Diophantus of Alexandria

Arithmetica

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Notation

Diophantus's *Arithmetica*[1] is a list of over 200 algebraic problems with solutions, dating from about ∼250AD. In it he introduced algebraic manipulations on equations including a symbol for one unknown (probably following other authors in Alexandria). Although most of his methods were known to his Babylonian and Egyptian predecessors (including Heron of Alexandria), his symbols allowed for clearer solutions and to consider more complicated problems. His notation was as follows:

	Diophantus	Modern
unknown	$\grave{\varsigma}$	x
square	Δ	x^2
cube	K	x^3
etc.	$\Delta\Delta$	x^4
	ΔK	x^5
reciprocal	$\grave{\varsigma}^{\times}$	$1/x$
fraction	$\beta^{\times}\gamma$ or $\overset{\beta}{\gamma}$	$\frac{\gamma}{\beta}$
units	M	
equals	ι^{σ}	$=$
subtract	$\pitchfork$	$-$

Like all Greeks at the time, Diophantus used the (extended) Greek alphabet to denote numbers, namely, $\grave{\alpha}$ 1, $\grave{\beta}$ 2,..., $\grave{\theta}$ 9, $\grave{\iota}$ 10, $\grave{\kappa}$ 20, ..., up to $\grave{\text{ϡ}}$' 900. So, with his notation, a quadratic equation like $2x^2 - 3x - 1 = 4$ would be written as

$$\Delta\grave{\beta}\pitchfork\grave{\varsigma}\grave{\gamma}\text{M}\grave{\alpha}\ \ \iota^{\sigma}\ \text{M}\grave{\delta}$$

Note that juxtaposed terms like $\grave{\varsigma}\grave{\gamma}\,\text{M}\grave{\alpha}$ are assumed to be added. For ease of reading, the modern equivalent symbols are used here, except for squares and cubes, so the above equation is written as

$$2\Delta - 3x - 1 = 4.$$

Only positive numbers "made sense" to the Greeks (because they denoted 'physical' quantities like length and area), so equations had to be 'balanced'; $x^2 + 3x = x$ was not a 'realistic' equation.

[1] There are multiple versions of the *Arithmetica* with unmatching problem numbers.

1 Book I

Problem 1 To split a given number (100) in two parts having a given difference (40).

Problem 2 To split a given number (60) in two parts having a given ratio (3:1).

Problem 3 To split a given number (80) in two parts, the larger of which exceeds a given ratio (3:1) of the smaller by a given number (4).

Problem 4 To find two numbers in a given ratio (5:1) and having a given difference (20).

Problem 5 To split a given number (100) in two parts such that given fractions of them (1/3, 1/5) have a given sum (30).

Problem 6 To split a given number (100) in two parts such that given fractions of them (1/4, 1/6) have a given difference (20).

Problem 7 To find a number such that when two given numbers (100,20) are subtracted from it, the remaining parts have a given ratio (1:3).

Problem 8 To find a number such that when two given numbers (100,20) are added to it, the sums have a given ratio (3:1).

Problem 9 To find a number which when subtracted from two given numbers (100,20), the remaining parts have a given ratio (6:1).

Problem 10 To find a number such that when added to a given number (20) and subtracted from another (100), the results have a given ratio (1:4).

Problem 11 To find a number such that when a given number (20) is added to it, and another (100) subtracted from it, the results have a given ratio (3:1).

Problem 12 To split a given number (100) twice into two pairs such that the first of each pair have a given ratio (2:1) as do the seconds (3:1).

Problem 13 To split a given number (100) as three pairs such that the second number of each pair and the first of the next pair (in cyclic order) have given ratios (3:1, 2:1, 4:1).

Problem 14 To find two numbers whose product and sum have a given ratio (3:1).

Problem 15 To find two numbers such that when a given number (30) is transferred from the second to the first, and when another number (50) is transferred from the first to the second, the resulting pairs have given ratios (2:1, 3:1).

Problem 16 To find three numbers such that the sum of any two are given (20,30,40).

Problem 17 To find four numbers such that the sums of any three of them are given (20,22,24,27).

Problem 18 To find three numbers such that the sum of any two is greater than the third by given amounts (20,30,40).

Problem 19 is the same as 18, solved with a different method.

Problem 20 To find four numbers such that the sum of any three is greater than the fourth by given amounts (20,30,40,50).

Problem 21 is the same as 20, solved with a different method.

Problem 22 To split a given number (100) in three parts such the first two together have a given ratio to the third (3:1) and the last two together have a given ratio to the first (4:1).

Problem 23 To find three numbers such that the third exceeds the second by a given fraction (1/3) of the first; the second exceeds the first by a fraction (1/3) of the third; and the first exceeds a fraction (1/3) of the second by a given amount (10).

Problem 24 is the same as 23, solved with a different method.

Problem 25 To find three numbers such that if each transfers a given fraction of itself (1/3,1/4,1/5) to the next (cyclically), the results are all equal.

Problem 26 To find four numbers such that if each transfers a given fraction of itself (1/3,1/4,1/5,1/6) to the next (cyclically), the results are all equal.

Problem 27 To find three numbers such that if each receives a given fraction of the sum of the remaining numbers (1/3,1/4,1/5), the results are all equal.

Problem 28 To find four numbers such that if each receives a given fraction of the sum of the remaining numbers (1/3,1/4,1/5,1/6), the results are all equal.

Problem 29 To find a number which when multiplied by two given numbers (200,5) makes one a square and the other its side.

Problem 30 To find two numbers whose sum and product are given (20,96).

Problem 31 To find two numbers whose sum and sum of squares are given (20,208).

Problem 32 To find two numbers whose sum and difference of squares are given (20,80).

Problem 33 To find two numbers whose difference and product are given (4,96).

Problem 34 To find two numbers in a given ratio (3:1) such that their sum and the sum of their squares have a given ratio (1:5).

Problem 35 To find two numbers in a given ratio (3:1) such that their difference and the sum of their squares have a given ratio (1:10).

Problem 36 To find two numbers in a given ratio (3:1) such that their sum and the difference of their squares have a given ratio (1:6).

Problem 37 To find two numbers in a given ratio (3:1) such that their difference and the difference of their squares have a given ratio (1:12).

Problem 38 To find two numbers in a given ratio (3:1) such that the smaller square and the larger number have a given ratio (6:1).

Problem 39 To find two numbers in a given ratio (3:1) such that the smaller square and its side have a given ratio (6:1).

Problem 40 To find two numbers in a given ratio (3:1) such that the smaller square and their sum have a given ratio (2:1).

Problem 41 To find two numbers in a given ratio (3:1) such that the smaller square and their difference have a given ratio (6:1).

Problem 42 restates problems 38 to 41 for the larger square.

Problem 43 To find a number, together with two given numbers (3,5), such that when any two are added together then multiplied with the third, the result is three numbers which have equal differences.

1.1 Solutions

Note: Solutions are not exact copies of the original ones; some were written much later, during the Byzantine and Islamic periods.

Problem 1 $x + y = 100$, $y - x = 40$
Solution:

Numbers	x	$x + 40$
sum	$2x + 40 = 100$	
subtract	$2x = 60$	
divide	$x = 30$	
solution	30	70

Problem 2 $x + y = 60$, $y = 3x$
Solution:

Numbers	x	$3x$
sum	$4x = 60$	
divide	$x = 15$	
solution	15	45

Problem 3 $x + y = 80$, $y = 3x + 4$
Solution:

Numbers	x	$3x + 4$
sum	$4x + 4 = 80$	
subtract	$4x = 76$	
divide	$x = 19$	
solution	19	61

Problem 4 $y = 5x$, $y - x = 20$
Solution:

Numbers	x	$5x$
difference	$4x = 20$	
divide	$x = 5$	
solution	5	25

Problem 5 $x + y = 100$, $\frac{x}{5} + \frac{y}{3} = 30$
Solution:

Numbers	$5x$	$90 - 3x$
sum	$2x + 90 = 100$	
subtract	$2x = 10$	
divide	$x = 5$	
solution	25	75

Problem 6 $x+y=100$, $\frac{y}{4}-\frac{x}{6}=20$
Solution:

Numbers	$6x$ $\quad 4x+80$
sum	$10x+80=100$
subtract	$10x=20$
divide	$x=2$
solution	12 $\quad$ 88

Problem 7 $3(x-100)=x-20$
Solution:

Number	x
remainders	$x-100$ $\quad x-20$
condition	$x-20=3x-300$
add/subtract	$2x=280$
divide	$x=140$

Problem 8 $x+100=3(x+20)$
Solution:

Number	x
sums	$x+100$ $\quad x+20$
condition	$x+100=3x+60$
add	$2x=40$
divide	$x=20$

Problem 9 $100-x=6(20-x)$
Solution:

Number	x
remainders	$100-x$ $\quad 20-x$
condition	$100-x=120-6x$
add	$5x=20$
divide	$x=4$

Problem 10 $4(20+x)=100-x$
Solution:

Number	x	
new numbers	$x+20$	$100-x$
condition	$100-x=4x+80$	
add	$5x=20$	
divide	$x=4$	

Problem 11 $x+20=3(x-100)$
Solution:

Number	x	
new numbers	$x+20$	$x-100$
condition	$x+20=3x-300$	
add	$2x=320$	
divide	$x=160$	

Problem 12 $x+y=100=a+b$, $a=2x$, $b=3y$
Solution:

First pair	x	$100-x$
Second pair	$2x$	$100-2x$
condition	$100-x=300-6x$	
add	$5x=200$	
divide	$x=40$	
solution	40,60	20,80

Problem 13 $x+y=100=a+b=u+v$, $y=3a$, $b=2u$, $v=4x$
Solution:

First pair	x	$24x-300$
Second pair	$4x-100$	$200-8x$
Third pair	$100-4x$	$4x$
sum	$25x-300=100$	
add	$25x=400$	
divide	$x=16$	
solution	(16,84),(28,72),(36,64)	

Problem 14 $xy=3(x+y)$
Solution:

Try numbers	x	2
condition	$2x=3x+6$	no solution
Modify	x	4
condition	$4x=3x+12$	
add	$x=12$	
solution	12	4

Problem 15 $y+30=2(x-30)$, $3(y-50)=x+50$
Solution:

2nd number	x	
after first transfer		
2nd number	$x-30$	
1st number	$2x-60$	
original 1st number	$2x-90$	
after second transfer		
1st number	$2x-140$	
2nd number	$x+50$	
condition	$x+50=6x-420$	
add	$5x=470$	
divide	$x=94$	
solution	94	98

Problem 16 $x+y=20$, $x+z=30$, $y+z=40$
Solution:

Sum of numbers	x		
numbers	$x-40$	$x-30$	$x-20$
condition	$2x-70=20$		
add	$2x=90$		
divide	$x=45$		
solution	5	15	25

Problem 17 $a+b+c=20$, $a+b+d=22$, $a+c+d=24$, $b+c+d=27$
Solution:

Sum of numbers	x			
numbers	$x-27$	$x-24$	$x-22$	$x-20$
sum of first three	$3x-73=20$			
add	$3x=93$			
divide	$x=31$			
solution	4	7	9	11

Problem 18 $x+y=z+20$, $x+z=y+30$, $y+z=40+x$
Solution:

Sum of numbers	x		
2nd and 3rd together	$x-$1st		
condition	$x-$1st = 1st + 40		
add	Twice 1st= $x-40$		
numbers (similarly)	$2^{\times}x-20$	$2^{\times}x-15$	$2^{\times}x-10$
sum	$x=2^{\times}3x-45$		
multiply	$2x=3x-90$		
add	$x=90$		
solution	25	30	35

Another solution:

3rd number	x		
1st and 2nd	3rd + 20		
3rd and 2nd	1st + 30		
so, 1st and 3rd and twice 2nd	1st + 3rd + 50		
twice 2nd	50		
2nd	25		
1st	$x-5$		
condition	$2x-5=65$		
add	$2x=70$		
divide	$x=35$		
solution	30	25	35

Problem 20 $a+b+c=20+d$, $a+b+d=30+c$, $a+c+d=40+b$, $b+c+d=50+a$
Solution:

Sum of numbers	x			
sum of 2nd, 3rd, 4th	$x-1$st			
condition	$x-1\text{st}=1\text{st}+50$			
add	Twice 1st $=x-50$			
numbers (similarly)	$2^{\times}x-25$	$2^{\times}x-20$	$2^{\times}x-15$	$2^{\times}x-10$
sum	$2x-70=x$			
add	$x=70$			
solution	10	15	20	25

Problem 22 $x+y+z=100$, $x+y=3z$, $y+z=4x$
Solution:

Smallest number	x		
sum of others	$100-x$		
condition	$100-x=4x$		
add	$5x=100$		
divide	$x=20$		
solution	20	55	25

Problem 23 $z-y=\frac{x}{3}$, $y-x=\frac{z}{3}$, $x-\frac{y}{3}=10$
Solution:

Smallest number	$x+10$		
2nd number	$3x$		
2nd minus 1st	$2x-10$		
3rd number	$6x-30$		
condition	$9x-90=x+10$		
add	$8x=100$		
divide	$x=2^{\times}25$		
solution	$22\frac{1}{2}$	$37\frac{1}{2}$	45

Problem 25 $x-\frac{x}{3}+\frac{z}{5}=y-\frac{y}{4}+\frac{x}{3}=z-\frac{z}{5}+\frac{y}{4}$
Solution:

1st number	$3x$		
2nd number	4 units		
2nd after transfer	$x+3$		
transfer from 3rd to 1st	$x+3+x-3x$		
3rd	$15-5x$		
condition	$x+3=15-5x-(3-x)+1$		
	$5x=10$		
	$x=2$		
solution	6	4	5

Problem 26 $a-\frac{a}{3}+\frac{d}{6}=b-\frac{b}{4}+\frac{a}{3}=c-\frac{c}{5}+\frac{b}{4}=d-\frac{d}{6}+\frac{c}{5}$
Solution:

1st number	$3x$
2nd number	4 units
2nd after transfer	$x+3$
transfer 4th to 1st	$3-x$
4th	$18-6x$
transfer 3rd to 4th	$6x-12$
3rd	$30x-60$
condition	$24x-47=x+3$
add	$23x=50$
divide	$x=50/23$
solution	150 92 120 114

Problem 27 $x+\frac{y+z}{3}=y+\frac{x+z}{4}=z+\frac{x+y}{5}$
Solution:

First number	x
transfer from 2nd and 3rd	1 unit
total sum	$x+3$
1st (or 2nd) after transfer	$x+1$
four 2nd, and rest	$4x+4$
three 2nd	$3x+1$
2nd	$x+\frac{1}{3}$
3rd	$\frac{8}{3}-x$
fifth of 1st and 2nd	$\frac{2}{5}x+\frac{1}{15}$
condition	$\frac{41}{15}-\frac{3}{5}x=x+1$
solve	$x=\frac{13}{12}$
numbers	$\frac{13}{12}$ $\frac{17}{12}$ $\frac{19}{12}$
solution	13 17 19

Problem 28 $a+\frac{b+c+d}{3}=b+\frac{a+c+d}{4}=c+\frac{a+b+d}{5}=d+\frac{a+b+c}{6}$
Solution:

First number	x
transfer from rest	1
total sum	$x+3$
1st after transfer	$x+1$
four 2nd, and rest	$4x+4$
three 2nd	$3x+1$
2nd	$x+\frac{1}{3}$
similarly, 3rd,4th	$x+\frac{1}{2}$ $x+\frac{3}{5}$
1st,2nd,3rd together	$3x+\frac{5}{6}$
sum	$4x+\frac{43}{30}=x+3$
solve	$x=\frac{47}{90}$
solution	47 77 92 101

Problem 29 $200x = y^2$, $5x = y$
Solution:

Number	x	
products	$5x$	$200x$
condition	$200x = 25\Delta$	
divide	$\Delta = 8x$	
solution	8	

Problem 30 $x + y = 20$, $xy = 96$
Solution:

Numbers	$x + 10$	$10 - x$
product	$100 - \Delta = 96$	
add	$\Delta = 4$	
root	$x = 2$	
solution	12	8

Problem 31 $x + y = 20$, $x^2 + y^2 = 208$
Solution:

