The Analytical Theory of Heat
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26 Nov 2009

Summarized from "Théorie analytique de la chaleur" (1822) (Cosimo Books Inc.) and `http://visualiseur.bnf.fr/CadresFenetre?O=NUMM-29061`

1 Introduction

1.1 Object of the Work

The laws of mechanics, applicable to all phenomena of motion, have been elucidated by Archimedes, Galileo, and Newton and perfected by later philosophers. Heat is as pervasive as gravity, but is not subject to these laws. Although ingenious instruments for its measurement are now available, its laws have not yet been laid out. The object of this work is to discover the laws of heat, how it attains an equilibrium distribution from some known initial temperature.

These laws are written as mathematical equations, for there cannot be a language more universal and more simple, more free from the errors, variations and obscurities of the human mind. Mathematics is ideal to express the relations of nature, it brings together diverse phenomena and discovers the hidden analogies that unite them. No region of nature is inaccessible to it, whether separated from us by the immensity of space or by a great number of centuries, or hidden in the Earth's interior. It is a faculty of the mind destined to supplement the shortness of life and the imperfection of the senses. Remarkably, such diverse phenomena are all interpreted in the same language as if to attest to the unity, the simplicity and the unchangeable order of the plan of the universe. The resulting theory is novel and fundamentally different from dynamics and mechanics. Although the general solution of partial differential equations is not possible, practical solutions can be found using convergent series.

We will see that the laws depend on three numerical quantities of a substance: its capacity to contain heat (heat capacity c), to conduct it (conductivity k), and to transmit it across the surface (external conductivity h).

A beautiful application of the theory is the determination of the internal temperature of the Earth. Apart from the daily and yearly variations in the top few metres, there is an unchanging temperature distribution which can be deduced from the laws.

1.2 Preliminary Definitions and General Notions

The laws will be based on one basic principle of conduction supported by common observations of heat, and not on the precise nature of heat, such as whether heat flow is a flow of a substance or of a motion.

Temperature is taken to be measured by a thermometer in perfect contact with the body, where 0 denotes the temperature of melting ice and 1 the temperature of boiling water (at atmospheric pressure). A quantity of heat C is taken to be measured relative to the standard quantity required to heat one kilogram of ice at 0 to water at 0. The specific heat capacity is the quantity of heat needed to raise the temperature of a substance from 0 to 1. Heat expands bodies: a homogeneous body which is heated equally throughout retains a uniform density. It is observed that if a quantity of heat zC is added to a volume V at 0, then the new volume is $V + z\delta$. Indeed this ratio z is what is termed the temperature. (This is not accurately valid at or near to changes of state.)

The heat which passes from a unit surface to air is proportional to their temperature difference and depends on the external conductivity (as well as the velocity of the air current and its density). The conductivity is diminished if the surface is polished and metallic, but increased if black. As the air is heated, the layer nearest the surface becomes lighter and rises, establishing a current that depends on the temperature. A small part of the emitted heat is refractible 'heat' rays similar to light, most is given out by contact (except in a vacuum). The internal conduction of heat proceeds from molecule to molecule, except that in liquids, the molecules can move about. Most physicists believe that heat is transmitted between molecules solely by heat rays, but whether this is the case is not necessary for what follows.

A small body acquires the temperature of a much larger body, whether touching it or if placed completely inside it. Incoming heat rays are partly reflected and partly absorbed; and some is re-emitted and some reflected back in. The net difference goes into heating or cooling the body. Thus increasing the reflectivity of the surface works both ways: it reflects both incoming and outgoing rays. Heat rays cannot penetrate solids and liquids, only gases. Hence only particles at the surface of an object emit rays; deep or oblique rays are reabsorbed, so emitted rays are predominantly normal to the surface, in fact proportional to the sine of the angle.

Heat is the origin of elasticity. It is the repulsive force which keeps solids and liquids from collapsing. Only heat prevents molecules from getting any closer; thus increasing the temperature expands a body and vice versa. There is a stable equilibrium between molecular attraction and heat repulsion.

1.3 Principle of Heat Flow

From experiments, the heat passed from one point to another is proportional to the difference in temperature. If two molecules have temperatures v' and v at a distance p, then the amount of heat transferred in a time dt is $(v' - v)\phi(p)dt$ where $\phi(p)$ is a function that decreases to nothing as p increases, at least for solids and liquids.

Similarly the heat lost from a surface area σ at a temperature v to its surroundings at an ambient temperature a is $\sigma h(v-a)dt$ where h is a measure of the external conductivity. Both laws depend on temperature differences, not the actual temperature.

It follows that if a body loses heat to its surroundings but its temperature remains homogeneous, then its temperature follows a logarithmic curve[1]. Also, if the ambient temperature is, say, 0, and the initial temperature α is increased to $g\alpha$ (g any number) then the temperatures will be v and gv respectively. All these observations have been confirmed by experiments with accurate thermometers.

1.4 Uniform Linear Heat Flow

Consider a homogeneous solid between two parallel planes A and B, separated by a distance e and maintained at temperatures a and b respectively, with b less than a. We show that the final temperature is that of an arithmetic progression $v = a + \frac{b-a}{e}z$, where z is the distance from A. For, given a particle m on any plane A' parallel to A and at a distance z from it, and another particle m' separated by z' vertically from m; then the heat transmitted from m to m' depends on their temperature difference, namely $\frac{b-a}{e}z'$, which does not depend on z. So the heat passing across any parallel plane is the same; in particular as much heat is lost from one face as is received from the opposite face, so its temperature remains constant.

Indeed the heat flow across such a plane is proportional to $\frac{a-b}{e}$. To establish the constant of proportionality, called the specific conductivity of the substance, one needs to take a large slab of the material of unit thickness (1 metre), keep one side at 0, and the other at 1, and measure the heat flow K in a unit time (1 minute) by weighing how much ice is turned to water with that heat. Then the actual heat flow is $F = K\frac{a-b}{e} = -K\frac{dv}{dz}$.

Suppose now that plane B is in contact with air at a temperature of b. Plane B will now heat up to an unknown temperature β. The heat lost is then $h(\beta-b)$ where h is the external conductivity of B, and this must equal the heat inflow $K\frac{a-\beta}{e}$, so that β can be determined and $v = a - \frac{hz(a-b)}{he+K}$.

1.5 The Permanent Temperature in a Thin Long Prism

Consider a $2l \times 2l$ square prism of infinite length with its face at $x = 0$ kept at a fixed temperature and the other faces exposed to air at 0. We will assume l to be small enough that there is no appreciable difference between the surface and interior temperatures of a slice, so that the temperature v depends only the distance x from the face. In the final permanent state of the system, as much heat enters a thin cross-section, namely $-4l^2K\frac{dv}{dx}$, as escapes, that is,

$$-4l^2K\left(\frac{dv}{dx} + d\left(\frac{dv}{dx}\right)\right) + 8hlvdx = -4l^2K\frac{dv}{dx}$$

[1] Meaning $a + be^{-kt}$

which gives $Kl\frac{d^2v}{dx^2} = 2hv$. Note, in passing, that this equation depends on l: the experiments that were performed to determine that heat cannot flow more than 6 feet through iron is valid only for the thickness used.

