid of the same volume, but it is forcibly immersed in it, then there will be a thrust upwards equal to the difference in weight between the solid and fluid displaced.**

Proof: Let the weight $V$ be just completely immersed, let $V + W$ be the weight of the displaced fluid. Add a weight $A$ to the solid, projecting out of the fluid, until it is at rest. Then by the previous proposition, $V + A = V + W$, so $A = W$.

□

*Proposition* 8.4
**If a solid is heavier than a fluid of the same volume, it will sink in the fluid to the bottom. If it is weighed there, it will be lighter by the weight of the fluid displaced.**

Proof: The first part is obvious, since the lower boundaries of the pyramids are compressed by different amounts, so can never be at rest and will give way until the solid reaches the bottom.

Let the solid's weight be $V$ be the weight of the fluid with the same volume as the solid. Let $V + W$ be the weight of the solid. Consider a solid, lighter than the fluid, such that its weight is $V$ and that of its displaced fluid is $V+W$. Join the solids together, so their combined weight is $V+W+V$ and that of the displaced fluid is $V+V+W$. Since they are the same, the solids will be at rest. The thrust which causes the first solid to sink is balanced by the upward thrust on the second solid, which is equal to $W$ by the previous proposition.

$\square$