Numbers	$10 + x$	$10 - x$
squares	$\Delta + 20x + 100$	$\Delta + 100 - 20x$
sum of squares	$2\Delta + 200 = 208$	
divide	$\Delta = 4$	
root	$x = 2$	
solution	12	8

Problem 32 $x + y = 20$, $x^2 - y^2 = 80$
Solution:

Numbers	$10 + x$	$10 - x$
difference of squares	$40x = 80$	
divide	$x = 2$	
solution	12	8

Problem 33 $x - y = 4$, $xy = 96$
Solution:

Numbers	$x + 2$	$x - 2$
product	$\Delta - 4 = 96$	
add	$\Delta = 100$	
root	$x = 10$	
solution	12	8

Problem 34 $y = 3x$, $x^2 + y^2 = 5(x + y)$
Solution:

Numbers	x	$3x$
sums	$4x$	10Δ
ratio	$10\Delta = 20x$	
divide	$x = 2$	
solution	2	6

Problem 35 $y = 3x$, $x^2 + y^2 = 10(y - x)$
Solution:

Numbers	x	$3x$
sum and difference	10Δ	$2x$
ratio	$10\Delta = 20x$	
divide	$x = 2$	
solution	2	6

Problem 36 $y = 3x$, $y^2 - x^2 = 6(x + y)$
Solution:

Numbers	x	$3x$
sum,diff	8Δ	$4x$
ratio	$8\Delta = 24x$	
divide	$x = 3$	
solution	3	9

Problem 37 $y = 3x$, $y^2 - x^2 = 12(y - x)$
Solution:

Numbers	x	$3x$
differences	$2x$	8Δ
ratio	$8\Delta = 24x$	
divide	$x = 3$	
solution	3	9

Problem 38 $y = 3x$, $x^2 = 6y$
Solution:

Numbers	x	$3x$
condition	$\Delta = 18x$	
divide	$x = 18$	
solution	18	54

Problem 39 $y = 3x$, $x^2 = 6x$
Solution:

Numbers	x	$3x$
condition	$\Delta = 6x$	
divide	$x = 6$	
solution	6	18

Problem 40 $y = 3x$, $x^2 = 2(x + y)$
Solution:

Numbers	x	$3x$
condition	$\Delta = 8x$	
divide	$x = 8$	
solution	8	24

Problem 41 $y = 3x$, $x^2 = 6(y - x)$
Solution:

Numbers	x	$3x$
condition	$\Delta = 12x$	
divide	$x = 12$	
solution	12	36

Problem 43 $5(x + 3) - 3(x + 5) = (5 + 3)x - 5(x + 3)$, etc.
Solution:

Number	x		
sum of pairs	$x + 5$	$x + 3$	8
multiplied by third	$3x + 15$	$5x + 15$	$8x$
Possibility 1	$3x + 15$	$5x + 15$	$8x$
difference	$2x = 3x - 15$		
	$x = 15$		
Possibility 2	$3x + 15$	$8x$	$5x + 15$
	$5x - 15 = 15 - 3x$		
	$8x = 30$		
	$x = 15/4$		
Possibility 3	$8x$	$3x + 15$	$5x + 15$
	$15 - 5x = 2x$		
	$x = 15/7$		
solutions	60	90	120
	$105/4$	$135/4$	30
	$150/7$	$180/7$	$120/7$

2 Book II

Problem 1 restates problem 34 of Book 1.
Problem 2 restates problem 37 of Book 1.
Problem 3 restates problem 14 of Book 1.
Problem 4 restates problem 31 of Book 1.
Problem 5 restates problem 32 of Book 1.

Problem 6 To find two numbers with a given difference (2) which is less than than the difference of their squares by a given number (20).

Problem 7 To find two numbers with a given difference (2) whose difference of squares is greater than a given multiple (3) of their difference by a given number (10).

Problem 8 To split a given square (16) into two squares.

Problem 9 To split a given sum of squares ($13 = 4 + 9$) into two other squares.

Problem 10 To find two squares with a given difference (60).

Problem 11 To find a number such that when added to two given numbers (2,3) makes each a square.

Problem 12 To find a number such that when subtracted from two given numbers (9,21) makes each a square.

Problem 13 To find a number such that when two given numbers (6,7) are subtracted from it, each becomes a square.

Problem 14 To split a given number (20) into two parts and to find a square such that when added to each part makes each a square.

Problem 15 To split a given number (20) into two parts and to find a square such that the square minus each part is each a square.

Problem 16 To find two numbers with a given ratio (3:1) such that when added to a given square (9) makes each a square.

Problem 17 To find three numbers such that when each gives a given fraction of itself ($\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$) and given numbers (6,7,8) to the next number (cyclically), the

results are all equal.

Problem 18 To split a given number (80) into three parts such that when each gives a given fraction of itself ($\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$) and given numbers (6,7,8) to the next number (cyclically), the results are all equal.

Problem 19 To find three squares such that the difference between the largest two and between the smallest two have a given ratio (3:1).

Problem 20 To find two numbers whose squares added to the other number is each a square.

Problem 21 To find two numbers whose squares minus the other number is each a square.

Problem 22 To find two numbers whose squares added to their sum is each a square.

Problem 23 To find two numbers whose squares minus their sum is each a square.

Problem 24 To find two numbers such that each added to the square of their sum gives a square.

Problem 25 To find two numbers such that each subtracted from the square of their sum gives a square.

Problem 26 To find two numbers such that added to their product give squares, whose sides have a given sum (6).

Problem 27 To find two numbers such that subtracted from their product give squares, whose sides have a given sum (5).

Problem 28 To find two squares such that each added to their product is a square.

Problem 29 To find two squares such that when subtracted from their product each gives a square.

Problem 30 To find two numbers whose sum added or subtracted from their product each gives a square.

Problem 31 To find two numbers whose sum is a square, and when this is

added or subtracted from their product, each gives a square.

Problem 32 To find three numbers whose squares added to the following is each a square.

Problem 33 To find three numbers whose squares minus the following is each a square.

Problem 34 To find three numbers whose squares added to their sum is each a square.

Problem 35 To find three numbers whose squares minus their sum is each a square.

2.1 Solutions

Problem 6 $y - x = 2$, $y^2 - x^2 - 2 = 20$
Solution:

numbers	x	$x+2$
diff. of sq.	$4x + 4 = 22$	
	$4x = 18$	
	$x = \frac{18}{4}$	
solution	$\frac{9}{2}$	$\frac{13}{2}$

Problem 7 $y - x = 2$, $y^2 - x^2 - 3(y - x) = 10$
Solution:

numbers	x	$x+2$
diff. of sq.	$4x + 4 = 3 \times 2 + 10 = 16$	
	$4x = 12$	
	$x = 3$	
solution	3	5

Problem 8 $x^2 + y^2 = 16$
Solution:

Squares	Δ	$16 - \Delta$
try square on $x - 4$	$\Delta + 16 - 8x = 16 - \Delta$	
	$2\Delta = 8x$	
	$x = 4$	
try square on $2x - 4$	$4\Delta + 16 - 16x = 16 - \Delta$	
	$5\Delta = 16x$	
	$x = 16/5$	
solution	$\frac{256}{25}$	$\frac{144}{25}$

It is impossible for a cube to be a sum of two cubes, a fourth power to be a sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly remarkable proof which this margin is too small to contain.
– Pierre de Fermat 1637

Problem 9 $x^2 + y^2 = 13$
Solution:

try square on $x + 2$	$\Delta + 4x + 4$	
try square on $x - 3$		$\Delta + 9 - 6x$
try square on $2x - 3$		$4\Delta + 9 - 12x$
sum of squares	$5\Delta + 13 - 8x = 13$	
	$5x = 8$	
solution	$\frac{324}{25}$	$\frac{1}{25}$

Problem 10 $y^2 - x^2 = 60$
Solution:

numbers	x	$x+3$
diff. of sq.	$6x + 9 = 60$	
	$6x = 51$	
	$x = \frac{51}{6}$	
	$\frac{17}{2}$	$\frac{23}{2}$
solution	$\frac{289}{4}$	$\frac{529}{4}$

Problem 11 $x+2=a^2$, $x+3=b^2$
Solution:

squares	$x+2 \qquad x+3$
diff. of sq.	$1 = \text{sum} \times \text{diff}$
try sum= 2, diff= $\frac{1}{2}$	$y \qquad 2-y$
try sum= 4, diff= $\frac{1}{4}$	$y \qquad 4-y$
	$2y-4=\frac{1}{4}$
	$y=\frac{17}{4}$
	$\frac{17}{8} \qquad \frac{15}{8}$
sum	$2x+5=\frac{289}{64}+\frac{225}{64}=\frac{257}{32}$
solution	$x=\frac{97}{64}$

Problem 12 $9-x=a^2$, $21-x=b^2$
Solution:

1st square	Δ
number	$9-\Delta$
try 2nd square on $x-4$	$\Delta+16-8x$
	$\Delta+16-8x=\Delta+12$
	$8x=4$
	$x=1/2$
solution	$\frac{35}{4}$

Problem 13 $x-6=a^2$, $x-7=b^2$
Solution:

squares	$x-6 \quad x-7$
diff. of sq.	1
try sum, diff.	$2 \qquad \frac{1}{2}$
	$y \qquad 2-y$
difference	$2y-2=\frac{1}{2}$
	$y=\frac{5}{4}$
	$x-6=\frac{25}{16}$
solution	$\frac{121}{16}$

Problem 14 $x+y=20$, $x+z^2=a^2$, $y+z^2=b^2$
Solution:

square on $x+1$	$\Delta+2x+1$
square on $x+2$	$\Delta+4x+4$
parts	$2x+1 \qquad 4x+4$
sum	$6x+5=20$
	$x=\frac{15}{6}$
solution	$6 \qquad 14 \qquad \frac{25}{4}$

Problem 15 $x+y=20$, $z^2-x=a^2$, $z^2-y=b^2$
Solution:

square on $x+1$	$\Delta+2x+1$
square on $x-1$	$\Delta+1-2x$
parts	$2x+1 \quad 4x$
sum	$6x+1=20$
	$x=\frac{19}{6}$
solution	$\frac{22}{3} \quad \frac{38}{3} \quad \frac{625}{36}$

Problem 16 $y=3x$, $x+9=a^2$, $y+9=b^2$
Solution:

square on $x+3$	$\Delta+6x+9$
try square on $x-3$	$\Delta+9-2x$
square on $2x-3$	$4\Delta+9-12x$
numbers	$\Delta+6x \quad 4\Delta-12x$
ratio	$3\Delta+18x=4\Delta-12x$
	$\Delta=30x$
	$x=30$
solution	1080 3240

Problem 17 $x-\frac{x}{5}-6+\frac{z}{7}+8=y-\frac{y}{6}-7+\frac{x}{5}+6=z-\frac{z}{7}-8+\frac{y}{6}+7$
Solution:

numbers	$5x$	$6x$	$7y$
2nd gives $x+7$, receives $x+6$		$6x-1$	
1st gives $x+6$	$4x-6$		
so 1st receives from 3rd	$2x+5$		
y			$2x-3$
3rd number			$14x-21$
3rd gives $2x+5$, receives $x+7$			$13x-19=6x-1$
			$x=\frac{18}{7}$
solution	$\frac{90}{7}$	$\frac{108}{7}$	$\frac{105}{7}$

Problem 18 $x+y+z=80$, $x-\frac{x}{5}-6+\frac{z}{7}+8=y-\frac{y}{6}-7+\frac{x}{5}+6=z-\frac{z}{7}-8+\frac{y}{6}+7$
Solution:

final parts	80/3	80/3	80/3
numbers	$5x$	$6y$	$7z$
2nd gives $y+7$, receives $x+6$		$5y+x-1=\frac{80}{3}$	
		$5y+x=\frac{83}{3}$	
1st gives $x+6$	$4x-6$		
so 1st receives from 3rd	$\frac{98}{3}-4x$		
z			$\frac{74}{3}-4x$
3rd number			$\frac{518}{3}-28x$
sum		$5x+6y+\frac{518}{3}-28x=80$	
		$y=24x-\frac{361}{3}$	
numbers	$5x$	$144x-722$	$802-149x$
		$121x-\frac{1805}{3}=\frac{83}{3}$	
		$x=\frac{1888}{363}$	
solution	$\frac{9440}{363}$	$\frac{9786}{363}$	$\frac{9814}{363}$

Problem 19 $z^2-y^2=3(y^2-x^2)$
Solution:

1st square on x	Δ		
2nd square on $x+1$		$\Delta+2x+1$	
try 3rd square on $x+2$			$\Delta+4x+4$
3rd square on $x+3$			$\Delta+6x+9$
diff. of smallest sq.	$2x+1$		
diff. of largest sq.		$4x+8$	
ratio	$6x+3=4x+8$		
	$x=\frac{5}{2}$		
solution	$\frac{25}{4}$	$\frac{49}{4}$	$\frac{121}{4}$

Problem 20 $x^2+y=a^2$, $y^2+x=b^2$
Solution:

square on $x+1$	$\Delta+2x+1$	
numbers	x	$2x+1$
sum of sq. and number	$4\Delta+5x+1$	
try square on $2x-1$	$4\Delta+1-4x$	
square on $2x-2$	$4\Delta+4-8x$	
condition	$4\Delta+5x+1=4\Delta+4-8x$	
	$13x=3$	
solution	$\frac{3}{13}$	$\frac{19}{13}$

Problem 21 $x^2 - y = a^2$, $y^2 - x = b^2$
Solution:

final square	Δ	
square on $x+1$	$\Delta + 2x + 1$	
numbers	$x+1$	$2x+1$
square on $2x+1$	$4\Delta + 4x + 1$	
try condition	$4\Delta + 3x = 9\Delta$	
	$5\Delta = 3x$	
	$x = \frac{3}{5}$	
solution	$\frac{8}{5}$	$\frac{11}{5}$

Problem 22 $x^2 + x + y = a^2$, $y^2 + x + y = b^2$
Solution:

square on $x+1$	$\Delta + 2x + 1$	
numbers	x	$x+1$
sum of 2nd sq. and sum	$\Delta + 4x + 2$	
try square on $x-1$	$\Delta + 1 - 2x$	
try square on $x-2$	$\Delta + 4 - 4x$	
condition	$\Delta + 4x + 2 = \Delta + 4 - 4x$	
	$8x = 2$	
	$x = \frac{1}{4}$	
solution	$\frac{1}{4}$	$\frac{5}{4}$

Problem 23 $x^2 - x - y = a^2$, $y^2 - x - y = b^2$
Solution:

square on $x+1$	$\Delta + 2x + 1$	
numbers	x	$x+1$
1st square minus sum	$\Delta - 2x - 1$	
try square on $x-1$	$\Delta + 1 - 2x$	
try square on $x-2$	$\Delta + 4 - 4x$	
condition	$\Delta - 2x - 1 = \Delta + 4 - 4x$	
	$2x = 5$	
solution	$\frac{5}{2}$	$\frac{7}{2}$

Problem 24 $x + (x+y)^2 = a^2$, $y + (x+y)^2 = b^2$
Solution:

square of sum	Δ	
try final squares	4Δ	9Δ
numbers	3Δ	8Δ
condition	$11\Delta = x$	
	$x = \frac{1}{11}$	
solution	$\frac{3}{121}$	$\frac{8}{121}$

Problem 25 $(x+y)^2-x=a^2$, $(x+y)^2-y=b^2$
Solution:

square of sum	16Δ	
try final squares	4Δ	9Δ
numbers	12Δ	7Δ
sum	$19\Delta=4x$	
	$x=\frac{4}{19}$	
solution	$\frac{192}{361}$	$\frac{112}{361}$

Problem 26 $x+xy=a^2$, $y+xy=b^2$, $a+b=6$
Solution:

1st square	4Δ	
product	$4\Delta-x$	
numbers	x	$4x-1$
condition	$4\Delta+3x-1$ is a square	
try square on $6-2x$	$4\Delta+36-24x$	
	$27x=37$	
solution	$\frac{37}{27}$	$\frac{121}{27}$

Problem 27 $xy-x=a^2$, $xy-y=b^2$, $a+b=5$
Solution:

1st square	4Δ	
product	$4\Delta+x$	
numbers	x	$4x+1$
condition	$4\Delta-3x-1$ is a square	
try square on $5-2x$	$4\Delta+25-20x$	
	$17x=26$	
solution	$\frac{26}{17}$	$\frac{121}{17}$

Problem 28 $x^2+x^2y^2=a^2$, $y^2+x^2y^2=b^2$
Solution:

try squares in some unit	Δ	1
product in square units	Δ	
condition	Δ sq.units $+\Delta$ units is a square	
taking the unit as the unknown	$\Delta\Delta+\Delta$ is a square	
	$\Delta+1$ is a square	
try square on $2-x$	$\Delta+4-4x=\Delta+1$	
	$4x=3$	
unit of squares is $\frac{9}{16}$	$\frac{9}{16}\Delta$	$\frac{9}{16}$
product	$\frac{81}{256}\Delta$	
from above	$\frac{81}{256}\Delta+\frac{9}{16}\Delta$ is a square	
remaining condition	$\frac{81}{256}\Delta+\frac{9}{16}$ is a square	
	$9\Delta+16$ is a square	
try square on $5-3x$	$9\Delta+16=9\Delta+25-30x$	
	$30x=9$	
solution	$\frac{81}{1600}$	$\frac{9}{16}$

Problem 29 $x^2y^2 - x^2 = a^2$, $x^2y^2 - y^2 = b^2$
Solution:

squares in some unit	Δ	1
product in square units	Δ	
condition	Δ sq.units $-\Delta$ units is a square	
taking the unit as the unknown	$\Delta\Delta - \Delta$ is a square	
try $9/16$	$\Delta - 1 = \frac{9}{16}$	
squares	$\frac{25}{16}\Delta$	$\frac{25}{16}$
remaining condition	$\frac{625}{256}\Delta - \frac{25}{16}$ is a square	
	$25\Delta - 16$ is a square	
try square on $5x - 1$	$25\Delta + 1 - 10x = 25\Delta - 16$	
	$10x = 17$	
solution	$\frac{289}{64}$	$\frac{25}{16}$

Problem 30 $xy + x + y = a^2$, $xy - x - y = b^2$
Solution:

square on $3 + 2$	$9 + 4 + 12$	
square on $3 - 2$	$9 + 4 - 12$	
product, sum	13Δ	12Δ
numbers	x	$13x$
sum	$12\Delta = 14x$	
	$x = \frac{7}{6}$	
solution	$\frac{7}{6}$	$\frac{91}{6}$

Problem 31 $x + y = a^2$, $xy + a^2 = b^2$, $xy - a^2 = c^2$
Solution:

square on $4 + 2$	$16 + 4 + 16$	
square on $4 - 2$	$16 + 4 - 16$	
product, sum	20Δ	16Δ
numbers	$2x$	$10x$
sum	$16\Delta = 12x$	
	$x = \frac{12}{16}$	
solution	$\frac{3}{2}$	$\frac{15}{2}$

Problem 32 $x^2 + y = a^2$, $y^2 + z = b^2$, $z^2 + x = c^2$
Solution:

square on $x + 1$	$\Delta + 2x + 1$		
square on $2x + 1$	$4\Delta + 4x + 1$		
numbers	x	$2x + 1$	$4x + 3$
square on $4x + 3$	$16\Delta + 24x + 9$		
condition	$16\Delta + 25x + 9$ is a square		
try square on $4x - 1$	$16\Delta + 1 - 8x$		
try square on $4x - 4$	$16\Delta + 16 - 32x$		
	$25x + 9 = 16 - 32x$		
	$57x = 7$		
solution	$\frac{7}{57}$	$\frac{71}{57}$	$\frac{199}{57}$

Problem 33 $x^2 - y = a^2$, $y^2 - z = b^2$, $z^2 - x = c^2$
Solution:

square on $x+1$	$\Delta + 2x + 1$
square on $2x+1$	$4\Delta + 4x + 1$
numbers	$x+1 \qquad 2x+1 \qquad 4x+1$
square on $4x+1$	$16\Delta + 8x + 1$
condition	$16\Delta + 7x$ is a square
try 25Δ	$16\Delta + 7x = 25\Delta$
	$x = 7/9$
solution	$\frac{16}{9} \qquad \frac{23}{9} \qquad \frac{37}{9}$

Problem 34 $s = x + y + z$, $x^2 + s = a^2$, $y^2 + s = b^2$, $z^2 + s = c^2$
Solution:

integer with three factorizations	$12 = 1 \times 12 = 2 \times 6 = 3 \times 4$
square on $12+1$	$169 = 145 + 24 = 121 + 48$
square on $6+2$	$64 = 40 + 24 = 16 + 48$
square on $4+3$	$49 = 25 + 24 = 1 + 48$
	$\frac{121}{4} + 12 = \frac{169}{4}$
	$\frac{16}{4} + 12 = \frac{64}{4}$
	$\frac{1}{4} + 12 = \frac{49}{4}$
numbers	$\frac{11}{2}x \qquad 2x \qquad \frac{1}{2}x$
sum	$8x = 12\Delta$
	$x = 8/12$
solution	$\frac{11}{3} \qquad \frac{4}{3} \qquad \frac{1}{3}$

Problem 35 $s = x + y + z$, $x^2 - s = a^2$, $y^2 - s = b^2$, $z^2 - s = c^2$
Solution:

as above	$\frac{169}{4} - 12 = \frac{121}{4}$
	$\frac{64}{4} - 12 = \frac{16}{4}$
	$\frac{49}{4} - 12 = \frac{1}{4}$
numbers	$\frac{13}{2}x \qquad 4x \qquad \frac{7}{2}x$
sum	$14x = 12\Delta$
	$x = 14/12$
solution	$\frac{91}{12} \qquad \frac{14}{3} \qquad \frac{49}{12}$

3 Book III

Problem 1 To find three numbers such that their squares subtracted from their sum each gives a square.

Problem 2 To find three numbers such that when added to their sum squared each gives a square.

Problem 3 To find three numbers such that when subtracted from their sum squared each gives a square.

Problem 4 To find three numbers such that each minus their sum squared gives a square.

Problem 5 To find three numbers whose sum is a square and the sum of any two minus the third is a square.

Problem 6 To find three numbers whose sum is a square and the sum of any two is a square.

Problem 7 To find three numbers in progression such that the sum of any two is a square.

Problem 8 To find three numbers such that any two, as well as their sum, added to a given number (3) is a square.

Problem 9 To find three numbers such that any two, as well as their sum, minus a given number (3) is a square.

Problem 10 To find three numbers such that the product of any two added to a given number (12) is a square.

Problem 11 To find three numbers such that the pdouct of any two minus a given number (10) is a square.

Problem 12 To find three numbers such that the product of any two added to the third is a square.

Problem 13 To find three numbers such that the product of any two minus the third is a square.

Problem 14 To find three numbers such that the product of any two added

to the square on the third is a square.

Problem 15 To find three numbers such that for any two of them, their sum added to their product is a square.

Problem 16 To find three numbers such that for any two of them, their sum subtracted from their product is a square.

Problem 17 To find two numbers such that their product added to any of them, and to their sum, is each a square.

Problem 18 To find two numbers such their product minus any of them, and minus their sum, is each a square.

Problem 19 To find four numbers such that when added or subtracted from their sum squared is a square.

Problem 20 is the same as Book II Problem 15.
Problem 21 is the same as Book II Problem 14.

3.1 Solutions

Problem 1 $s = x + y + z$, $s - x^2 = a^2$, $s - y^2 = b^2$, $s - z^2 = c^2$
Solution:

split 5 into squares in two ways	$5 = 1 + 4 = \frac{4}{25} + \frac{121}{25}$
numbers	$x \qquad 2x \qquad \frac{2}{5}x$
let sum	5Δ
condition 1	$5\Delta - \Delta = 4\Delta$ is a square
condition 2	$5\Delta - 4\Delta = \Delta$ is a square
condition 3	$5\Delta - \frac{4}{25}\Delta = \frac{121}{25}\Delta$ is a square
sum	$x + 2x + \frac{2}{5}x = 5\Delta$
	$\frac{17}{5}x = 5\Delta$
solution	$\frac{17}{25} \qquad \frac{34}{25} \qquad \frac{34}{125}$

Problem 2 $s = x + y + z$, $x + s^2 = a^2$, $y + s^2 = b^2$, $z + s^2 = c^2$
Solution:

square of sum	Δ
final squares	$4\Delta \qquad 9\Delta \qquad 16\Delta$
numbers	$3\Delta \qquad 8\Delta \qquad 15\Delta$
sum	$3\Delta + 8\Delta + 15\Delta = x$
	$x = \frac{1}{26}$
solution	$\frac{3}{676} \qquad \frac{8}{676} \qquad \frac{15}{676}$

Problem 3 $s = x + y + z$, $s^2 - x = a^2$, $s^2 - y = b^2$, $s^2 - z = c^2$
Solution:

square of sum	16Δ
final squares	$\Delta \qquad 4\Delta \qquad 9\Delta$
numbers	$15\Delta \qquad 12\Delta \qquad 7\Delta$
sum	$15\Delta + 12\Delta + 7\Delta = 4x$
	$x = \frac{2}{17}$
solution	$\frac{28}{289} \qquad \frac{48}{289} \qquad \frac{60}{289}$

Problem 4 $s = x + y + z$, $x - s^2 = a^2$, $y - s^2 = b^2$, $z - s^2 = c^2$
Solution:

square of sum	Δ
final squares	$\Delta \qquad 4\Delta \qquad 9\Delta$
numbers	$2\Delta \qquad 5\Delta \qquad 10\Delta$
sum	$2\Delta + 5\Delta + 10\Delta = x$
	$x = \frac{1}{17}$
solution	$\frac{2}{289} \qquad \frac{5}{289} \qquad \frac{10}{289}$

Problem 5 $x+y+z=a^2$, $x+y-z=b^2$, $y+z-x=c^2$, $z+x-y=d^2$
Solution:

let sum	$\Delta+2x+1$
condition 1	1st + 2nd − 3rd is a square, 1
condition 2	2nd + 3rd − 1st is a square, Δ
numbers	$x+\frac{1}{2}$ $\quad$ $\frac{1}{2}\Delta+\frac{1}{2}$ $\quad$ $\frac{1}{2}\Delta+x$
3rd+1st−2nd	$\frac{1}{2}\Delta+x+x+\frac{1}{2}-\frac{1}{2}\Delta-\frac{1}{2}$ is a square, say 9
	$2x=9$
solution	5 $\quad$ $\frac{85}{8}$ $\quad$ $\frac{117}{8}$

Problem 6 $x+y+z=a^2$, $x+y=b^2$, $y+z=c^2$, $z+x=d^2$
Solution:

let sum	$\Delta+2x+1$
condition 1	1st + 2nd is a square, Δ
condition 2	2nd + 3rd is a square, $\Delta-2x+1$
numbers	$4x$ $\quad$ $\Delta-4x$ $\quad$ $2x+1$
condition 3	1st + 2nd is a square, say, 49 or 121
	$6x+1=121$
	$x=20$
solution	80 $\quad$ 320 $\quad$ 41

Problem 7 $y=x+d$, $z=y+d$, $x+y=a^2$, $y+z=b^2$, $z+x=c^2$
Solution:

	Pairs are also in progression
squares	Δ $\quad$ $\Delta+2x+1$ $\quad$ $\Delta+4x+2$
try square on $x-2$	$\Delta+4x+2=\Delta-4x+4$
try square on $x-8$	$\Delta+4x+2=\Delta-16x+64$
	$x=31/10$
squares	$\frac{961}{100}$ $\quad$ $\frac{1681}{100}$ $\quad$ $\frac{2401}{100}$
1st + 2nd	961
2nd+3rd	1681
1st+3rd	2401
sum	$\frac{5043}{2}$
solution	$\frac{241}{2}$ $\quad$ $\frac{1681}{2}$ $\quad$ $\frac{3121}{2}$

Problem 8 $x+y+3=a^2$, $y+z+3=b^2$, $z+x+3=c^2$, $x+y+z+3=d^2$
Solution:

let 1st + 2nd + 3	$\Delta+4x+4$
let 2nd + 3rd + 3	$\Delta+6x+9$
let sum + 3	$\Delta+8x+16$
numbers	$2x+7$ $\quad$ $\Delta+2x-6$ $\quad$ $4x+12$
1st + 3rd + 3	$6x+22=100$, any square
	$x=13$
solution	33 $\quad$ 189 $\quad$ 64

Problem 9 $x+y-3=a^2$, $y+z-3=b^2$, $z+x-3=c^2$, $x+y+z-3=d^2$
Solution:

let 1st + 2nd − 3	Δ		
let 2nd + 3rd − 3	$\Delta+2x+1$		
let sum − 3	$\Delta+4x+4$		
numbers	$2x+3$	$\Delta-2x$	$4x+4$
1st + 3rd − 3	$6x+4=64$, say		
	$x=10$		
solution	23	80	44

Problem 10 $xy+12=a^2$, $yz+12=b^2$, $zx+12=c^2$
Solution:

try squares	25	16	$\frac{49}{4}$
products	13	4	$\frac{1}{4}$
numbers	$x^\times$	$4x$	$\frac{1}{4}x$
condition	$\Delta+12$ is a square		
try sq. on $x+3$	$\Delta+6x+9$		
	$6x+9=12$		
	$x=\frac{1}{2}$		
solution	2	2	$\frac{1}{8}$

Problem 11 $xy-10=a^2$, $yz-10=b^2$, $zx-10=c^2$
Solution:

try squares	4	$\frac{9}{4}$	$\frac{81}{4}$
products	14	$\frac{49}{4}$	$\frac{121}{4}$
numbers	$x^\times$	$\frac{49}{4}x$	$\frac{121}{4}x$
condition	$\frac{5929}{16}\Delta-10$ is a square		
	$5929\Delta-160$ is a square		
try sq. on $77x-2$	$5929\Delta-308x+4$		
	$308x=164$		
	$x=41/77$		
solution	$\frac{77}{41}$	$\frac{451}{28}$	$\frac{287}{44}$

Problem 12 $xy+z=a^2$, $yz+x=b^2$, $zx+y=c^2$
Solution:

square on $x+3$	$\Delta+6x+9$		
numbers	x	$x+6$	9
conditions	$10x+6$ and $10x+54$ are squares		
diff. of sq.	48		
sum, diff.	12, 4		
sides	$6+2$, $6-2$		
	$10x+6=16$		
	$x=1$		
solution	1	7	9

Problem 13 $xy - z = a^2$, $yz - x = b^2$, $zx - y = c^2$
Solution:

numbers	x $x+4$ $4x$
condition	$\Delta + 4x - 4x$ is a square
conditions	$4\Delta - x - 4$ and $4\Delta + 15x$ are squares
diff. of sq.	$16x + 4$
sum, diff.	$4x + 1$, 4
sides	$2x + \frac{5}{2}$, $2x - \frac{3}{2}$
	$4\Delta + 10x + \frac{25}{4} = 4\Delta + 15x$
	$5x = \frac{25}{4}$
	$x = \frac{5}{4}$
solution	$\frac{5}{4}$ $\frac{21}{4}$ 5

Problem 14 $xy + z^2 = a^2$, $yz + x^2 = b^2$, $zx + y^2 = c^2$
Solution:

square on $x + 2$	$\Delta + 4x + 4$
square on $2x + 1$	$4\Delta + 4x + 1$
numbers	$4x + 4$ 1 x
condition	$16\Delta + 33x + 16$ is a square
try square on $4x - 5$	$16\Delta + 25 - 40x$
	$33x + 16 = 25 - 40x$
	$x = 9/73$
solution	328 73 9

Problem 15 $x + y + xy = a^2$, $y + z + yz = b^2$, $z + x + zx = c^2$
Solution:

numbers	4 9 x
condition	$36 + 4 + 9 = 49$ a square
conditions	$5x + 4$, $10x + 9$ are squares
diff. of sq.	$5x + 5$
sum, diff.	$x + 1$, 5
sides	$\frac{1}{2}x + 3$, $\frac{1}{2}x - 2$
square	$\frac{1}{4}\Delta + 3x + 9 = 10x + 9$
	$x = 28$
solution	4 9 28