The integral of this equation is

$$v = Ae^{-x\sqrt{\frac{2h}{kl}}} + Be^{x\sqrt{\frac{2h}{kl}}} = Ae^{-x\sqrt{\frac{2h}{kl}}}$$

where A is an arbitrary constant and $B = 0$ if v remains small as x becomes infinite. This logarithmic law has been observed by several physicists. By measuring the temperature at two points, one can find the ratio h/k. Notice that two equal bars of different materials but the same h would have the same temperature at two points with $\frac{x_1^2}{x_2^2} = \frac{k_1}{k_2}$. The flow of heat $-4kl^2\frac{dv}{dx}$ is proportional to $\sqrt{l^3}$.

1.6 Heating of Closed Spaces

Imagine a space of air enclosed by a thin boundary of surface area s and thickness e, and maintained throughout at a temperature n. It is now heated by a source at temperature α and area σ. Eventually the air reaches a temperature m and the boundary has inner and outer temperatures a and b.

As the boundary surfaces have fixed temperatures, there is no transfer of heat tangential to the boundary, only perpendicular to it. The flow of heat entering the air is $\sigma(\alpha - m)g$ (where g is the conductivity of the heat source), that leaving it at the boundary is $s(m - a)h$, through the boundary $s\frac{a-b}{e}K$, and out to the surrounding air $s(b - n)H$. These four expressions are all equal. Adding the expressions for $m - a$, $a - b$ and $b - n$ gives

$$m - n = (\alpha - m)P = (\alpha - n)\frac{P}{1 + P}$$

where P is $\frac{\sigma}{s}(\frac{g}{h} + \frac{ge}{K} + \frac{g}{H}) = \frac{\sigma}{s}p$. It is directly proportional to the source temperature $\alpha - n$, depends on h, H and $\frac{K}{e}$ in equivalent ways, and is independent of the form or volume of the enclosure. Bad conductors of heat increase m until $m = \alpha$ when $K = 0$ (or $H = 0$ or $h = 0$). The value of p may be determined experimentally by measuring $m - n$, $\alpha - n$ and σ/s. For example, flames and animated bodies are constant sources of heat; taking these measurements in one case of a room with people in it, one can estimate the increase in temperature when more people are present, although in practice many of the assumptions made above are not satisfied.

One can repeat these calculations for the case when there is a second heat source of temperature, or to the case when one space is enclosed by a second space and so on; we find that such enclosures greatly aid in the amount of heating. In contrast to the heat lost $hs(b - n)$, an enclosure would raise the intermediate air temperature to n' and the heat loss would be less $hs(b - n')$. More precisely, the heat which enters the enclosure $hs(b - n')$, equals the heat which enters the shell's inner surface, $hs(n' - a')$, equals the heat which passes through the shell $Ks\frac{a'-b'}{e}$, equals the heat which passes to the air $hs(b'-n)$. Adding gives $b-n=(b'-n)(3+\frac{he}{K})$; the heat lost is now $\frac{hs(b-n)}{3+\frac{he}{K}}$, at least 3 times less than originally. However the effect of heat radiation has not been included. Suppose then, that a number j of parallel laminae separate the source at temperature b from the air at temperature n with no air in between. The heat transmitted from one plate to the next is proportional to their temperature difference. The heat quantities crossing each lamina is now $Hs(b_{i-1}-a_i)=\frac{Ks}{e}(a_i-b_i)=Hs(b_i-a_{i+1})$ etc. Adding gives $b_0-n=(b_0-a_1)j(1+\frac{He}{K})+1$. Thus the heat lost now is $Hs(b_0-a_1)=\frac{Hs(b_0-n)}{j(1+\frac{He}{K})+1}$; for thin laminae this is $\frac{Hs(b_0-n)}{j+1}$. In both cases inserting enclosures greatly assists in the retention of heat, explaining the results of experimenters who have enclosed thermometers by several layers of glass sheets. Similarly the temperature at high altitudes is very much less than at the Earth's surface.

1.7 Uniform Heat Flow in Three Dimensions

Suppose a cuboid to have internal permanent temperature $v=A+ax+by+cz$. The difference in temperature between particles m and m' at coordinates x,y,z and x',y',z' is $a(x-x')+b(y-y')+c(z-z')$. Summing up all such heat transfers, the heat flow across any parallel planes is the same. So any small cuboid of a solid receives as much heat across one plane as it loses in the opposite face, and its temperature is maintained. In summary,

Theorem I. If a homogeneous solid has a temperature determined by $v=A-ax-by-cz$, including the surface, then the distribution is stationary.

Corollary I. In a solid enclosed by infinite parallel planes, the temperature $v=1-z$ is stationary.

Corollary II. In the same solid, the heat flow across parallel planes is the same (otherwise the temperature changes).

Lemma. If the temperature is now $v=g-gz$, then the heat flow is also multiplied by g (because the temperature differences, and hence the heat flow, are also scaled by g).

Theorem II. If a bounded prism has temperature $v=A-ax-by-cz$ then the heat flow across a plane is the same as in Corollary II.

Corollary. The flow of heat in the z-direction per unit area and unit time is cK or $-K\frac{dv}{dz}$, and similarly for the other directions.

1.8 Measure of Heat Flow at a Point

Consider a solid with temperature $v=f(x,y,z,t)$ at a point m. Imagine a plane parallel to the xy-plane through the point and an infinitely small circle ω on it, centered at m. In an infinitely small instant dt, the points below the circle will send an amount of heat equal to $-K\frac{dv}{dz}\omega dt$ to the points on the other side. For a plane in a general direction, the temperature distribution in the vicinity of m with coordinates $x+\xi,y+\eta,z+\zeta$ will be indistinguishable from $v = A + a\xi + b\eta + c\zeta$, so the heat flow would be $-K\frac{dv}{dz}$, $-K\frac{dv}{dy}$ or $-K\frac{dv}{dx}$ depending on the direction of the plane.

Theorem III. If $v = f(x, y, z, t)$ then near to any point, the temperature is $v + \xi\frac{dv}{dx} + \eta\frac{dv}{dy} + \zeta\frac{dv}{dz}$ so the heat flow in the z-direction is $-K\frac{dv}{dz}$ per unit area and time.

As an example, if the temperature of a cube centered at the origin is $v = e^{-gt}\cos x \cos y \cos z$, the heat flow across a face would be

$$-K \iint \frac{dv}{dz}\omega\, dt = Ke^{-gt}dt \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \cos x\, \cos y\, dxdy = 4Ke^{-gt}\sin z\, dt$$

where we denote an integral from $x = a$ to $x = b$ by $\int_a^b$. The total heat loss across all six faces after infinite time is $\frac{4K}{g}$.