Problem 16 $xy - x - y = a^2$, $yz - y - z = b^2$, $zx - z - x = c^2$
Solution:

numbers	$4x + 1$ $x + 1$
product	$4\Delta + 5x + 1$
condition	$4\Delta - 1$ is a square
try square on $2x - 2$	$4\Delta + 4 - 8x$
	$8x = 5$
	$x = 5/8$
numbers	$\frac{7}{2}$ $\frac{13}{8}$ x
conditions	$\frac{5}{2}x - \frac{7}{2}$, $\frac{5}{8}x - \frac{13}{8}$ are squares
	$10x - 14$, $10x - 26$ are squares
diff. of sq.	12
sum, diff.	6, 2
sides	$3 + 1$, $3 - 1$
square	$10x - 14 = 16$
	$x = 3$
solution	$\frac{7}{2}$ $\frac{13}{8}$ 3

Problem 17 $xy + x = a^2$, $xy + y = b^2$, $xy + x + y = c^2$
Solution:

numbers	x $4x - 1$
prod. plus 1st	$4\Delta - x + x$ is a square
conditions	$4\Delta + 4x - 1$ and $4\Delta + 3x - 1$ are squares
diff. of sq.	x
sum, diff.	$4x$, $\frac{1}{4}$
sides	$2x + \frac{1}{8}$, $2x - \frac{1}{8}$
square	$4\Delta + 4x - 1 = 4\Delta + \frac{1}{2}x + \frac{1}{64}$
	$x = \frac{65}{224}$
solution	$\frac{65}{224}$ $\frac{36}{224}$

Problem 18 $xy - x = a^2$, $xy - y = b^2$, $xy - x - y = c^2$
Solution:

numbers	$x + 1$ $4x$
prod. $-$ 2nd	$4\Delta + 4x - 4x$ is a square
conditions	$4\Delta + 3x - 1$, $4\Delta - x - 1$ are squares
diff. of sq.	$4x$
sum, diff.	$4x$, 1
sides	$2x + \frac{1}{2}$, $2x - \frac{1}{2}$
square	$4\Delta + 3x - 1 = 4\Delta + 2x + \frac{1}{4}$
	$x = 5/4$
solution	$\frac{9}{4}$ 5

Problem 19 $s = x_1 + x_2 + x_3 + x_4$, $s^2 + x_i = u_i^2$, $s^2 - x_i = v_i^2$
Solution:

	Find a square that splits into squares in four ways.
two triangles	$3 \quad 4 \quad 5$
	$5 \quad 12 \quad 13$
scaled	$39 \quad 52 \quad 65$
	$25 \quad 60 \quad 65$
but	$65 = 49 + 16 = 64 + 1$
diff. of sq. on 49 ± 16	is square on $2 \times 7 \times 4$ $(= 56)$
diff. of sq. on 64 ± 1	is square on $2 \times 8 \times 1$ $(= 16)$
so sq. on 65	$4225 = 1521 + 2704 = 625 + 3600 = 1089 + 3136 = 3969 + 256$
sq. on $52 - 39$	sq. on 65 minus $2 \times 52 \times 39$ $(= 4056)$
sq. on $60 - 25$	sq. on 65 minus $2 \times 60 \times 25$ $(= 3000)$
sq. on $56 - 33$	sq. on 65 minus $2 \times 33 \times 56$ $(= 3696)$
sq. on $63 - 16$	sq. on 65 minus $2 \times 16 \times 63$ $(= 2016)$
sum of squares	$65x$
numbers	$4056\Delta \quad 3000\Delta \quad 3696\Delta \quad 2016\Delta$
	$12768\Delta = 65x$
	$x = 65/12768$
solution	$\frac{17136600}{163021824} \quad \frac{12675000}{163021824} \quad \frac{15615600}{163021824} \quad \frac{8517600}{163021824}$

4 Book IV

Problem 1 To split a given number (370) into two cubes whose sides add up to a given number (10).

Problem 2 To find two numbers whose difference is a given number (6) and whose cubes have a given difference (504).

Problem 3 To find a number such that when multiplied by a side and its square gives a cube and its side, respectively.

Problem 4 To find a number such that when added to a square and its side gives another square and its side, respectively.

Problem 5 To find a number such that when added to a square and its side gives a side and its square, respectively.

Problem 6 To find a square such that when added to a cube and a square gives another cube and a square.

Problem 7 To find a square such that when added to a cube and a square gives another square and cube.

Problem 8 To find a number such that when added to a cube and its side makes another cube and its side.

Problem 9 To find a number such that when added to a cube and its side makes another side and its cube.

Problem 10 To find two numbers whose sum equals the sum of their cubes.

Problem 11 To find two numbers whose difference equals the difference of their cubes.

Problem 12 To find two numbers such that when each is added to the other's cube, the result is the same.

Problem 13 To find two numbers such that when unity (1) is added to each of them, to their sum, and to their difference, all give squares.

Problem 14 To find three squares whose sum equals the sum of their differences.

Problem 15 To find three numbers such that the sum of any two multiplied by the third are given numbers (27, 32, 35).

Problem 16 To find three numbers whose sum is a square, such that each number squared added to the following number is also a square.

Problem 17 To find three numbers whose sum is a square, such that each number squared minus the following number is also a square.

Problem 18 To find two numbers such that the first cubed added to the second is a cube, and the second squared added to the first is a square.

Problem 19 To find three numbers (in general) such that the product of any two plus 1 is a square.

Problem 20 To find four numbers such that the product of any two plus 1 is a square.

Problem 21 To find three numbers in continued proportion[2] such that the difference of any two is a square.

Problem 22 To find three numbers such that the product of all three added to any of them is a square.

Problem 23 To find three numbers such that the product of all three minus any of them is a square.

Problem 24 To split a given number (6) in two parts such that their product is a cube minus its side.

Problem 25 To split a given number (4) in three parts such that their product is a cube whose side equals the sum of the differences of the parts.

Problem 26 To find two numbers whose product added to each of them is a cube.

Problem 27 To find two numbers whose product minus each of them is a cube.

Problem 28 To find two numbers whose sum added or subtracted from their product is each a cube.

[2] geometric progression

Problem 29 To find four numbers whose sum added to the sum of their squares is a given number (12).

Problem 30 To find four squares whose sum minus the sum of their sides is a given number (4).

Problem 31 To split 1 in two parts such that when they are added to two given numbers (3, 5), their product is a square.

Problem 32 To split a given number (6) in three parts such that when the third is added or subtracted from the product of the first two, the results are squares.

Problem 33 To find two numbers such that the first with a fraction of the second has a given ratio (3:1) to the remainder of the second, while the second with the same fraction of the first has a given ratio (5:1) to the remainder of the first.

Problem 34 To find three numbers such that the product of any two added to the sum of the two make given numbers (8, 15, 24).

Problem 35 To find three numbers such that product of any two minus the sum of the two make given numbers (8, 15, 24).

Problem 36 To find three numbers such that the product of any two has a given ratio (3:1, 4:1, 5:1) to their sum.

Problem 37 To find three numbers such that the product of any two has a given ratio (3:1, 4:1, 5:1) to the sum of all three.

Problem 38 To find three numbers such that when their sum is multiplied with the first gives a triangular number, with the second gives a square, and with the third gives a cube.

Problem 39 To find three numbers such that the difference between the first and second has a given ratio (3:1) to the difference between the second and third, and also the sum of any two is a square.

Problem 40 To find three numbers such that the difference between the squares of the first and second has a given ratio (3:1) to the difference between the second and third, and also the sum of any two is a square.

4.1 Solutions

Problem 1 $x^3+y^3=370$, $x+y=10$
Solution:

part sides	$5+x$	$5-x$
cubes	$125+75x+15\Delta+K$	$125-75x+15\Delta-K$
sum	$30\Delta+250=370$	
	$\Delta=4$	
	$x=2$	
solution	343	27

Problem 2 $x-y=6$, $x^3-y^3=504$
Solution:

numbers	$x+3$	$x-3$
cubes	$K+9\Delta+27x+27$	$K-9\Delta+27x-27$
diff. of cubes	$18\Delta+54=504$	
	$\Delta=25$	
	$x=5$	
solution	512	8

Problem 3 $xy=z^3$, $x^2y=z$
Solution:

cube on 2	8		
numbers	x	$8x^\times$	2
products	$8x$	8	
condition	$8x=2$		
	$x=1/4$		
solution	$\frac{1}{4}$	32	2

Problem 4 $x+y^2=z^2$, $x+y=z$
Solution:

squares	Δ	4Δ
number	3Δ	
condition	$3\Delta+x=2x$	
	$3\Delta=x$	
	$x=1/3$	
solution	$\frac{1}{3}$	$\frac{1}{9}$

Problem 5 $x^2+y=z$, $x+y=z^2$
Solution:

squares	Δ	4Δ	
numbers	x	$4\Delta-x$	$2x$
condition	$5\Delta-x=2x$		
	$5\Delta=3x$		
	$x=3/5$		
solution	$\frac{3}{5}$	$\frac{21}{25}$	$\frac{6}{5}$

Problem 6 $x^2+y^3=a^3$, $x^2+z^2=b^2$
Solution:

square, square, cube	16Δ	9Δ	K
condition	$16\Delta+9\Delta$ is a square		
	$16\Delta+K$ is a cube, say, $8K$		
	$7K=16\Delta$		
	$x=16/7$		
solution	$\frac{4096}{49}$	$\frac{2304}{49}$	$\frac{4096}{343}$

Problem 7 $x^2+y^3=a^2$, $x^2+z^2=b^3$
Solution:

	let cubes be the same		
square on $x+2x$	$\Delta+4\Delta+4\Delta$		
let	$\Delta+4\Delta=K$ a cube		
so	$K+4\Delta$ is a square		
	$5\Delta=K$		
	$x=5$		
solution	100	125	25

Problem 8 $x+y^3=z^3$, $x+y=z$
Solution:

try numbers	x	$2x$
conditions	$x+8K$ is a cube	
	$x+2x$ is its side	
	$27K=8K+x$	
	$19\Delta=1$	
	need two cubes whose difference is a square	
	K	$K+3\Delta+3y+1$
diff. of cubes	$3\Delta+3y+1$	
try square on $1-2y$	$4\Delta-4y+1$	
	$\Delta=7y$	
	$y=7$	
numbers	x	$7x$
conditions	$x+343K$ is a cube	
	$x+7x$ is its side	
	$512K=343K+x$	
	$169\Delta=1$	
	$x=\frac{1}{13}$	
solution	$\frac{1}{13}$	$\frac{7}{13}$

Problem 9 $x+y^3=z$, $x+y=z^3$
Solution:

try sides	$2x$	$3x$
cubes	$8K$	$27K$
number	$27K-2x$	
condition	$27K-2x+8K=3x$	
	$35\Delta=5$	
	need two cubes such that	
	sum of cubes : sum of sides is a square	
try sides	y	$2-y$
cubes	K	$8-12y+6\Delta-K$
sum of sides	2	
sum of cubes	$6\Delta-12y+8$	
ratio	$3\Delta-6y+4$	
try square on $2-4y$	$16\Delta-16y+4$	
	$13\Delta=10y$	
	$y=10/13$	
sides	$\frac{10}{13}$	$\frac{16}{13}$
try sides	$5x$	$8x$
cubes	$125K$	$512K$
number	$512K-5x$	
condition	$512K-5x+125K=8x$	
	$637\Delta=13$	
	$x=\frac{1}{7}$	
solution	$\frac{267}{343}$	$\frac{5}{7}$

Problem 10 $x+y=x^3+y^3$
Solution:

try sides	$2x$	$3x$
cubes	$8K$	$27K$
condition	$35K=5x$	
	as in previous problem	
solution	$\frac{5}{7}$	$\frac{8}{7}$

Problem 11 $y-x=y^3-x^3$
Solution:

try sides	$2x$	$3x$
cubes	$8K$	$27K$
condition	$19K=x$	
	need two cubes such that	
	diff. of cubes : diff. of sides is a square	
	y	$y+1$
cubes	K	$K+3\Delta+3y+1$
ratio	$3\Delta+3y+1$	
try square on $1-2y$	$4\Delta-4y+1$	
	$y=7$	
try sides	$7x$	$8x$
cubes	$343K$	$512K$
condition	$169K=x$	
	$x=\frac{1}{13}$	
solution	$\frac{7}{13}$	$\frac{8}{13}$

Problem 12 $x+y^3=y+x^3$
Solution:

try sides	$2x$	$3x$
cubes	$8K$	$27K$
condition	$27K+2x=8K+3x$	
	$19K=x$	
	as in previous problem	
sides	$7x$	$8x$
condition	$512K+7x=343K+8x$	
	$x=\frac{1}{13}$	
solution	$\frac{7}{13}$	$\frac{8}{13}$

Problem 13 $x+1=a^2$, $y+1=b^2$, $x+y+1=c^2$, $y-x+1=d^2$
Solution:

first square on $3x+1$	$9\Delta+6x+1$	
	diff of 2nd and 3rd squares is a square	
	$9\Delta+6x$ is a square	
sum, diff	$9x+6$	x
sides	$\frac{9}{2}x+6+\frac{1}{2}x$	$\frac{9}{2}x+6-\frac{1}{2}x$
	$4x+3$	$5x+3$
squares	$16\Delta+24x+9$	$25\Delta+30x+9$
numbers	$9\Delta+6x$	$16\Delta+24x+8$
condition	$7\Delta+18x+9$ is a square	
try square on $3-3x$	$9\Delta-18x+9$	
	$2\Delta=36x$	
	$x=18$	
solution	3024	5624

Problem 14 $x^2+y^2+z^2=(x^2-y^2)+(y^2-z^2)+(x^2-z^2)$
Solution:

1st, 3rd squares	$\Delta+2x+1$	1	
sum of diff.	= twice of 1st − 3rd		
2nd square	$2\Delta+4x-\Delta-2x-2$		
	$\Delta+2x-2$		
try square on $x-4$	$\Delta-8x+16$		
	$x=9/5$		
squares	$\frac{196}{25}$	$\frac{121}{25}$	1
solution (scaled)	196	121	25

Problem 15 $x(y+z)=27$, $y(x+z)=32$, $z(x+y)=35$
Solution:

try numbers	$10x^\times$	$25x^\times$	x
conditions	$35x^\times$ times x is 35		
	$10x^\times+x$ times $25x^\times$ is 32		
	$25x^\times+x$ times $10x^\times$ is 27		
	last two are impossible		
	need to split 35 such that diff. is 5		
	$\frac{35}{2}-\frac{5}{2}$	$\frac{35}{2}+\frac{5}{2}$	
	15	20	
numbers	$15x^\times$	$20x^\times$	x
conditions	$15x^\times+x$ times $20x^\times$ is 32		
	$20x^\times+x$ times $15x^\times$ is 27		
	$300\Delta^\times+20=32$		
	$x=5$		
solution	3	4	5

Problem 16 $x+y+z=a^2$, $x^2+y=b^2$, $y^2+z=c^2$, $z^2+x=d^2$
Solution:

	each number is a difference of squares		
try 2nd $=4x$	$4x$ is a diff. of squares		
sum, diff.	$2x$	2	
	$x+1$	$x-1$	
1st	$x-1$		
try square on $4x+1$	$16\Delta+8x+1$		
3rd	$8x+1$		
numbers	$x-1$	$4x$	$8x+1$
condition on sum	$13x$ is a square		
	x is 13 times a square		
new numbers	$13\Delta-1$	52Δ	$104\Delta+1$
condition on sum	169Δ is a square		
conditions	$169\Delta\Delta+26\Delta+1$ is a square (on $13\Delta+1$)		
	$2704\Delta\Delta+104\Delta+1$ is a square (on $52\Delta+1$)		
	$10816\Delta\Delta+221\Delta$ is a square		
	$10816\Delta+221$ is a square		
try square on $104x+1$	$10816\Delta+208x+1$		
	$208x=220$		
	$x=55/52$		
solution	$\frac{2817}{208}$	$\frac{12100}{208}$	$\frac{24408}{208}$