2 The Equations of Heat Flow

2.1 Ring

We now turn to the more general problem of finding the variable state of temperature v in a solid body as a function of x and t. We start with a solid ring of radius R, cross-section area S and perimeter l exposed to air at 0; assume the temperature is constant across a cross-section but otherwise given arbitrarily initially.

A thin slice will gain heat, $KS\frac{d^2v}{dx^2}dxdt$ as in the introduction, and loses $hlv\, dx\, dt$ to the air. This amount of heat divided by $CDSdx$, where C is the specific heat capacity and D the density, gives the increase in temperature dv,

$$\frac{dv}{dt} = \frac{K}{CD}\frac{d^2v}{dx^2} - \frac{hl}{CDS}v$$

If there are sources and the temperature has become stationary, then the equation is $\frac{d^2v}{dx^2} = \frac{hl}{KS}v$, whose solution in between any pair of sources is $v = M\alpha^{-x} + N\alpha^{+x}$, where $\alpha = e^{-\sqrt{\frac{hl}{KS}}}$; the constants are determined by the temperature of the sources. If x_1, x_2, x_3 are equally spaced points (with common distance λ) then their temperature satisfies $\frac{v_1+v_3}{v_2} = \alpha^{\lambda} + \alpha^{-\lambda} = q$, independent of the position or sources. This has been confirmed by experiment. If one measures q, then α^{λ} and the conductivities ratio $\frac{h}{K} = \frac{S}{l}(\log\alpha)^2$ can be found.

2.2 Solid Sphere

We study here the variation of temperature of a solid homogeneous sphere initially at temperature 1 and then exposed to air at 0. The amount of heat entering a spherical shell of radius x is $-4K\pi x^2\frac{dv}{dx}dt$, that leaving is $-4K\pi x^2\frac{dv}{dx}dt - 4K\pi d(x^2\frac{dv}{dx})dt$, and the net heat gain goes into raising the temperature by $CD4\pi x^2 dx$, so

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{2}{x}\frac{dv}{dx}\right).$$

The other conditions on the temperature $v = \phi(x,t)$ is that the surface $x = X$ is kept at 0 for all t, so $\phi(X,t) = 0$, while initially $\phi(x,0) = F(x)$. Alternatively, if the surrounding medium is air, the radiated heat is $4h\pi X^2 V dt$, where V is the surface temperature, so $-4K\pi X^2 \frac{dV}{dx} dt = 4h\pi X^2 V dt$, i.e., $\frac{dV}{dx} = -\frac{h}{K} V$. These equations will be solved in a later chapter.

2.3 Solid Cylinder

We repeat the analysis for an infinite cylinder. The net heat gained is now $2K\pi\, dt\, d(x\frac{dv}{dx})$ which raises the temperature by $2CD\pi x\, dx$, so

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx}\right)$$

The equation at the surface becomes $-2K\pi X \frac{dv}{dx} dt = 2\pi X h v dt$, i.e., $\frac{dv}{dx} = -\frac{h}{K} v$ at $x = X$, and $v = F(x)$ at $t = 0$.

2.4 General Equation of Heat Flow

Consider now the general problem with a variable temperature $v = \phi(x,y,z,t)$. In a prismatic molecule bounded by points x, y, z and $x + dx, y + dy, z + dz$, the net heat gained across opposite planes is $-K dydzd(\frac{dv}{dx})$, that is $-K dxdydz \frac{d^2v}{dx^2}$. Adding the contributions due to the other faces, the net increase in heat is $-K dxdydz(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2})dt$ which goes into raising the temperature by $CDdxdydz$, so

Theorem IV. The most general equation of heat propagation applicable to all solids is

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) \tag{1}$$

If the temperature is stationary, it must be the case that $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0$. The heat flow can be interpreted as the sum of three flows in perpendicular directions. Initially, $\phi(x,y,z,0) = F(x,y,z)$ for some known function.

(The specific heat c has been shown by Prof. Dulong and Petit to increase slowly with temperature. The other coefficients h and k also depend on the temperature in principle but we have assumed them constant because the variation is small; correction terms must be added for more precision. We also neglect the small thermal expansion of solids.)

This equation is the same as that given for the sphere and cylinder. For suppose the temperature v in a cylinder depends only on r and t, independent of x. Now $\frac{dv}{dy} = \frac{dv}{dr}\frac{dr}{dy}$ and $\frac{d^2v}{dy^2} = \frac{d^2v}{dr^2}(\frac{dr}{dy})^2 + \frac{dv}{dr}(\frac{d^2r}{dy^2})$, and similarly for z, so

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2}\left(\left(\frac{dr}{dy}\right)^2 + \left(\frac{dr}{dz}\right)^2 + \frac{dv}{dr}\left(\frac{d^2r}{dy^2} + \frac{d^2r}{dz^2}\right)\right)$$

Since $r^2 = z^2 + y^2$, it follows that $(\frac{dr}{dy})^2 + (\frac{dr}{dz})^2 = 1$ since $y = r\frac{dr}{dy}$, $1 = (\frac{dr}{dy})^2 + r\frac{d^2r}{dy^2}$, and $r^2 = y^2 + z^2 = r^2\left((\frac{dr}{dy})^2 + (\frac{dr}{dz})^2\right)$. Also $\frac{d^2r}{dy^2} + \frac{d^2r}{dz^2} = \frac{1}{r}$ since $2 = 1 + r(\frac{d^2r}{dy^2} + \frac{d^2r}{dz^2})$. Substituting gives

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr}$$

which was derived separately above.

For a sphere, $x^2 + y^2 + z^2 = r^2$, so $r^2 = r^2((\frac{dr}{dx})^2 + (\frac{dr}{dy})^2 + (\frac{dr}{dz})^2)$, and $\frac{d^2r}{dx^2} + \frac{d^2r}{dy^2} + \frac{d^2r}{dz^2} = \frac{1}{r}$. Again we get the previously derived

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr}.$$

Elsewhere I have also derived the heat equation for a fluid

$$C\frac{d\theta}{dt} = K\left(\frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{d^2\theta}{dz^2}\right) - C\left(\frac{d}{dx}(u\theta) + \frac{d}{dy}(v\theta) + \frac{d}{dz}(w\theta)\right)$$

2.5 General Equation of Outflow at the Surface

The equations at the surface depend on the conditions there; for example, if it is maintained at a constant temperature of 0, then $\phi(x, y, z, t) = 0$ for all x, y, z on the surface; if heat is lost to air then $-K dxdy\frac{dv}{dz} = h\,dxdy\,v$, that is $hv + K\frac{dv}{dy} = 0$. In other cases, as in the problem of terrestrial temperatures, the surface temperature varies in a known manner with time.

In general, if the solid surface $f(x, y, z) = 0$ loses heat to the air maintained at a constant temperature, say 0, a third condition must be added to (1) and the initial condition. Let μ be a point on the surface, at x, y, z, having temperature $v = \phi(x, y, z, t)$ and let ν be a point at $x + dx, y + dy, z + dz$, normal to the surface and having temperature w. By theorem III, the heat flow across an infinitesimal surface area ω is $-K\frac{w-v}{\alpha}\omega dt$ where $\alpha = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$ is the distance between μ and ν. Differentiating $f(x, y, z) = 0$ gives $mdx+ndy+pdz = 0$, so $p\delta x = m\delta z$, $p\delta y = n\delta z$, and $\alpha = \frac{q}{p}\delta z$, where $q = (m^2 + n^2 + p^2)^{\frac{1}{2}}$, so

$$\begin{aligned} w = v + \delta v &= v + \frac{dv}{dx}\delta x + \frac{dv}{dy}\delta y + \frac{dv}{dz}\delta z \\ &= v + \frac{\alpha}{q}\left(m\frac{dv}{dx} + n\frac{dv}{dy} + p\frac{dv}{dz}\right). \end{aligned}$$

This heat flow $-K\frac{w-v}{\alpha}\omega dt$ is lost to the air, and equals $hv\omega dt$. The surface condition therefore becomes

$$-K\left(m\frac{dv}{dx} + n\frac{dv}{dy} + p\frac{dv}{dz}\right) = hvq. \tag{2}$$

2.6 General Remarks

Five quantity units have been used: length, time, temperature, weight, and heat. The heat capacity C is always accompanied in the equations by the density D, so we can define $c = CD$, which is the heat capacity of a material of unit volume. If this is done, then the unit of weight is unnecessary.

In the equation of heat, if the length quantity x is redefined as mx, then h, K, c become $\frac{h}{m^2}, \frac{K}{m}, \frac{c}{m^3}$ respectively. These represent the dimensions of length in the respective quantities. The dimensions of h, K, c are

	length	time	temperature
K	-1	-1	-1
h	-2	-1	-1
c	-3	0	1

The equations are homogeneous for each dimension, i.e., they are dimensionally correct; e.g. each term of $\frac{dv}{dt} = \frac{K}{CD}\frac{d^2v}{dx^2} - \frac{hl}{CDS}v$ has dimensions 0, -1, 1 for length, time, and temperature.

3 Heat Flow in an Infinite Rectangular Solid

3.1 Statement of the Problem

As an illustration of the method to be used for solving the preceding differential equations, consider a homogeneous infinite solid bounded by three planes A, B, C, with B and C parallel to each other and perpendicular to A. The face A is kept at a constant temperature of 1, while B and C have a constant temperature of 0. Heat will flow across face A out through B and C, raising the temperature of the solid until it approaches a steady state. Take the width of A to be π, and the x-axis to be parallel to B from the midpoint of A. It is required to find the steady temperature v as a surface $v = \phi(x, y)$ over the x-y plane.

The heat equation in this case is $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$, with $\phi(x, \pm\frac{1}{2}\pi) = 0$, $\phi(0, y) = 1$, and ϕ ought to be small for large x. We could solve for v, variable in time, and then take t infinite, but the adopted procedure is more direct.

We first seek a simple function that satisfies the equation, and then generalize it so it satisfies the conditions. Functions of two variables often reduce to simpler expressions, such as $F(x)f(y)$ when one or both values are infinite. Substituting such a function in the heat equation we find

$$\frac{F''(x)}{F(x)} + \frac{f''(y)}{f(y)} = 0$$

so that $\frac{F''(x)}{F(x)} = m$, $\frac{f''(y)}{f(y)} = -m$, constant, hence $F(x) = e^{-mx}$, $f(y) = \cos my$. The constant m cannot be negative else $F(x)$, and thus v, would become infinite when x is infinite; in order for v to be 0 at $y = \pm\frac{1}{2}\pi$, m must be taken to be one of $1, 3, 5, 7$,&c. A more general solution is easily obtained by adding such simple functions,

$$v = ae^{-x}\cos y + be^{-3x}\cos 3y + ce^{-5x}\cos 5y + \&\text{c}.$$

Notice that the solution represented by the first term, $ae^{-x}\cos y$, has positive $\frac{d^2v}{dx^2}$ and negative $\frac{d^2v}{dy^2}$; the heat flow is thus in the positive x-direction and either of the y-directions.

The third condition, $\phi(0, y) = 1$ $(-\frac{1}{2}\pi < y < \frac{1}{2}\pi)$ is then

$$1 = a\cos y + b\cos 3y + c\cos 5y + \&\text{c} \tag{3}$$

where the coefficients a, b, c,&c, are to be determined. Any doubts as to whether such a series can possibly give a constant 1 will be cleared presently.

3.2 Trigonometric Series

Differentiating (3) repeatedly, at $y = 0$, gives the infinite number of equations

$$\begin{aligned}1 &= a + b + c + \&\text{c}\\ 0 &= a + 3^2b + 5^2c + \&\text{c}\\ 0 &= a + 3^4b + 5^4c + \&\text{c}\end{aligned}$$

If we solve the first two equations in two unknowns, then three, &c, we get different values for a, b, c, d, &c which converge to limiting values. The first equation is $a = 1$, then two give $a = \frac{3^2}{3^2-1}$, $b = -\frac{1}{3^2-1}$, and so on. With each new equation a term is added to the series and we obtain

$$\begin{aligned}a &= \frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{9^2}{9^2-1}\cdot\frac{11^2}{11^2-1}\cdot\&\text{c}\\ &= \frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{11\cdot 11}{10\cdot 12}\cdot\&\text{c}\\ &= \frac{4}{\pi} \qquad \text{by Wallis' theorem.}\\ b &= \frac{1^2}{1^2-3^2}\cdot\frac{5^2}{5^2-3^2}\cdot\frac{7^2}{7^2-3^2}\cdot\frac{9^2}{9^2-3^2}\&\text{c}\\ &= \frac{1\cdot 1}{2\cdot 4}\cdot\frac{5\cdot 5}{2\cdot 8}\cdot\frac{7\cdot 7}{4\cdot 10}\cdot\&\text{c} = -\frac{6}{3\cdot 3}\frac{2}{\pi} = -2\cdot\frac{2}{3\pi}\\ c &= \frac{1^2}{1^2-5^2}\cdot\frac{3^2}{3^2-5^2}\cdot\frac{7^2}{7^2-5^2}\cdot\frac{9^2}{9^2-5^2}\&\text{c}\\ &= \frac{1\cdot 1}{4\cdot 6}\cdot\frac{3\cdot 3}{2\cdot 8}\cdot\frac{7\cdot 7}{2\cdot 12}\cdot\&\text{c} = \frac{10}{5\cdot 5}\frac{2}{\pi} = 2\cdot\frac{2}{5\pi}\end{aligned}$$

Hence

$$\frac{\pi}{4} = \cos y - \frac{1}{3}\cos 3y + \frac{1}{5}\cos 5y - \frac{1}{7}\cos 7y + \&\text{c}. \tag{4}$$

The series obviously converges, albeit not rapidly, for any value of y between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. In the range $\frac{1}{2}\pi < y < \frac{3}{2}\pi$, the constant value of the series is $-\frac{1}{4}\pi$

since each term of the series changes sign. At $y=\pm\frac{1}{2}\pi,\pm\frac{3}{2}\pi$, &c, the series is 0. The case $y=0$ gives Leibnitz' series $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$&c. Putting $y=\frac{1}{2}\frac{\pi}{2}$, we find $\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+$&c., and similarly several other series due to Euler can be obtained. Integrating equation (4)

$$\frac{\pi}{4}y=\sin y-\frac{1}{3^2}\sin 3y+\frac{1}{5^2}\sin 5y-\&\text{c},$$

and putting $y=\frac{\pi}{2}$ gives $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+$&c.