Problem 17 $x+y+z=a^2$, $x^2-y=b^2$, $y^2-z=c^2$, $z^2-x=d^2$
Solution:

	each number is a difference of squares		
try 2nd $=4x$	$4x$ is a diff. of squares		
sum, diff.	$2x$	2	
	$x+1$	$x-1$	
1st	$x+1$		
try square on $4x-1$	$16\Delta-8x+1$		
3rd	$8x-1$		
numbers	$x+1$	$4x$	$8x-1$
condition on sum	$13x$ is a square		
	x is 13 times a square		
new numbers	$13\Delta+1$	52Δ	$104\Delta-1$
condition	$10816\Delta\Delta-221\Delta$ is a square		
	$10816\Delta-221$ is a square		
try square on $104x-1$	$10816\Delta-208x+1$		
	$208x=222$		
	$x=111/104$		
solution	$\frac{13153}{832}$	$\frac{49284}{832}$	$\frac{97736}{832}$

Problem 18 $x^3 + y = a^3$, $y^2 + x = b^2$
Solution:

numbers	x $8-K$
condition	$KK - 16K + 64 + x$ is a square
try square on $K+8$	$KK + 16K + 64$
	$32K = x$
	need to replace 8 by a cube such that four times cube is a square
	cube needs to be a square
	64 is both a cube and a square
numbers	x $64-K$
condition	$KK - 128K + 4096 + x$ is a square
sq. on $K+64$	$KK + 128K + 4096$
	$256\Delta = 1$
	$x = 1/16$
solution	$\frac{1}{16}$ $\frac{262143}{4096}$

Problem 19 $xy + 1 = a^2$, $yz + 1 = b^2$, $zx + 1 = c^2$
Solution:

	any product is one less than a square
first square on $x+1$	$\Delta + 2x + 1$
second square on $2x+1$	$4\Delta + 4x + 1$
numbers	$x+2$ x $4x+4$
condition	$4\Delta + 12x + 9$ is a square (on $2x+3$)
solution	$x+2$ x $4x+4$

Problem 20 $x_i x_j + 1 = u_{ij}^2$ $(i \neq j)$
Solution:

square on $x+1$	$\Delta + 2x + 1$
square on $2x+1$	$4\Delta + 4x + 1$
square on $3x+1$	$9\Delta + 6x + 1$
numbers	x $x+2$ $4x+4$ $9x+6$
2nd times 3rd + 1	$4\Delta + 12x + 9$ is a square on $2x+3$
3rd time 4th + 1	$36\Delta + 60x + 25$ is a square on $6x+5$
2nd times 4th + 1	$9\Delta + 24x + 13$ is a square
try square on $3x-4$	$9\Delta - 24x + 16$
	$48x = 3$
	$x = 1/16$
solution	$\frac{1}{16}$ $\frac{33}{16}$ $\frac{68}{16}$ $\frac{105}{16}$

Problem 21 $y = rx$, $z = ry$, $y - x = a^2$, $z - y = b^2$, $z - x = c^2$
Solution:

numbers	x $x+9$ $x+25$
condition	$\Delta + 25x = \Delta + 18x + 81$
	$7x = 81$
	$x = 81/7$
solution	$\frac{81}{7}$ $\frac{144}{7}$ $\frac{256}{7}$

Problem 22 $xyz + x = a^2$, $xyz + y = b^2$, $xyz + z = c^2$
Solution:

square on $x+1$	$\Delta + 2x + 1$
product, 1st	$\Delta + 2x$ $\quad$ 1
square on $x+2$	$\Delta + 4x + 4$
product, 2nd	$\Delta + 2x$ $\quad$ $2x + 4$
3rd	$\Delta + 2x$ divided by $2x + 4$
numbers	1 $\quad$ $2x + 4$ $\quad$ $\frac{1}{2}x$
condition	$2\Delta + \frac{5}{2}x$ is a square, say, 4Δ
	$3\Delta = \frac{5}{2}x$
	$x = 5/6$
solution	1 $\quad$ $\frac{17}{3}$ $\quad$ $\frac{5}{12}$

Problem 23 $xyz - x = a^2$, $xyz - y = b^2$, $xyz - z = c^2$
Solution:

product, 1st	$\Delta + x$ $\quad$ x
product of 2nd and 3rd	$x + 1$
numbers	x $\quad$ 1 $\quad$ $x+1$
conditions	$\Delta + x - 1$, $\Delta - 1$ are squares
diff. of squares	x
sum, diff.	$2x$ $\quad$ $\frac{1}{2}$
sides	$x + \frac{1}{4}$ $\quad$ $x - \frac{1}{4}$
squares	$\Delta + \frac{1}{2}x + \frac{1}{16}$ $\quad$ $\Delta - \frac{1}{2}x + \frac{1}{16}$
	$\Delta - 1 = \Delta - \frac{1}{2}x + \frac{1}{16}$
	$\frac{1}{2}x = \frac{17}{16}$
solution	$\frac{17}{8}$ $\quad$ 1 $\quad$ $\frac{25}{8}$

Problem 24 $x + y = 6$, $xy = z^3 - z$
Solution:

numbers	x $\quad$ $6 - x$
product	$6x - \Delta$
try cube on $2x-1$	$8K - 12\Delta + 6x - 1$
cube $-$ side	$8K - 12\Delta + 4x = 6x - \Delta$
	$8K = 2x + 11\Delta$
	need to replace $2x-1$ so there are no x terms
cube on $3x-1$	$27K - 27\Delta + 9x - 1$
condition	$27K - 27\Delta + 6x = 6x - \Delta$
	$27K = 26\Delta$
	$x = 26/27$
solution	$\frac{26}{27}$ $\quad$ $\frac{136}{27}$

Problem 25 $x+y+z=4$, $xyz=a^3$, $a=(y-x)+(z-y)+(z-x)$
Solution:

cube	$8K$		
1st − 3rd	x		
try 1st, 3rd	$2x$	$3x$	
2nd	$8K$ divided by 6Δ, that is, $\frac{4}{3}x$ but $\frac{4}{3}$ is smaller than 2; need to replace 2, 3 by two consecutive numbers such that 8 divided by their product lies between them.		
	y	$y+1$	
product	$\Delta+y$		
	8 needs to lie between $K+\Delta$ and $K+2\Delta+y$		
	take 8 to be cube on $y+\frac{1}{3}$		
	$y+\frac{1}{3}=2$		
	$y=5/3$		
numbers	$\frac{5}{3}x$	$\frac{9}{5}x$	$\frac{8}{3}x$
scaling	$25x$	$27x$	$40x$
sum	$92x=4$		
	$x=1/23$		
solution	$\frac{25}{23}$	$\frac{27}{23}$	$\frac{40}{23}$

Problem 26 $xy+x=a^3$, $xy+y=b^3$
Solution:

numbers	$8x$	$\Delta-1$
product	$8K-8x$	
condition	$8K-8x+\Delta-1$ is a cube	
try cube on $2x-1$	$8K-12\Delta+6x-1$	
	$13\Delta=14x$	
	$x=14/13$	
solution	$\frac{112}{13}$	$\frac{27}{169}$

Problem 27 $xy-x=a^3$, $xy-y=b^3$
Solution:

numbers	$8x$	$\Delta+1$
product	$8K+8x$	
condition	$8K+8x-\Delta-1$ is a cube	
try cube on $2x-1$	$8K-12\Delta+6x-1$, stuck	
numbers	$8x+1$	Δ
product	$8K+\Delta$	
condition	$8K+\Delta-8x-1$ is a cube	
	$13\Delta=14x$	
	$x=14/13$	
solution	$\frac{125}{13}$	$\frac{196}{169}$

Problem 28 $xy + x + y = a^3$, $xy - x - y = b^3$
Solution:

try cubes on 4, 2	64	8
diff. of cubes	56	
sum, product	28	36
numbers	$14 + x$	$14 - x$
	$196 - \Delta = 36$	
	$\Delta = 160$	
	need to replace 64 and 8 by two cubes such that	
	the square on half their sum minus the product is a square	
try cubes on $y + 1$, $y - 1$	$K + 3\Delta + 3x + 1$	$K - 3\Delta + 3x - 1$
diff. of cubes	$6\Delta + 2$	
sum, product	$3\Delta + 1$	$K + 3x$
	$\frac{9}{4}\Delta\Delta + \frac{3}{2}\Delta + \frac{1}{4} - K - 3y$ is a square	
	$9\Delta\Delta + 6\Delta - 4K - 12y + 1$ is a square	
try square on $3\Delta + 1 - 6y$	$9\Delta\Delta + 42\Delta - 36K - 12y + 1$	
	$36\Delta = 32K$	
	$y = 9/8$	
cubes	$\frac{4913}{512}$	$\frac{1}{512}$
sum, product	$\frac{2456}{512}$	$\frac{2457}{512}$
numbers	$\frac{1228}{512} + x$	$\frac{1228}{512} - x$
	$\frac{94249}{16384} - \Delta = \frac{2457}{512}$	
	$\Delta = \frac{15625}{16384}$	
	$x = \frac{125}{128}$	
solution	$\frac{27}{8}$	$\frac{91}{64}$

Problem 29 $a + b + c + d + a^2 + b^2 + c^2 + d^2 = 12$
Solution:

square on $x + \frac{1}{2}$	$\Delta + x + \frac{1}{4}$			
	square plus side plus fourth is a square			
	so 13 is a sum of four squares			
	$13 = 4 + 9 = \frac{64}{25} + \frac{36}{25} + \frac{144}{25} + \frac{81}{25}$			
squares on $x + \frac{1}{2}$	$\frac{64}{25}$	$\frac{36}{25}$	$\frac{144}{25}$	$\frac{81}{25}$
solution	$\frac{11}{10}$	$\frac{7}{10}$	$\frac{19}{10}$	$\frac{13}{10}$

Problem 30 $a^2+b^2+c^2+d^2-a-b-c-d=4$
Solution:

square on $x-\frac{1}{2}$	$\Delta-x+\frac{1}{4}$
	square minus side plus fourth is a square
	so 5 is a sum of four squares
	$5=4+1=\frac{64}{25}+\frac{36}{25}+\frac{9}{25}+\frac{16}{25}$
squares on $x-\frac{1}{2}$	$\frac{64}{25}$ $\frac{36}{25}$ $\frac{9}{25}$ $\frac{16}{25}$
solution	$\frac{21}{10}$ $\frac{17}{10}$ $\frac{11}{10}$ $\frac{13}{10}$

Problem 31 $x+y=1$, $(x+3)(y+5)=a^2$
Solution:

numbers	1 $1-x$
condition	$x+3$ times $6-x$ is a square
	$18+3x-\Delta$ is a square
try 4Δ	$5\Delta=3x+18$
	need to replace 4 by a square whose 'determinant' is a square
	$\Delta+1$ times 18 plus $\frac{9}{4}$ is a square
	$18\Delta+\frac{81}{4}$ is a square
	$72\Delta+81$ is a square
try square on $8y+9$	$64\Delta+144y+81$
	$8\Delta=144y$
	$y=18$
	$18+3x-\Delta=324\Delta$
	$325\Delta=18+3x$
times 325 times 4	$422500\Delta=23400+3900x$
square on $650x-3$	$422500\Delta-3900x+9=23409$
	$650x-3=153$
	$x=\frac{6}{25}$
solution	$\frac{6}{25}$ $\frac{19}{25}$

Problem 32 $x+y+z=6$, $xy+z=a^2$, $xy-z=b^2$
Solution:

numbers	$2 \qquad 4-x \qquad x$
conditions	$8-x$, $8-3x$ are squares
	$8-x$, $8-3x$ are squares
	need to replace 2 by y such that ratio of $y+1$ to $y-1$ is a square
	$y+1=4y-4$
	$y=5/3$
numbers	$\frac{5}{3} \qquad \frac{13}{3}-x \qquad x$
	$\frac{65}{9}-\frac{8}{3}x$, $\frac{65}{9}-\frac{2}{3}x$ are squares
	$65-24x$, $65-6x$ are squares
	$65-24x$, $260-24x$ are squares
diff. of squares	195
sum, diff.	$15, 13$
	$14, 1$
	$65-24x=1$
	$x=8/3$
solution	$\frac{5}{3} \qquad \frac{5}{3} \qquad \frac{8}{3}$

Problem 33 $x+ry=3(y-ry)$, $y+rx=5(x-rx)$
Solution:

	let 2nd give 1 to 1st
parts	$3x-1 \qquad x+1$
	let 1st give y to 2nd
	1st + y is 5 times 1st $-y$
	1st + 2nd is 6 times 1st $-y$
	$4x/6$ is 1st$-y$
	y is $\frac{7}{3}x-1$
ratios	$\frac{7}{3}x-1 : 3x-1$ is same as $1 : x+1$
	$7\Delta+4x-3=9x-3$
	$7\Delta=5x$
	$x=5/7$
solution	$8 \qquad 12 \qquad 7$

Problem 34 $xy+x+y=8$, $yz+y+z=15$, $zx+z+x=24$
Solution:

let 2nd	$x-1$
1st	$8-x+1$ divided by x
	$9x^{\times}-1$
3rd	$15-x+1$ divided by x
	$16x^{\times}-1$
condition	$144\Delta^{\times}-1=24$
	$\Delta=144/25$
	$x=12/5$
solution	$\frac{11}{4} \qquad \frac{7}{5} \qquad \frac{17}{3}$

Problem 35 $xy - x - y = 8$, $yz - y - z = 15$, $zx - z - x = 24$
Solution:

let 2nd	$x + 1$
1st	$8 + x + 1$ divided by x
	$9x^{\times} + 1$
3rd	$15 + x + 1$ divided by x
	$16x^{\times} + 1$
condition	$144\Delta^{\times} - 1 = 24$
	$\Delta = 144/25$
	$x = 12/5$
solution	$\frac{19}{4}$ $\frac{17}{5}$ $\frac{23}{3}$

Problem 36 $xy = 3(x + y)$, $yz = 4(y + z)$, $zx = 5(z + x)$
Solution:

numbers[3]	y x z
sum : product	$x + y : xy$ is $3 : 1$
	$xy = 3x + 3y$
	y is the ratio $3x : x - 3$
similarly	z is the ratio $4x : x - 4$
product of 1st and 3rd	$12\Delta : \Delta - 7x + 12$
sum of 1st and 3rd	$7\Delta - 24x : \Delta - 7x + 12$
condition	$12\Delta = 35\Delta - 120x$
	$23\Delta = 120x$
	$x = 120/23$
solution	$\frac{360}{51}$ $\frac{120}{23}$ $\frac{480}{28}$

Problem 37 $xy = 3(x + y + z)$, $yz = 4(x + y + z)$, $zx = 5(x + y + z)$
Solution:

let sum	y
numbers	$3yx^{\times}$ x $4yx^{\times}$
condition	$12\Delta^{\times}y = 5$
	need $60y$ to be a square
	$60y = 900\Delta$
	$y = 15$
	$2700\Delta^{\times} = 75$
	$x = 6$
numbers	$\frac{15}{2}$ 6 10
rescale	$\frac{15}{2}x$ $6x$ $10x$
sum	$\frac{47}{2}x = 15\Delta$
	$x = \frac{47}{30}$
solution	$\frac{47}{4}$ $\frac{47}{5}$ $\frac{47}{3}$

[3]Diophantus does not use two unknowns; but by considering a specific number in a generic way, in effect he does so, especially since letters signified numbers in Greece.