3.3 Remarks on these Series

Another derivation of (4) is to take the finite series

$$y=\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x+\&\text{c}-\frac{1}{2m-1}\cos(2m-1)x \qquad (m\text{ even}).$$

Differentiating gives $-\frac{dy}{dx}=\sin x-\sin 3x+\sin 5x-\sin 7x+\&\text{c}-\sin(2m-1)x$ so

$$\begin{aligned}-2\frac{dy}{dx}\sin 2x &= (\cos x-\cos 3x)-(\cos x-\cos 5x)+(\cos 3x-\cos 7x)+\&\text{c}\\ &\quad -(\cos(2mx-3x)-\cos(2mx+x))\\ &= \cos(2m+1)x-\cos(2m-1)x\\ &= -2\sin 2mx\sin x\end{aligned}$$

hence $y=\frac{1}{2}\int dx\frac{\sin 2mx}{\cos x}$. Integrating by parts, we get

$$2y=\text{const.}-\frac{1}{2m}\cos 2mx\sec x+\frac{1}{2^2m^2}\sin 2mx\sec' x+\&\text{c}.$$

from which it is evident that y takes a constant value as m increases. This constant is found to be $\frac{\pi}{4}$ by taking $x=0$ in the original series.

This type of analysis can lead to other series. For m even, let

$$y=\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\&\text{c}.-\frac{1}{m}\sin mx$$

Then $2\sin x\frac{dy}{dx}=(\sin 2x-0)-(\sin 3x-\sin x)+\&\text{c}+(\sin(mx)-\sin(m-2)x)$, so dividing by $2\sin x$ and integrating gives $y=\frac{1}{2}x-\int dx\frac{\cos(mx+\frac{1}{2}x)}{2\cos\frac{1}{2}x}$ which can be integrated by parts as before, and letting m be infinite, we deduce the known series

$$\frac{1}{2}x=\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\&\text{c} \quad (0<x<\pi).$$

Let now

$$\begin{aligned}y &= \frac{1}{2}\cos 2x-\frac{1}{4}\cos 4x+\frac{1}{6}\cos 6x-\&\text{c}-\frac{1}{2m}\cos 2mx\\ &= c-\frac{1}{2}\int dx\tan x+\frac{1}{2}\int dx\frac{\sin(2m+1)x}{\cos x}\\ &= \frac{1}{2}\log 2+\frac{1}{2}\log\cos x, \quad (\text{since } y=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\&\text{c}=\frac{1}{2}\log 2 \text{ at } x=0)\end{aligned}$$

so $\log(2\cos\frac{1}{2}x) = \cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x -$ &c as given by Euler.

Repeating for $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x +$ &c we find the unnoticed series $\frac{1}{4}\pi = \sin x + \frac{1}{3}\sin 3x +$ &c.

One final derivation of (4) is as follows: $\frac{1}{2}\pi = \arctan u + \arctan\frac{1}{u}$ because angles of normal slopes add to $\frac{1}{2}\pi$. Hence $\frac{\pi}{2} = u + \frac{1}{u} + \frac{1}{3}(u^3 + \frac{1}{u^3}) + \frac{1}{5}(u^5 + \frac{1}{u^5}) + \&c.$ Putting $u = e^{x\sqrt{-1}}$ gives $\frac{\pi}{4} = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \&c.$

3.4 General Solution

The complete solution of the problem is the exceedingly convergent series

$$\frac{\pi v}{4} = e^{-x}\cos y - \frac{1}{3}e^{-3x}\cos 3y + \frac{1}{5}e^{-5x}\cos 5y - \&\text{c}$$

If we take points farther away from the face A, as x becomes infinite, the expression will become first the sum of the first few terms, and then nearly $e^{-x}\cos y$, which is a trigonometric line in the y-axis and a logarithmic curve in the x-axis. Indeed, each term of the series is a stationary solution, but are here superposed in order to satisfy the special condition. It is clear that the general system of temperatures is the sum of a multitude of simple systems, characteristic of the given problem, whose coefficients depend on the conditions.

We can write the solution, using imaginary numbers as

$$\begin{aligned}\frac{\pi v}{2} &= e^{-(x-y\sqrt{-1})} - \frac{1}{3}e^{-3(x-y\sqrt{-1})} + \&\text{c} + e^{-(x+y\sqrt{-1})} - \frac{1}{3}e^{-3(x+y\sqrt{-1})} + \&\text{c}\\ &= \arctan e^{-(x+y\sqrt{-1})} + \arctan e^{-(x-y\sqrt{-1})}\\ &= \arctan\frac{2\cos y}{e^x - e^{-x}}\end{aligned}$$

If one were to calculate the total heat flow across face A, $-2K\int_0^{\frac{\pi}{2}}\frac{dv}{dx}dy$ per unit time, one would find an infinite answer: this is due to the fact that at the ends of A there are points at temperatures of 0 and 1 close to each other, creating a cataract of heat.

Remarks: If the base length is $2l$ instead of π and the face temperature is A instead of 1, we must replace y with $\frac{1}{2}\pi\frac{y}{l}$, x with $\frac{1}{2}\pi\frac{x}{l}$, and v with $\frac{v}{A}$. Hence we have

$$v = \frac{4A}{\pi}(e^{-\frac{\pi x}{2l}}\cos\frac{\pi y}{2l} - \frac{1}{3}e^{-\frac{3\pi x}{2l}}\cos\frac{3\pi y}{2l} + \&\text{c.})$$

The problem admits no other solution: For let $v = \phi(x, y)$ be the above solution. Now suppose the temperature of A is brought to 0, then the heat of the body will flow out until the temperature takes its final value of 0. If the initial temperature was $v = -\phi(x, y)$ instead, then the heat will flow in and the temperature will increase to 0. Now consider three cases: (i) $v = \phi(x, y, t)$ initially at $f(x, y)$ with the initial temperature of A being 1, (ii) $v = \Phi(x, y, t)$ initially at $F(x, y)$ with the initial temperature of A being 0, and (iii) the initial temperature is $f(x, y) + F(x, y)$ and that of A is 1. Then we show that the solution of (iii) is $\phi(x, y, t) + \Phi(x, y, t)$. For when a molecule of volume M

acquires a quantity of heat Δ in an instant dt, its temperature increases by $\frac{\Delta}{CM}dt$. For solids (i) and (ii) it would be $\frac{d}{CM}dt$ and $\frac{D}{CM}dt$ respectively, say. The heat acquired by M in (i) is in fact $d = \sum q_1(f_1 - f)dt$ where f_1 is the initial temperature of another point, and q_1 depends on the distance between them, and the sum is taken over all points. Similarly in the second solid, $D = \sum q_1(F_1 - F)dt$. Hence $\Delta = \sum q_1\{f_1 + F_1 - (f + F)\}dt = d + D$. So the variation of temperature in (iii) is the sum of (i) and (ii) in the first instant, but then we are in the same situation in subsequent instances. This shows that a fundamental characteristic of the heat equation is that its flow can be decomposed as a superposition of several others, each acting independently of the others.

Now let $v = \psi(x, y)$ be another conceived stationary solution. It can be decomposed into two states, one in which the initial temperature is $\psi(x, y) - \phi(x, y)$ with A maintained at 0, and a second in which the initial temperature is $\phi(x, y)$ with A maintained at 1. In the latter there will be no change in temperature for it satisfies the stationary heat equation, while in the first the temperature becomes null, as remarked earlier. Hence the final state is the stated solution.

3.5 Trigonometric Series of an Arbitrary Function

We now consider the problem of writing an arbitrary function as a trigonometric series. Let $\phi(x) = a\sin x + b\sin 2x + c\sin 3x + \&c$. We can also write $\phi(x) = x\phi'(0) + \frac{x^2}{\underline{|2}}\phi''(0) + \frac{x^3}{\underline{|3}}\phi'''(0) + \frac{x^4}{\underline{|4}}\phi^{\text{iv}}(0) + \&c$[2]. Comparing the two equations we find, $\phi''(0) = 0$, $\phi^{\text{iv}}(0) = 0$, &c, as well as

$$\begin{aligned} A &= \phi'(0) = a + 2b + 3c + \&c \\ B &= \phi'''(0) = a + 2^3 b + 3^3 c + \&c \\ C &= \phi^{\text{v}}(0) = a + 2^5 b + 3^5 c + \&c \end{aligned}$$

Looking at a finite number of unknowns and equations, first one equation gives $a_1 = A$, then two $a_2 = (2^2 A - B)/(2^2 - 1)$, $b_2 = (B - A)/(2^3 - 2)$ &c. To solve a set of 3 equations in 3 unknowns, eliminate the last unknown to reduce the set of equations to

$$\begin{gathered} A_2 = 3^2 A_3 - B_3 = (3^2 - 1)a_3 + 2(3^2 - 2^2)b_3 \\ B_2 = 3^2 B_3 - C_3 = (3^2 - 1)a_3 + 2^3(3^2 - 2^2)b_3 \end{gathered}$$

which is precisely of the same form as for two unknowns. This gives a way of generating the next set of $m + 1$ equations from m equations, namely replace a_1 by $(2^2 - 1)a_2$, then a_2 by $a_3(3^2 - 1)$ etc. In the limit,

$$a_1 = (2^2 - 1)a_2 = (2^2 - 1)(3^2 - 1)a_3 = \cdots = a(2^2 - 1)(3^2 - 1)\ldots,$$

$$b_2 = (3^2 - 2^2)b_3 = \cdots = b(3^2 - 2^2)(4^2 - 2^2)\ldots$$

[2] $\underline{|n}$ is $n!$

Similarly,

$$A_1 = A_2 2^2 - B_2 = A_3 2^2 3^2 - B_3(2^2+3^2) + C_3 \ \&\text{c}$$

The law is readily noticed, namely

$$\frac{a_1}{2^2 3^2 \ldots} = A - B\left(\frac{1}{2^2}+\frac{1}{3^2}+\&\text{c}\right) + C\left(\frac{1}{2^2.3^2}+\frac{1}{2^2.4^2}+\&\text{c}\right)+\&\text{c}$$

Similarly,

$$2b_2\frac{1-2^2}{1^2.3^2.4^2\ldots} = A - B\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&\text{c}\right) + C\left(\frac{1}{1^2.3^2}+\frac{1}{1^2.4^2}+\frac{1}{1^2.5^2}+\cdots\right)\&\text{c}$$

and so on. Placing the known values of a_1, b_2,&c, we note that the fractions on the left simplify greatly to $\frac{1}{1}\cdot\frac{1}{2}$, $-\frac{2}{2}\cdot\frac{2}{4}$, &c. These sums can be found from

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \&\text{c} = x\left(1-\frac{x^2}{1^2\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\&\text{c}$$

After making the necessary substitutions and simplifying, one finds that

$$\begin{aligned}\frac{1}{2}\phi(x) \;=\; & \sin x\left\{\phi'(0)+\phi'''(0)\left(\frac{\pi^2}{\underline{|3}}-\frac{1}{1^2}\right)+\phi^{\text{v}}(0)\left(\frac{\pi^4}{\underline{|5}}-\frac{1}{1^2}\frac{\pi^2}{\underline{|3}}+\frac{1}{1^4}\right)+\&\text{c}\right\}\\ & -\frac{1}{2}\sin 2x\left\{\phi'(0)+\phi'''(0)\left(\frac{\pi^2}{\underline{|3}}-\frac{1}{2^2}\right)+\phi^{\text{v}}(0)\left(\frac{\pi^4}{\underline{|5}}-\frac{1}{2^2}\frac{\pi^2}{\underline{|3}}+\frac{1}{2^4}\right)+\&\text{c}\right\}\\ & +\frac{1}{3}\sin 3x\left\{\phi'(0)+\phi'''(0)\left(\frac{\pi^2}{\underline{|3}}-\frac{1}{3^2}\right)+\phi^{\text{v}}(0)\left(\frac{\pi^4}{\underline{|5}}-\frac{1}{3^2}\frac{\pi^2}{\underline{|3}}+\frac{1}{3^4}\right)+\&\text{c}\right\}\\ & \&\text{c}\end{aligned}$$

For example, taking $\phi(x) = x^3$ gives

$$\frac{1}{2}x^3 = \left(\pi^2 - \frac{\underline{|3}}{1^2}\right)\sin x - \left(\pi^2-\frac{\underline{|3}}{2^2}\right)\frac{1}{2}\sin 2x + \left(\pi^2-\frac{\underline{|3}}{3^2}\right)\frac{1}{3}\sin 3x + \&\text{c}$$