Problem 38 $s = x + y + z$, $sx = \frac{n(n+1)}{2}$, $sy = a^2$, $sz = b^3$
Solution:

sum	Δ
try numbers	$6\Delta^\times$ $\quad$ $4\Delta^\times$ $\quad$ $8\Delta^\times$
sum	$18\Delta^\times = \Delta$
	$18 = \Delta\Delta$
	need to replace the 6, 4, 8, so that their sum is a fourth power
let square, cube	$\Delta\Delta - 2\Delta + 1$ $\quad$ 8
triangular	$2\Delta - 9$
	8 triangular + 1 is always a square
	$16\Delta - 71$ is a square
try square on $4x - 1$	$16\Delta - 8x + 1$
	$8x = 72$
	$x = 9$
solution	$\frac{17}{9}$ $\quad$ $\frac{6400}{81}$ $\quad$ $\frac{8}{81}$

Problem 39 $x - y = 3(y - z)$, $x + y = a^2$, $y + z = b^2$, $z + x = c^2$
Solution:

numbers	$7x + 2$ $\quad$ $2 + x$ $\quad$ $2 - x$
condition	$8x + 4$, $6x + 4$, 4 are squares
diff. of sq.	$2x$
sum, diff.	$\frac{1}{2}x$, 2
sides	$\frac{1}{4}x + 2$, $\frac{1}{4}x - 2$
square	$\frac{1}{16}\Delta + x + 4 = 8x + 4$
	$\frac{1}{16}\Delta = 7x$
	$x = 112$, which is too large
try 2nd, 3rd squares	$\Delta + 4y + 4$, 4
condition	$\Delta + 4y$ is three times 1st $-$ 2nd
1st square	$\frac{4}{3}\Delta + \frac{16}{3}y + 4$
condition	$3\Delta + 12y + 9$ is a square
try sq. on $2y - 3$	$4\Delta - 12y + 9$
	$\Delta = 24y$
	$y = 24$, too large
try sq. on $5y - 3$	$25\Delta - 30y + 9$
	$22\Delta = 42y$
	$y = \frac{21}{11}$
squares	$\frac{2304}{121}$ $\quad$ $\frac{1849}{121}$ $\quad$ 4
	$6x + 4 = \frac{1849}{121}$
	$x = \frac{455}{242}$
solution	$\frac{3669}{242}$ $\quad$ $\frac{939}{242}$ $\quad$ $\frac{29}{242}$

Problem 40 $x^2 - y^2 = 3(y^2 - z^2)$, $x + y = a^2$, $y + z = b^2$, $z + x = c^2$
Solution:

try numbers	$8\Delta + 2$ $8\Delta - 2$ $\Delta - 2$
conditions	16Δ, 9Δ, $9\Delta - 4$ are squares
	but $64\Delta : 7\Delta$ is not $3 : 1$
	need to replace 2 by a number such that
	when multiplied by 32 gives 21
numbers	$8\Delta + \frac{21}{32}$ $8\Delta - \frac{21}{32}$ $\Delta - \frac{21}{32}$
condition	$9\Delta - \frac{21}{16}$ is a square
	$144\Delta - 21$ is a square
try sq. on $12x - 1$	$144\Delta - 24x + 1$
	$24x = 22$
	$x = 11/12$
solution	$\frac{2125}{288}$ $\frac{1747}{288}$ $\frac{53}{288}$

5 Book V

Problem 1 To find three numbers in continued proportion such that each minus a given number (12) gives a square.

Problem 2 To find three numbers in continued proportion such that when added to a given number (20) each gives a square.

Problem 3 To find three numbers such that when each, or the product of any two, is added to a given number (5), the result is a square.

Problem 4 To find three numbers such that when each, or the product of any two, minus a given number (6) gives a square.

Problem 5 To find three squares such that the product of any two added to the sum of the two, or to the other square, give squares.

Problem 6 To find three numbers such that each minus 2, or the product of any two minus their sum or minus the remaining number, all give squares.

Problem 7 To find three numbers such that their sum added to or subtracted from any of their squares is another square.

Problem 8 To find three numbers such that their sum added to or subtracted from the product of any two gives a square.

Problem 9 To split one into two parts such that each added to a given number (6) gives a square.

Problem 10 To split one into two parts such that when added to two given numbers (2, 6) give squares.

Problem 11 To split one into three parts such that each added to a given number (3) gives a square.

Problem 12 To split one into three parts such that when added to three given numbers (2, 3, 4) give squares.

Problem 13 To split a given number (10) in three parts such that any two together form a square.

Problem 14 To split a given number (10) in four parts such that any three

together form a squares.

Problem 15 To find three numbers such that each added to the cube of their sum is a cube.

Problem 16 To find three numbers such that each subtracted from the cube of their sum is a cube.

Problem 17 To find three numbers such that each minus the cube of their sum is a cube.

Problem 18 To find three numbers whose sum is a square and such that each added to the cube of their sum is a square.

Problem 19 To find three numbers whose sum is a square and such that each subtracted from the cube of their sum is a square.

According to T. Heath, there are three missing problems following Problem 19:

19a To find three numbers such that their sum is a square and each minus the cube of their sum is a square.

19b To find three numbers such that their sum is a given number and each added to the cube of their sum is a square.

19c To find three numbers such that their sum is a given number and each subtracted from the cube of their sum is a square.

Problem 20 To split a given fraction ($\frac{1}{4}$) into three parts such that each minus the cube of their sum is a square.

Problem 21 To find three squares such that each added to their product is a square.

Problem 22 To find three squares such that their product minus each is a square.

Problem 23 To find three squares such each minus their product is a square.

Problem 24 To find three squares such that the product of any two plus 1 is a square.

Problem 25 To find three squares such that the product of any two minus 1 is a square.

Problem 26 To find three squares such that 1 minus the product of any two is a square.

Problem 27 To find three squares such that any two added to a given number (15) is a square.

Problem 28 To find three squares such that any two minus a given number (13) is a square.

Problem 29 To find three squares such that the sum of their squares is a square.

Problem 30 A man buys two amounts of wine, one at a cost of 8 drachmas each, and the other at 5 drachmas each. The total cost is a square number, and if 60 is added to this number, the result is a square, the side of which is the total amount of wine.

5.1 Solutions

Problem 1 $x-12=a^2$, $rx-12=b^2$, $r^2x-12=c^2$
Solution:

diff. of sq.	12		
sum, diff.	$4,\ 3$		
sides	$2+\frac{3}{2},\ 2-\frac{3}{2}$		
squares	$\frac{49}{4},\ \frac{1}{4}$		
numbers	$\frac{49}{4}$	$\frac{7}{2}x$	Δ
squares	$\frac{1}{4}$	$\frac{7}{2}x-12$	$\Delta-12$
diff. of sq.	$\Delta-\frac{7}{2}x$		
sum, diff.	$x,\ x-\frac{7}{2}$		
sides	$x-\frac{7}{4},\ \frac{7}{4}$		
	$\frac{7}{2}x-12=\frac{49}{16}$		
	$x=241/56$		
numbers	$\frac{49}{4}$	$\frac{241}{16}$	$\frac{58081}{3136}$

Problem 2 $x+20=a^2$, $rx+20=b^2$, $r^2x+20=c^2$
Solution:

try numbers	16	$4x$	Δ	
condition	36	$4x+20$	$\Delta+20$	are squares
diff. of sq.	$\Delta-4x$			
sum, diff.	$x,\ x-4$			
sides	$x-2,\ 2$			
	but $4x+20=4$ is impossible			
	need to replace 16 by a square greater than 80			
try sq. on $9+y$	$\Delta+18y+81$			
condition	$\Delta+18y+101$ is a square			
try sq. on $y-11$	$\Delta-22y+121$			
	$40y=20$			
	$y=1/2$			
numbers	$\frac{361}{4}$	$\frac{19}{2}x$	Δ	
condition	$\frac{441}{4}$	$\frac{19}{2}x+20$	$\Delta+20$	are squares
diff. of sq.	$\Delta-\frac{19}{2}x$			
su, diff.	x	$x-\frac{19}{2}$		
sides	$x-\frac{19}{4}$	$\frac{19}{4}$		
	$\frac{19}{2}x+20=\frac{361}{16}$			
	$x=\frac{41}{152}$			
solution	$\frac{361}{4}$	$\frac{41}{16}$	$\frac{1681}{23104}$	

Problem 3 $x_i+5=a_i^2$, $x_ix_j+5=b_{ij}^2$ $(i\neq j)$
Solution:

	the squares must be of consecutive numbers		
squares	$\Delta+6x+9$	$\Delta+8x+16$	
numbers	$\Delta+6x+4$	$\Delta+8x+11$	$4\Delta+28x+29$
condition	$4\Delta+28x+34$ is a square		
try sq. on $2x-6$	$4\Delta-24x+36$		
	$52x=2$		
	$x=1/26$		
solution	$\frac{2861}{676}$	$\frac{7645}{676}$	$\frac{5084}{169}$

Problem 4 $x_i-6=a_i^2$, $x_ix_j-6=b_{ij}^2$ $(i\neq j)$
Solution:

squares	Δ	$\Delta+2x+1$	
numbers	$\Delta+6$	$\Delta+2x+7$	$4\Delta+4x+25$
condition	$4\Delta+4x+19$ is a square		
try sq. on $2x-6$	$4\Delta-24x+36$		
	$28x=17$		
	$x=17/28$		
solution	$\frac{4993}{784}$	$\frac{6729}{784}$	$\frac{5665}{196}$

Problem 5 $x_i^2x_j^2+x_i^2+x_j^2=a_{ij}^2$, $x_i^2x_j^2+x_k^2=b_{ijk}^2$
Solution:

squares	$\Delta+2x+1$	$\Delta+4x+4$	$4\Delta+12x+12$
condition	$4\Delta+12x+12$ is a square		
	$\Delta+3x+3$ is a square		
try sq. on $x-3$	$\Delta-6x+9$		
	$9x=6$		
	$x=2/3$		
solution	$\frac{25}{9}$	$\frac{64}{9}$	$\frac{196}{9}$

Problem 6 $x_i-2=a_i^2$, $x_ix_j-x_i-x_j=b_{ij}^2$, $x_ix_j-x_k=c_{ijk}^2$
Solution:

numbers	$\Delta+2$	$\Delta+2x+3$	$4\Delta+4x+6$
conditions	are all satisfied, except		
	$4\Delta+4x+4$ is a square		
	$\Delta+x+1$ is a square		
try sq. on $x-2$	$\Delta-4x+4$		
	$5x=3$		
	$x=3/5$		
solution	$\frac{59}{25}$	$\frac{114}{25}$	$\frac{246}{25}$

Problem 7 $x_i^2 + (x_1 + x_2 + x_3) = a_i^2$, $x_i^2 - (x_1 + x_2 + x_3) = b_i^2$
Solution:

	In right-angled triangles, the square on the hypotenuse plus or minus four times the area is another square; so need to find three right-angled trianges with the same area. need two numbers whose product added to their sum of squares is a square
numbers	y 1
try sq. on $y-2$	$y + \Delta + 1$ is a square $\Delta - 4y + 4$ $5y = 3$ $y = 3/5$
rescaling	3, 5 are such numbers with $15 + 9 + 25 = 49$
Pythagorean triangles	from 7, 3; 7, 5; 7, 8; $(8 = 3 + 5)$
triangle sides	40, 42, 58; 24, 70, 74; 15, 112, 113;
numbers	$58x$ $74x$ $113x$
condition	$245x = 3360\Delta$ $x = 7/96$
solution	$\frac{203}{48}$ $\frac{259}{48}$ $\frac{791}{96}$

Problem 8 $x_i x_j + (x_1 + x_2 + x_3) = a_{ij}^2$, $x_i x_j - (x_1 + x_2 + x_3) = b_{ij}^2$
Solution:

	to find three right-angled triangles with equal areas		
hypotenus	58	74	113
squares	3364	5476	12769
numbers	y	$3364y^{\times}$	$5476y^{\times}$
condition	$18421264\Delta^{\times} = 12769$ $y = \frac{4292}{113}$		
scaled numbers	$\frac{4292}{113}x$	$\frac{3277}{37}x$	$\frac{4181}{29}x$
sum	$\frac{32824806}{121249}x = 3360\Delta$ $x = \frac{781543}{9699920}$		
solution	$\frac{781543}{255380}$	$\frac{781543}{109520}$	$\frac{781543}{67280}$

Problem 9 $x+y=1$, $x+6=a^2$, $y+6=b^2$
Solution:

	twice six plus one needs to split in two squares;	
	$13=9+4$ but need squares larger than 6;	
	find square just larger than 13/2	
	$\frac{13}{2}+\frac{1}{4}\Delta^{\times}$ is a square	
	$26\Delta+1$ is a square	
try sq. on $5y+1$	$25\Delta+10y+1$	
	$\Delta=10y$	
	$y=10$	
square, side	$\frac{2601}{400}$	$\frac{51}{20}$
differ from sides 3, 2	$\frac{9}{20}$	$\frac{11}{20}$
sq. on $2+11x$	$121\Delta+44x+4$	
sq. on $3-9x$	$81\Delta-54x+9$	
sum	$202\Delta-10x+13=13$	
	$202\Delta=10x$	
	$x=\frac{5}{101}$	
sides	$\frac{257}{101}$	$\frac{258}{101}$
solution	$\frac{4843}{10201}$	$\frac{5358}{10201}$

Problem 10 $x+y=1$, $x+2=a^2$, $y+6=b^2$
Solution:

	2 plus 6 plus 1 needs to split into two squares	
squares	Δ	$9-\Delta$
pick squares bet. 2 and 3	$\frac{289}{144}$	$\frac{361}{144}$
sides	$\frac{17}{12}$	$\frac{19}{12}$
	x is between $\frac{17}{12}$ and $\frac{19}{12}$	
try sq. on $3-3x$	$9-18x+9\Delta=9-\Delta$	
	$18x=10\Delta$	
	$x=9/5$ too large	
try sq. on $3-\frac{7}{2}x$	$9-21x+\frac{49}{4}\Delta=9-\Delta$	
	$21x=\frac{53}{4}\Delta$	
	$x=\frac{84}{53}$	
squares	$\frac{7056}{2809}$	$\frac{18225}{2809}$
solution	$\frac{1438}{2809}$	$\frac{1371}{2809}$

Problem 11 $x+y+z=1$, $x+3=a^2$, $y+3=b^2$, $z+3=c^2$
Solution:

	thrice 3 plus 1 needs to split in three squares
	$\frac{10}{3}+\frac{1}{9}\Delta^{\times}$ is a square
	$30\Delta+1$ is a square
try sq. on $5y+1$	$25\Delta+10y+1$
	$5\Delta=10y$
	$y=2$
square, side	$\frac{121}{36}$ $\frac{11}{6}$
	$10=9+\frac{16}{25}+\frac{9}{25}$ of sides $\frac{90}{30}$, $\frac{24}{30}$, $\frac{18}{30}$
	differ from required side $\frac{55}{30}$ by $\frac{35}{30}$, $\frac{31}{30}$, $\frac{37}{30}$
try sq. on $3-35x$	$9-210x+1225\Delta$
sq. on $\frac{4}{5}+31x$	$\frac{16}{25}+\frac{248}{5}x+961\Delta$
sq. on $\frac{3}{5}+37x$	$\frac{9}{25}+\frac{222}{5}x+1369\Delta$
sum	$10-116x+3555\Delta=10$
	$x=116/3555$
sides	$\frac{1321}{711}$ $\frac{1288}{711}$ $\frac{1285}{711}$
solution	$\frac{228478}{505521}$ $\frac{142381}{505521}$ $\frac{134662}{505521}$

Problem 12 $x+y+z=1$, $x+2=a^2$, $y+3=b^2$, $z+4=c^2$
Solution:

	2 plus 3 plus 4 plus 1 needs to split in three squares
	first need square just bigger than $2\frac{1}{2}$
	$\frac{5}{2}+\frac{1}{4}\Delta^{\times}$ is a square
	$10\Delta+1$ is a square
try sq. on $3y+1$	$9\Delta+6y+1$
	$\Delta=6y$
	$y=6$
square	$\frac{361}{144}$ square on $\frac{19}{12}$
	second need square just bigger than $7\frac{1}{2}$
	$\frac{15}{2}+\frac{1}{4}\Delta^{\times}$ is a square
	$30\Delta+1$ is a square
try sq. on $5y+1$	$25\Delta+10y+1$
	$5\Delta=10y$
	$y=2$
square	$\frac{121}{16}$ square on $\frac{11}{4}$
	but $10=9+1$
diff. of sq. from 3, 1	$\frac{3}{12}$ $\frac{7}{12}$
try sq. on $1+7x$	$1+14x+49\Delta$
sq. on $3-3x$	$9-18x+9\Delta$
sum	$10-4x+58\Delta=10$

	$x = 2/29$			
sides	$\frac{43}{29}$		$\frac{81}{29}$	
squares	$\frac{1849}{841}$		$\frac{6561}{841}$	
	split 6561/841 in two squares near 3, 4			
try sq. on $\frac{81}{29} - \frac{5}{2}x$	$\frac{6561}{841} - \frac{405}{29}x + \frac{25}{4}\Delta = \frac{6561}{841} - \Delta$			
	$\frac{29}{4}\Delta = \frac{405}{29}x$			
	$x = \frac{1620}{841}$			
solution	$\frac{167}{841}$		$\frac{502557}{707281}$	$\frac{64277}{707281}$

Problem 13 $x + y + z = 10$, $x + y = a^2$, $y + z = b^2$, $z + x = c^2$
Solution:

	sum of three squares is 20		
	$20 = 4 + 16$		
	so need to split 16 in two squares		
	this is problem 10		
squares	4	$\frac{6400}{841}$	$\frac{7056}{841}$
solution	6	$\frac{2010}{841}$	$\frac{1354}{841}$

Problem 14 $a + b + c + d = 10$, $a + b + c = u_1^2$, $a + b + d = u_2^2$, $a + c + d = u_3^2$, $b + c + d = u_4^2$
Solution:

	sum of four squares is 30			
	$30 = 16 + 9 + 4 + 1$			
	need to split 17 in two squares			
	this is problem 10			
squares	4	9	$\frac{1032256}{121801}$	$\frac{1038361}{121801}$
solution	6	1	$\frac{185754}{121801}$	$\frac{179649}{121801}$

Problem 15 $s = x + y + z$, $x + s^3 = a^3$, $y + s^3 = b^3$, $z + s^3 = c^3$
Solution:

cube of sum	K		
numbers	$7K$	$26K$	$63K$
	$96K = x$		
	$96\Delta = 1$		
	need to replace 7, 26, 63		
sides	$y + 1$	$2 - y$	2
cubes	$K + 3\Delta + 3y + 1$	$8 - 12y + 6\Delta - K$	8
repl. numbers	$K + 3\Delta + 3y$	$7 - 12y + 6\Delta - K$	7
sum	$9\Delta - 9y + 14$ is a square		
try sq. on $3y - 4$	$9\Delta - 24y + 16$		
	$15y = 2$		
	$y = 2/15$		
repl. numbers	$\frac{1538}{3375}$	$\frac{18577}{3375}$	7
numbers	$\frac{1538}{3375}K$	$\frac{18577}{3375}K$	$7K$
sum	$\frac{43740}{3375}K = x$		
	$x = 5/18$		
solution	$\frac{769}{78732}$	$\frac{18577}{157464}$	$\frac{875}{5832}$

Problem 16 $s = x + y + z$, $s^3 - x = a^3$, $s^3 - y = b^3$, $s^3 - z = c^3$
Solution:

cube of sum	K		
numbers	$\frac{7}{8}K$	$\frac{26}{27}K$	$\frac{63}{64}K$
sum	$\frac{4877}{1728}K = x$		
	need to replace numbers to give a square		
	let square be $9/4$		
	sum of cubes is $3 - \frac{9}{4} = \frac{3}{4}$		
multiply by 216	sum of cubes is 162		
	$162 = 125 + 64 - 27$		
	$162 = 125 + \frac{6859}{343} + \frac{5832}{343}$		
	$\frac{3}{4} = \frac{125}{216} + \frac{6859}{74088} + \frac{5832}{74088}$		
cubes	$\frac{125}{216}K$	$\frac{6859}{74088}K$	$\frac{5832}{74088}K$
numbers	$\frac{91}{216}K$	$\frac{67229}{74088}K$	$\frac{316}{343}K$
sum	$\frac{9}{4}K = x$		
	$x = 2/3$		
solution	$\frac{91}{729}$	$\frac{67229}{250047}$	$\frac{2528}{9261}$

Problem 17 $s = x + y + z$, $x - s^3 = a^3$, $y - s^3 = b^3$, $z - s^3 = c^3$
Solution:

cube of sum	K		
numbers	$2K$	$9K$	$28K$
sum	$39K = x$		
	need to replace the numbers		
sides	y	$3 - y$	1
cubes	K	$27 - 27y + 9\Delta - K$	1
add one	$K + 1$	$28 - 27y + 9\Delta - K$	2
sum	$9\Delta - 27y + 31$ needs to be a square		
try sq. on $3y - 7$	$9\Delta - 42y + 49$		
	$15y = 18$		
	$y = 6/5$		
cubes	$\frac{216}{125}$	$\frac{729}{125}$	1
numbers	$\frac{341}{125}K$	$\frac{854}{125}K$	$2K$
sum	$\frac{289}{25}K = x$		
	$x = 5/17$		
solution	$\frac{341}{4913}$	$\frac{854}{4913}$	$\frac{250}{4913}$

Problem 18 $x + y + z = a^2$, $x + a^6 = b^2$, $y + a^6 = c^2$, $z + a^6 = d^2$
Solution:

sum	Δ			
numbers	$3KK$	$8KK$	$15KK$	
sum	$26KK = \Delta$			
	need to replace the numbers			
squares	$\Delta\Delta - 2\Delta + 1$	$\Delta + 2y + 1$	$\Delta - 2y + 1$	
numbers	$\Delta\Delta - 2\Delta$	$\Delta + 2y$	$\Delta - 2y$	
sum	$\Delta\Delta$ take 3 as y	63	15	3
numbers	$63KK$	$15KK$	$3KK$	
sum	$91KK = \Delta$			
	$91\Delta\Delta = 1$			
	$x = 1/3$			
solution	$\frac{63}{729}$	$\frac{15}{729}$	$\frac{3}{729}$	

Problem 19 $x + y + z = a^2$, $a^6 - x = b^2$, $a^6 - y = c^2$, $a^6 - z = d^2$
Solution:

	solution is missing

Problem 20 $x + y + z = \frac{1}{4}$, $x - \frac{1}{4^3} = a^2$, $y - \frac{1}{4^3} = b^2$, $z - \frac{1}{4^3} = c^2$
Solution:

	each part minus $\frac{1}{64}$ is a square;		
	sum of squares is $\frac{1}{4} - \frac{3}{64}$, i.e., $\frac{13}{64}$;		
	problem reduces to splitting $\frac{13}{64}$ into three squares;		
squares	$\frac{9}{64}$ $\frac{1}{25}$		$\frac{9}{400}$
solution	$\frac{5}{32}$ $\frac{89}{1600}$		$\frac{61}{1600}$

Problem 21 $s = x^2y^2z^2$, $x^2 + s = a^2$, $y^2 + s = b^2$, $z^2 + s = c^2$
Solution:

product of three	Δ		
	need three squares, adding 1 gives a square		
	use right-angled triangles		
try squares	$\frac{9}{16}\Delta$	$\frac{25}{144}\Delta$	$\frac{64}{225}\Delta$
product	$\frac{1}{36}\Delta\Delta\Delta = \Delta$		
	$\Delta\Delta = 36$		
	need to replace the numbers		
triangles	3,4,5;	9,40,41;	8,15,17;
squares	$\frac{9}{16}\Delta$	$\frac{81}{1600}\Delta$	$\frac{64}{225}\Delta$
product	$\frac{6561}{65536}\Delta\Delta\Delta = \Delta$		
	$6561\Delta\Delta = 65536$		
	$x = 16/9$		
solution	$\frac{16}{9}$	$\frac{100}{9}$	$\frac{4}{25}$

Problem 22 $s = x^2y^2z^2$, $s - x^2 = a^2$, $s - y^2 = b^2$, $s - z^2 = c^2$
Solution:

product	Δ		
try squares	$\frac{16}{25}\Delta$	$\frac{25}{169}\Delta$	$\frac{64}{289}\Delta$
product	$\frac{1024}{48841}\Delta\Delta\Delta = \Delta$		
	need to replace the numbers		
triangles	3,4,5;	5,12,13;	3,4,5;
new triangles	3,4,5;	$\frac{13}{2}, \frac{60}{13}, \frac{119}{26}$	$\frac{5}{2}, \frac{12}{5}, \frac{7}{10}$
squares	$\frac{16}{25}\Delta$	$\frac{14400}{28561}\Delta$	$\frac{576}{625}\Delta$
product	$\frac{5308416}{17850625}\Delta\Delta\Delta = \Delta$		
	$5308416\Delta\Delta = 17850625$		
	$x = 65/48$		
solution	$\frac{169}{144}$	$\frac{625}{676}$	$\frac{169}{100}$

Problem 23 $s = x^2y^2z^2$, $x^2 - s = a^2$, $y^2 - s = b^2$, $z^2 - s = c^2$
Solution:

product	Δ		
triangles	3,4,5;	$\frac{13}{2}, \frac{60}{13}, \frac{119}{26}$	$\frac{5}{2}, \frac{12}{5}, \frac{7}{10}$
squares	$\frac{25}{16}\Delta$	$\frac{28561}{14400}\Delta$	$\frac{625}{576}\Delta$
product	$\frac{17850625}{5308416}\Delta\Delta\Delta = \Delta$		
	$x = 48/65$		
solution	$\frac{144}{169}$	$\frac{676}{625}$	$\frac{100}{169}$

Problem 24 $x^2y^2+1=a^2$, $y^2z^2+1=b^2$, $z^2x^2+1=c^2$
Solution:

	product of two plus 1 is a square;
	so, product of all three plus each is a square;
	same problem as 21.
solution	$\frac{16}{9}$ $\frac{100}{9}$ $\frac{4}{25}$

Problem 25 $x^2y^2-1=a^2$, $y^2z^2-1=b^2$, $z^2x^2-1=c^2$
Solution:

	same problem as 22.
solution	$\frac{169}{144}$ $\frac{625}{676}$ $\frac{169}{100}$

Problem 26 $1-x^2y^2=a^2$, $1-y^2z^2=b^2$, $1-z^2x^2=c^2$
Solution:

	same problem as 23.
solution	$\frac{144}{169}$ $\frac{676}{625}$ $\frac{100}{169}$

Problem 27 $x^2+y^2+15=a^2$, $y^2+z^2+15=b^2$, $z^2+x^2+15=c^2$
Solution:

	let one square be 9;
diff. of sq.	24
sum, diff.	$6x$ $4x^\times$
sides	$3x-2x^\times$ $3x+2x^\times$
sum, diff.	$8x$ $3x^\times$
sides	$4x-\frac{3}{2}x^\times$ $4x+\frac{3}{2}x^\times$
squares	9 $9\Delta-12+4\Delta^\times$ $16\Delta-12+\frac{9}{4}\Delta^\times$
condition	$25\Delta-24+\frac{25}{4}\Delta^\times+15$ is a square
	$100\Delta+25\Delta^\times-36$ is a square, say, 100Δ
	$25\Delta^\times=36$
	$x=5/6$
solution	9 $\frac{1}{100}$ $\frac{529}{225}$

Problem 28 $x^2+y^2-13=a^2$, $y^2+z^2-13=b^2$, $z^2+x^2-13=c^2$
Solution:

	let one square be 25;
diff. of sq.	12
sum, diff.	$3x$ $4x^\times$
sides	$\frac{3}{2}x-2x^\times$
sum, diff.	$4x$ $3x^\times$
sides	$2x-\frac{3}{2}x^\times$
squares	25 $\frac{9}{4}\Delta-6+4\Delta^\times$ $4\Delta-6+\frac{9}{4}\Delta^\times$
condition	$\frac{25}{4}\Delta-12+\frac{25}{4}\Delta^\times-13$ is a square, say, $\frac{25}{4}\Delta^\times$
	$\frac{25}{4}\Delta=25$
	$x=2$
solution	25 4 $\frac{169}{16}$

Problem 29 $x^2 + y^2 + z^2 = a^2$
Solution:

squares	4 9 Δ
condition	$\Delta\Delta + 97$ is a square
try sq. on $\Delta - 10$	$\Delta\Delta - 20\Delta + 100$
	$20\Delta = 3$
	need to replace 4 and 9
try squares	4 Δ
	$16 + \Delta\Delta$
sq. using $\Delta + 4$	$2\Delta + 8$ and $\Delta\Delta + 8\Delta + 16$ instead of 20, 100
	$2\Delta + 8$ and 8Δ instead of 20, 3
	their ratio needs to be a square;
	$\Delta + 4$ needs to be a square;
try sq. on $y + 1$	$\Delta + 2y + 1$
	$2y = 3$
	$y = 3/2$
squares (scaled)	16 9 Δ
condition	$\Delta\Delta + 337$ is a square
sq. on $\Delta - 25$	$\Delta\Delta - 50\Delta + 625$
	$50\Delta = 288$
	$x = 12/5$
solution	16 9 $\frac{144}{25}$

Problem 30 $5x + 8y = a^2$, $a + 60 = (x + y)^2$
Solution:

amount of wine	x
total cost	$\Delta - 60$
condition	fifth of first cost plus eighth of second cost is x
	$\Delta - 60$ must be between $5x$ and $8x$
	$\Delta - 5x$ is larger than 60, so x is larger than 10
	$\Delta - 8x$ is smaller than 60, so x is smaller than 13
try sq. on $x - 20$	$\Delta - 40x + 400$
	$40x = 460$
	$x = \frac{23}{2}$
first cost	y
second amount	$\frac{23}{2} - \frac{1}{5}y$
condition	$y + 92 - \frac{8}{5}y = \frac{529}{4} - 60$
	$\frac{3}{5}y = \frac{79}{4}$
	$y = \frac{395}{12}$
solution	$\frac{79}{12}$ $\frac{59}{12}$

6 Book VI

Problem 1 To find a right-angled triangle such that each side subtracted from the hypotenuse is a cube.

Problem 2 To find a right-angled triangle such that each side added to the hypotenuse is a cube.

Problem 3 To find a right-angled triangle whose area added to a given number (5) is a square.

Problem 4 To find a right-angled triangle whose area minus a given number (6) is a square.

Problem 5 To find a right-angled triangle such that its area subtracted from a given number (10) gives a square.

Problem 6 To find a right-angled triangle such that a side added to its area is a given number (7).

Problem 7 To find a right-angled triangle such that its side subtracted from its area is a given number (7).

Problem 8 To find a right-angled triangle such that the sum of the sides added to the area is a given number (6).

Problem 9 To find a right-angled triangle such that the sum of the sides subtracted from the area is a given number (6).

Problem 10 To find a right-angled triangle such that the sum of a side with the hypotenuse and the area is a given number (4).

Problem 11 To find a right-angled triangle such that the sum of a side with the hypotenuse subtracted from the area is a given number (4).

Problem 12 To find a right-angled triangle such that the area added to any side is a square.

Problem 13 To find a right-angled triangle such that the area minus any side is a square.

Problem 14 To find a right-angled triangle such that the area minus the

hypotenuse or minus a side are each squares.

Problem 15 To find a right-angled triangle such that the area added to the hypotenuse or a side are each squares.

Problem 16 To find a right-angled triangle such that the bisector of an acute angle cuts the opposite side rationally.

Problem 17 To find a right-angled triangle such that the area added to the hypotenuse is a square and the perimeter is a cube.

Problem 18 To find a right-angled triangle such that the area added to the hypotenuse is a cube and the perimeter is a square.

Problem 19 To find a right-angled triangle such that its area added to a side is a square and the perimeter is a cube.

Problem 20 To find a right-angled triangle such that its area added to a side is a cube and the perimeter is a square.

Problem 21 To find a right-angled triangle such that its area added to the perimeter is a cube and the perimeter is a square.

Problem 22 To find a right-angled triangle such that its area added to the perimeter is a square and the perimeter is a cube.

Problem 23 To find a right-angled triangle such that the square of the hypotenuse is the sum of a square and its side, while the square of the hypotenuse divided by a side is the sum of a cube and its side.

Problem 24 To find a right-angled triangle such that one side is a cube, the other side is the difference between a cube and its side, and the hypotenuse is the sum of a cube and its side.