We can simplify by noting that $\phi'(0) + \frac{\pi^2}{\underline{|3}}\phi'''(0) + \frac{\pi^4}{\underline{|5}}\phi^{\text{v}}(0) + \&\text{c} = \frac{1}{\pi}\phi(\pi)$ and similarly $\phi''(0) + \frac{\pi^2}{\underline{|3}}\phi^{\text{v}}(0) + \&\text{c} = \frac{1}{\pi}\phi''(\pi)$, &c, so that

$$\begin{aligned}\frac{1}{2}\pi\phi(x) \;=\; & \sin x\left\{\phi(\pi)-\frac{1}{1^2}\phi''(\pi)+\frac{1}{1^4}\phi^{\text{iv}}(\pi)-\&\text{c}\right\}\\ & -\frac{1}{2}\sin 2x\left\{\phi(\pi)-\frac{1}{2^2}\phi''(\pi)+\frac{1}{2^4}\phi^{\text{iv}}(\pi)-\&\text{c}\right\}\\ & +\&\text{c}\end{aligned}$$

Let s be the coefficients in the above expression,

$$s(\pi) = \phi(\pi) - \frac{1}{n^2}\phi''(\pi) + \frac{1}{n^4}\phi^{\text{iv}}(\pi) - \&\text{c}.$$

Differentiating with respect to π (as a variable) we find $s + \frac{1}{n^2}\frac{d^2 s}{d\pi^2} = \phi(\pi)$, so

$$s = a\cos nx + b\sin nx + n\sin nx\int \cos nx\phi(x)dx - n\cos nx\int \sin nx\phi(x)dx$$

and at $x = \pi$, $s = \pm n\int \phi(x)\sin nx dx$. Thus

$$\begin{aligned}\frac{1}{2}\pi\phi(x) &= \sin x\int_0^\pi \sin x\phi(x)dx + \sin 2x\int_0^\pi \sin 2x\phi(x)dx + \&\mathrm{c}\\ &\quad + \sin ix\int_0^\pi \sin ix\phi(x)dx + \&\mathrm{c}\end{aligned}$$

These results extend even to functions which are discontinuous and entirely arbitrary. This is because the integrals represent the area of a curve, which exists whether it is analytical or not. They are analogous to the center of gravity of a body, which exists however irregular the body is.

This can be verified: Let $\phi(x) = a_1\sin x + a_2\sin 2x + \&\mathrm{c} + a_j\sin jx + \&\mathrm{c}$. Multiply by $\sin ix$ and integrate

$$\begin{aligned}\int_0^\pi \frac{1}{2}\pi\phi(x)\sin ix dx &= a_1\int_0^\pi \sin x\sin ix dx + a_2\int_0^\pi \sin 2x\sin ix dx + \&\mathrm{c}\\ &\quad + a_j\int_0^\pi \sin jx\sin ix dx + \&\mathrm{c}\end{aligned}$$

It is easy to show that all the integrals on the right are zero,

$$\frac{1}{i-j}\sin(i-j)x - \frac{1}{i+j}\sin(i+j)x + c$$

except $\int_0^\pi \sin ix\sin ix\, dx = \frac{1}{2}\pi$, so that $a_i = \frac{2}{\pi}\int_0^\pi \phi(x)\sin ix dx$. Note that this equality is valid only for $x = 0$ to $x = \pi$.

The above derived series can be verified. For example, if $\phi(x) = 1$ between $x = 0$ and $x = \pi$, then $a_i = \int \sin ix dx = \frac{2}{i}$ if i is odd, 0 if even, as deduced previously. If $\phi(x) = x$, we get $\int_0^\pi x\sin ix dx = \pm\frac{\pi}{i}$. Even $\cos x$ can be so expanded,

$$\cos x = \frac{2}{\pi}\left\{\left(\frac{1}{1} + \frac{1}{3}\right)\sin 2x + \left(\frac{1}{3} + \frac{1}{5}\right)\sin 4x + \&\mathrm{c}\right\}$$

It is to be remarked that although the left-hand function is even, the right-hand series contains only odd powers.

A similar development of an arbitrary function in terms of cosines is possible

$$\phi(x) = a_0\cos 0x + a_1\cos x + a_2\cos 2x + \&\mathrm{c} + a_i\cos ix + \&\mathrm{c}$$

where $a_i = \frac{2}{\pi}\int_0^\pi \phi(x)\cos ix dx$ and $a_0 = \frac{1}{\pi}\int_0^\pi \phi(x)dx$.

For example, $\phi(x) = x$ gives (the third trigonometric series found for x)

$$x = \frac{1}{2}\pi - 4\frac{\cos x}{\pi} - 4\frac{\cos 3x}{3^2\pi} - 4\frac{\cos 5x}{5^2\pi} - \&\mathrm{c}.$$

Choosing $\phi(x) = \sin x$ gives

$$\frac{1}{4}\pi \sin x = \frac{1}{2} - \frac{\cos 2x}{1 \cdot 3} - \frac{\cos 4x}{3.5} - \frac{\cos 6x}{5.7} - \&\mathrm{c}$$

In particular, at $x = \frac{1}{2}\pi$, $\frac{1}{4}\pi = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \&\mathrm{c}$

It has always been held that a function which is discontinuous, or is zero outside a definite interval, cannot be developed as a series of sines or cosines — we have shown otherwise. Take, as an example, a function which takes the value $\frac{1}{2}\pi$ when x is between 0 and α and null when x is between α and π. Using the above formulas we find

$$\phi(x) = (1 - \cos\alpha)\sin x + \frac{1 - \cos 2\alpha}{2}\sin 2x + \frac{1 - \cos 3\alpha}{3}\sin 3x + \&\mathrm{c}$$

Similarly if $\phi(x) = \sin\frac{\pi x}{\alpha}$ for x between 0 and α and 0 otherwise, we get

$$\phi(x) = 2\alpha\left\{\frac{\sin\alpha\sin x}{\pi^2 - \alpha^2} + \frac{\sin 2\alpha\sin 2x}{\pi^2 - 2^2\alpha^2} + \frac{\sin 3\alpha\sin 3x}{\pi^2 - 3^2\alpha^2} + \&\mathrm{c}\right\}$$

It is remarkable that when α becomes equal to π, all the terms vanish except the first which becomes $\frac{0}{0}$ with a value of $\sin x$.

In the same manner we can find a sine or cosine series for a function represented by parabolic arcs and straight lines or contours of trapezia or triangles.

One must note in the expression $\phi(x) = a + b\cos x + c\cos 2x + \&\mathrm{c}$ that the agreement is valid for x between 0 and π. The right-hand side is periodic and remains the same for x negative. A sine series is also by necessity periodic and changes sign for x negative.