6.1 Solutions

Problem 1 $x^2+y^2=z^2$, $z-x=a^3$, $z-y=b^3$
Solution:

sides from x, 2[4]	$4x$	$\Delta-4$	$\Delta+4$
condition	$\Delta+4-4x$ is a cube		
	square on $x-2$ is a cube		
	so $x-2$ is a cube, say, 8		
	$x=10$		
solution	40	96	104

Problem 2 $x^2+y^2=z^2$, $x+z=a^3$, $y+z=b^3$
Solution:

sides from x, 2	$4x$	$4-\Delta$	$4+\Delta$
condition	$\Delta+4+4x$ is a cube		
	$x+2$ is a cube, say $\frac{27}{8}$		
	$x=\frac{11}{8}$		
triangle	$\frac{11}{2}$	$\frac{135}{64}$	$\frac{377}{64}$
solution	352	135	377

Problem 3 $x^2+y^2=z^2$, $\frac{1}{2}xy+5=a^2$
Solution:

sides	$3x$	$4x$	$5x$
area	6Δ		
condition	$6\Delta+5$ is a square, say, 9Δ		
	need to replace 3,4,5		
triangle from y, $y^\times$	2	$\Delta-\Delta^\times$	$\Delta+\Delta^\times$
area	$\Delta-\Delta^\times$		
condition	$\Delta-\Delta^\times+5$ is a square		
try sq. on $y+10y^\times$	$\Delta+20+100\Delta^\times$		
	$100\Delta+505$ is a square		
try sq. on $10y+5$	$100\Delta+100y+25$		
	$100y=480$		
	$y=24/5$		
triangle from $\frac{24}{5}$, $\frac{5}{24}$	$2x$	$\frac{331151}{14400}x$	$\frac{332401}{14400}x$
condition	$\frac{331151}{14400}\Delta+5$ is square on $\frac{413}{60}x$		
	$14045\Delta=2880$		
	$x=24/53$		
solution	$\frac{48}{53}$	$\frac{331151}{31800}$	$\frac{332401}{31800}$

[4] A triangle formed from m, n has sides m^2+n^2, m^2-n^2, $2mn$.

Problem 4 $x^2+y^2=z^2$, $\frac{1}{2}xy-6=a^2$
Solution:

sides	$3x$	$4x$	$5x$
condition	$6\Delta-6$ is a square		
	need to replace 3,4,5		
triangle from y, $y^\times$	2	$\Delta-\Delta^\times$	$\Delta+\Delta^\times$
area	$\Delta-\Delta^\times$		
try sq. on $y-3y^\times$	$\Delta-6+9\Delta^\times$		
	$36\Delta-60$ is a square		
try sq. on $6y-2$	$36\Delta-24y+4$		
	$24y=64$		
	$y=8/3$		
triangle from $\frac{8}{3}$, $\frac{3}{8}$	$2x$	$\frac{4015}{576}x$	$\frac{4177}{576}x$
condition	$\frac{4015}{576}\Delta-6$ is square on $\frac{37}{24}x$		
	$\frac{4015}{576}\Delta-6=\frac{1369}{576}\Delta$		
	$x=8/7$		
solution	$\frac{16}{7}$	$\frac{4015}{504}$	$\frac{4177}{504}$

Problem 5 $x^2+y^2=z^2$, $10-\frac{1}{2}xy=a^2$
Solution:

sides	$3x$	$4x$	$5x$
condition	$10-6\Delta$ is a square		
	need to replace 3,4,5		
triangle from y, $y^\times$	2	$\Delta-\Delta^\times$	$\Delta+\Delta^\times$
try sq. on $y^\times+5y$	$\Delta^\times+10+25\Delta$		
	$26\Delta+10$ is a tenth of a square		
	$260\Delta+100$ is a square		
	$65\Delta+25$ is a square		
try sq. on $8y+5$	$64\Delta+80y+25$		
	$\Delta=80y$		
	$y=80$		
triangle from 80, $\frac{1}{80}$	$2x$	$\frac{40959999}{6400}x$	$\frac{40960001}{6400}x$
condition	$10-\frac{40959999}{6400}\Delta=\frac{1024064001}{6400}\Delta$		
	$x=\frac{1}{129}$		
solution	$\frac{2}{129}$	$\frac{13653333}{275200}x$	$\frac{40960001}{825600}x$

Problem 6 $x^2+y^2=z^2$, $x+\frac{1}{2}xy=7$
Solution:

sides	$3x$	$4x$	$5x$
condition	$6\Delta+3x=7$		
	need to replace 3,4,5		
try numbers	1	y	
discriminant	$\frac{1}{4}+\frac{7}{2}y$ is a square		
	$14y+1$ is a square		
	and $\Delta+1$ is a square		
diff. of sq.	$\Delta-14y$		
sum, diff.	y, $y-14$		
sides	$y-7$, 7		
square	$49=14y+1$		
	$y=24/7$		
sides (scaled)	$24x$	$7x$	$25x$
condition	$84\Delta+7x=7$		
	$x=\frac{1}{4}$		
solution	6	$\frac{7}{4}$	$\frac{25}{4}$

Problem 7 $x^2+y^2=z^2$, $\frac{1}{2}xy-x=7$
Solution:

	need to replace 3,4,5		
sides	$7x$	$24x$	$25x$
condition	$84\Delta-7x=7$		
	$x=1/3$		
solution	$\frac{7}{3}$	8	$\frac{25}{3}$

Problem 8 $x^2+y^2=z^2$, $x+y+\frac{1}{2}xy=6$
Solution:

	need to replace 3,4,5		
try triangle	1	y	
discriminant	$\frac{1}{4}\Delta+\frac{7}{2}y+\frac{1}{4}$ is a square		
	$\Delta+14y+1$ and $\Delta+1$ are squares		
diff. of sq.	$14y$		
sum, diff.	$2y$	7	
sides	$y+\frac{7}{2}$, $y-\frac{7}{2}$		
square	$\Delta-7y+\frac{49}{4}=\Delta+1$		
	$7y=\frac{45}{4}$		
	$y=\frac{45}{28}$		
sides (scaled)	$45x$	$28x$	$53x$
condition	$630\Delta+45x+28x=6$		
	$x=1/18$		
solution	$\frac{5}{2}$	$\frac{14}{9}$	$\frac{53}{18}$

Problem 9 $x^2+y^2=z^2$, $\frac{1}{2}xy-x-y=6$
Solution:

sides	$28x$	$45x$	$53x$
condition	$630\Delta-73x=6$		
	$x=6/35$		
solution	$\frac{24}{5}$	$\frac{54}{7}$	$\frac{318}{35}$

Problem 10 $x^2+y^2=z^2$, $y+z+\frac{1}{2}xy=4$
Solution:

	need to replace 3,4,5		
try triangle on 1, $y+1$	$2y+2$	$\Delta+2y$	$\Delta+2y+2$
discriminant	$\Delta\Delta+8K+18\Delta+12y+1$ is a square		
try sq. on $6y+1-\Delta$	$\Delta\Delta-12K+34\Delta+12y+1$		
	$20K=16\Delta$		
	$y=4/5$		
sides	$90x$	$56x$	$106x$
condition	$2520\Delta+56x+106x=4$		
	$x=2/105$		
solution	$\frac{12}{7}$	$\frac{16}{15}$	$\frac{212}{105}$

Problem 11 $x^2+y^2=z^2$, $\frac{1}{2}xy-y-z=4$
Solution:

sides	$90x$	$56x$	$106x$
condition	$2520\Delta-56x-106x=4$		
	$x=1/12$		
solution	$\frac{15}{2}$	$\frac{14}{3}$	$\frac{53}{6}$

Problem 12 $x^2+y^2=z^2$, $\frac{1}{2}xy+x=a^2$, $\frac{1}{2}xy+y=b^2$
Solution:

try sides	$3x$	$4x$	$5x$	
condition	$6\Delta+4x$ and $6\Delta+3x$ are squares;			
	need to find a square times Δ such that 4 divided by this square minus 6			
	gives a square for the second expression;			
	$96+12\Delta-72$ is a square			
	$3\Delta+6$ is a square, for example, 81			
	needed square is 25			$6\Delta+4x=25\Delta$
	$19\Delta=4x$			
	$x=4/19$			
solution	$\frac{12}{19}$	$\frac{16}{19}$	$\frac{20}{19}$	

Problem 13 $x^2+y^2=z^2$, $\frac{1}{2}xy-x=a^2$, $\frac{1}{2}xy-y=b^2$
Solution:

try sides	$3x$ $4x$ $5x$
condition	$6\Delta-3x$ and $6\Delta-4x$ are squares;
	need a square Δ such that $96-72+12\Delta$ is a square;
	$6+3\Delta$ is a square, for example, 9
	needed square is 1
sides	3 4 5
	$6\Delta-4x=\Delta$
	$x=4/5$
solution	$\frac{12}{5}$ $\frac{16}{5}$ 4

Problem 14 $x^2+y^2=z^2$, $\frac{1}{2}xy-z=a^2$, $\frac{1}{2}xy-x=b^2$
Solution:

try sides	$8x$ $15x$ $17x$
condition	$60\Delta-17x$ and $60\Delta-8x$ are squares
diff. of sq.	$9x$
sum, diff.	$3x$, 3
sides	$\frac{3}{2}x+\frac{3}{2}$, $\frac{3}{2}x-\frac{3}{2}$
	$\frac{9}{4}\Delta+\frac{9}{2}x+\frac{9}{4}=60\Delta-8x$
	$231\Delta=50x+9$
	$x=1/3$
sides	$\frac{8}{3}$ 5 $\frac{17}{3}$

Problem 15 $x^2+y^2=z^2$, $\frac{1}{2}xy+z=a^2$, $\frac{1}{2}xy+x=b^2$
Solution:

try sides	$8x$ $15x$ $17x$
condition	$60\Delta+17x$ and $60\Delta+8x$ are squares
diff. of sq.	$9x$
sum, diff.	$3x$, 3
sides	$\frac{3}{2}x+\frac{3}{2}$, $\frac{3}{2}-\frac{3}{2}$
	$\frac{9}{4}\Delta-\frac{9}{2}x+\frac{9}{4}=60\Delta+8x$
	$231\Delta+50x=9$
	$x=9/77$
solution	$\frac{72}{77}$ $\frac{135}{77}$ $\frac{153}{77}$

Problem 16 $x^2+y^2=z^2$, $\frac{u}{z}=\frac{v}{x}$, $u+v=y$
Solution:

sides of smaller triangle ABD	$3x$ $4x$ $5x$
let bisected side BC	3
$AB:BD$ equals $AC:CD$	hypotenuse is to $3-3x$ as $4:3$
hypotenuse	$4-4x$
sum of sq. on sides	$16-32x+16\Delta=16\Delta+9$
	$32x=7$
	$x=7/32$
solution	3 $\frac{7}{8}$ $\frac{25}{8}$

Problem 17 $x^2+y^2=z^2$, $\frac{1}{2}xy+z=a^2$, $x+y+z=b^3$
Solution:

sides	2	x	$16-x$
condition	18 is a cube		
	need to replace 16 by a square less than a cube by 2		
square, cube	$\Delta+2y+1$	$K-3\Delta+3y-1$	
	$K-4\Delta+y=4$		
	$y=4$		
sides	2	x	$25-x$
condition	$625-50x+\Delta=4+\Delta$		
	$50x=621$		
	$x=621/50$		
solution	2	$\frac{621}{50}$	$\frac{629}{50}$

Problem 18 $x^2+y^2=z^2$, $\frac{1}{2}xy+z=a^3$, $x+y+z=b^2$
Solution:

sides	2	x	$27-x$
	need to replace 27 by a cube less than a square by 2		
square, cube	$\frac{9}{4}\Delta+3y+1$	$K-3\Delta+3y-1$	
	$K-3\Delta+3y+1=\frac{9}{4}\Delta+3y+1$		
	$K=\frac{21}{4}\Delta$		
	$y=21/4$		
sides	2	x	$\frac{4913}{64}-x$
condition	$4+\Delta=\frac{24137569}{4096}-\frac{4913}{32}x+\Delta$		
	$628864x=24121185$		
solution	2	$\frac{24121185}{628864}$	$\frac{24153953}{628864}$

Problem 19 $x^2+y^2=z^2$, $\frac{1}{2}xy+x=a^2$, $x+y+z=b^3$
Solution:

triangle[5] from $2x+1$	$2x+1$	$2\Delta+2x$	$2\Delta+2x+1$
condition	$4\Delta+6x+2$ is a cube		
	after dividing sides by $x+1$, need $4x+2$ to be a cube		
condition	$2K+3\Delta+x+2\Delta+3x+1$ is a square		
	$4x+2$ is a cube and $2x+1$ is a square		
	$x=3/2$		
solution	$\frac{8}{5}$	3	$\frac{17}{5}$

Problem 20 $x^2+y^2=z^2$, $\frac{1}{2}xy+x=a^3$, $x+y+z=b^2$
Solution:

triangle	$2x+2$	$2\Delta+2x$	$2\Delta+2x+1$
	as in solution 19		
	$4x+2$ is a square and $2x+1$ is a cube		
	$x=7/2$		
solution	$\frac{16}{9}$	7	$\frac{65}{9}$

[5] The triangle on an odd number $2n+1$ has sides $2n+1$, $2n(n+1)$, $2n^2+2n+1$.

Problem 21 $x^2+y^2=z^2$, $\frac{1}{2}xy+x+y+z=a^3$, $x+y+z=b^2$
Solution:

triangle on $1,x$	$2x$	$\Delta-1$	$\Delta+1$
condition	$2\Delta+2x$ is a square and $K+2\Delta+x$ is a cube		
	if y is a square minus 2 times Δ, then $x=2y^\times$		
condition in y	$8K^\times+8\Delta^\times+2y^\times$ is a cube		
	$8+8y+2\Delta$ is a cube		
	twice a square is a cube, powers of 27/8, say 531441/32768		
	$y=217/256$		
	$x=512/217$		
solution	$\frac{1024}{217}$	$\frac{215055}{47089}$	$\frac{309233}{47089}$

Problem 22 $x^2+y^2=z^2$, $\frac{1}{2}xy+x+y+z=a^2$, $x+y+z=b^3$
Solution:

try area, perimeter	60	40	
hypotenuse	x		
sum of sides	$40-x$		
squared	$\Delta+1600-80x=\Delta+240$		
	$x=17$		
sides	$23/2+y$	$23/2-y$	
product	$529/4-\Delta=120$		
solve	$y=7/2$		
triangle	15	8	17
	need to change 60 and 40		
area, perimeter	y	64	
sum of sides	$64-x$		
squared	$4096-128x+\Delta=\Delta+2y$		
	$x=32-\frac{1}{64}y$		
	fourth of square on $4096+4y$ minus twice the product of		
	4096 and $4y$ is a square		
	$4\Delta-24576y+4194304$ is a square		
	$\Delta-6144y+1048576$ is a square		
	and $y+64$ is a square		
scale by 16384	$16384y+1048576$ is a square		
diff. of squares	$\Delta-22528y$		
sum, diff.	$11y$, $\frac{1}{11}y-2048$		
sides	$\frac{243}{22}y-1024$, $\frac{243}{22}y+11264$		
	$y=39424/225$		
sides	$x^\times$	$\frac{78848}{225}x$	
	$78848\Delta+225=8432x$		
	$x=\frac{25}{448}$		
solution	$\frac{448}{25}$	$\frac{176}{9}$	$\frac{5968}{225}$

Problem 23 $x^2+y^2=z^2$, $z^2=a^2+a$, $z^2/x=b^3+b$
Solution:

sides	x Δ
sq. of hyp.	$\Delta+\Delta\Delta$, a sum of a square and its side
condition	$x+K$, a sum of a cube and its side
	need $\Delta+\Delta\Delta$ is a square
	$1+\Delta$ is a square
try sq. on $x-2$	$\Delta-4x+4$
	$4x=3$
	$x=3/4$
solution	$\frac{3}{4}$ $\frac{9}{16}$ $\frac{15}{16}$

Problem 24 $x^2+y^2=z^2$, $x=a^3$, $y=b^3-b$, $z=c^3+c$
Solution:

sides	2Δ	$K-x$	$K+x$
condition	$2\Delta=K$		
	$x=2$		
solution	6	8	10

References

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