Any function $F(x)$ can be divided into a function $\phi(x)$, such that $\phi(x) = \phi(-x)$, and another $\psi(x)$ such that $\psi(x) = -\psi(-x)$. Indeed, ϕ is the line passing equally between $F(x)$ and its reflection $F(-x)$, $\phi(x) = \frac{1}{2}F(x) + \frac{1}{2}F(-x)$, while $\psi(x) = \frac{1}{2}F(x) - \frac{1}{2}F(-x)$. The functions ϕ and ψ can be developed into series of cosines and sines respectively, valid on $-\pi$ to π,

$$\begin{aligned}
\pi F(x) &= \frac{1}{2}\int_{-\pi}^{\pi}\phi(x)\,dx + \cos x\int_{-\pi}^{\pi}\phi(x)\cos x\,dx + \&\mathrm{c} \\
&\quad + \sin x\int_{-\pi}^{\pi}\psi(x)\sin x dx + \sin 2x\int_{-\pi}^{\pi}\psi(x)\sin 2x dx + \&\mathrm{c} \\
&= \frac{1}{2}\int_{-\pi}^{\pi}F(x)dx + \cos x\int_{-\pi}^{\pi}F(x)\cos x dx + \&\mathrm{c} \\
&\quad + \sin x\int_{-\pi}^{\pi}F(x)\sin x dx + \&\mathrm{c}
\end{aligned}$$

since the integrals $\int_{-\pi}^{\pi}\psi(x)\cos ix dx$ and $\int_{-\pi}^{\pi}\phi(x)\sin ix dx$ vanish.

By replacing x by $\frac{\pi x}{r}$, X by $2r$, and letting $f(x)$ be $F(\frac{\pi x}{r})$ we find

$$\begin{aligned}
\frac{1}{2}Xf(x) &= \frac{1}{2}\int_0^X f(x)dx + \&\mathrm{c} + \cos\frac{2\pi ix}{X}\int_0^X f(x)\cos\frac{2\pi ix}{X}dx + \&\mathrm{c} \\
&\quad + \sin\frac{2\pi ix}{X}\int_0^X f(x)\sin\frac{2\pi ix}{X}dx + \&\mathrm{c}
\end{aligned} \tag{5}$$

and similarly for the series solely in cosines or sines.

More generally, a function $f(x)$ can be developed as a series of terms

$$f(x) = a_1\phi(\mu_1 x) + a_2\phi_2(\mu_2 x) + \&c$$

where $\int_0^X f(x)\phi(\mu_i x)\,dx = a_i \int_0^X \phi(\mu_i x)^2\,dx$ as long as $\int_0^X \phi(\mu_i x)\phi(\mu_j x)\,dx = 0$ when i and j are different.

3.6 Application to the Actual Problem

We can now solve the equation $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$ where $v(\pm\pi, y) = 0$, $v(x, 0) = f(x)$ (the width is taken to be 2π for convenience). The values $ae^{-my}\sin mx$ are solutions; if $m = i\frac{\pi}{r}$, then $v(\pm\pi, y) = 0$ is satisfied. If $f(x) = a_1 \sin\frac{x}{r} + a_2 \sin\frac{2x}{r} + \&c$ then

$$\frac{1}{2}\pi v = a_1 e^{-y}\sin x + a_2 e^{-2y}\sin 2x + \&c$$

The right side can be rewritten as

$$\begin{aligned}\int_0^\pi f(\alpha)d\alpha \sum e^{-iy}\sin ix \sin i\alpha &= \frac{1}{2}\int_0^\pi f(\alpha)d\alpha \sum e^{-iy}[\cos i(x-\alpha) - \cos i(x+\alpha)] \\ &= \frac{1}{2}\int_0^\pi f(\alpha)d\alpha \{F(y, x-\alpha) - F(y, x+\alpha)\}\end{aligned}$$

where $F(y,p) = \sum e^{-iy}\cos ip = \frac{e^{-(y+p\sqrt{-1})}}{1-e^{-(y+p\sqrt{-1})}} + \frac{e^{-(y-p\sqrt{-1})}}{1-e^{-(y-p\sqrt{-1})}} = \frac{\cos p - e^{-y}}{e^y - 2\cos p + e^{-y}}$.

4 Heat Flow in a Ring

4.1 General Solution

The equation of the heated ring cooling in air, $\frac{dv}{dt} = k\frac{d^2v}{dx^2} - hv$, has solutions of the type $v = e^{-ht}u$ where $\frac{du}{dt} = k\frac{d^2u}{dx^2}$. This in turn has solutions $ae^{mt}\sin nx$ and $be^{mt}\cos nx$ where $m = -kn^2$. The value of v must not change when x is increased by $2\pi r$, so $2\pi nr = 2\pi i$, i.e., $n = \frac{i}{r}$. Write the initial temperature $f(x)$ as

$$f(x) = \begin{cases} \quad\;\; +a_1 \sin 1\frac{x}{r} + a_2 \sin 2\frac{x}{r} + \&c \\ +b_0 \; +b_1 \cos 1\frac{x}{r} + b_2 \cos 2\frac{x}{r} + \&c \end{cases}$$

so

$$u = b_0 + \left.\begin{matrix} a_1 \sin\frac{x}{r} \\ b_1 \cos\frac{x}{r}\end{matrix}\right| e^{-k\frac{t}{r^2}} + \left.\begin{matrix} a_2 \sin\frac{2x}{r} \\ b_2 \cos\frac{2x}{r}\end{matrix}\right| e^{-k\frac{2^2 t}{r^2}} + \&c$$

As t increases, u and v tend towards the constant state $b_0 = \frac{1}{2\pi r}\int f(x)dx$, which is the mean initial temperature.

Example 1: $(h = 1, r = \pi)$ Suppose the initial temperature is that for a point source of heat, namely $v = be^{-\pi}(e^{-\pi+x} + e^{\pi-x})$ (see earlier) then

$$v = 2e^{-ht}M\left(\frac{1}{2} + \frac{\cos xe^{kt}}{1^2+1} + \frac{\cos 2xe^{2^2kt}}{2^2+1} + \&c\right)$$

Example 2: Half the ring has initial temperature 1, the other half 0. The solution is $\frac{1}{2}\pi v = e^{-ht}(\frac{1}{4}\pi + \sin x\, e^{-kt} + \frac{1}{3}\sin 3x\, e^{-3^2kt} + \&c)$. Note that the mean temperature, or v when k is infinite, decreases as $e^{-ht}M$.

After some time, as some elementary states decrease much more rapidly than others, the temperature is more or less $v = e^{-ht}(b_0 + (a_1 \sin\frac{x}{r} + b_1\cos\frac{x}{r})e^{-k\frac{t}{r^2}})$. If we follow the temperature v_1+v_2 of two points on a diameter, we find $\frac{v_1+v_2}{2} = b_0e^{-ht}$. Indeed there are precisely two points for which $v_1 = b_0e^{-ht} = v_2$.

4.2 Heat Exchange between Separate Masses

Consider two equal cubes, one with temperature a and the other b, of infinite conductivity. If they are brought in contact their temperature would instantly become the mean $\frac{1}{2}(a+b)$. Now suppose an infinitely thin layer of one face is detached and attaches to the second cube, then returns and these cycles repeated. At each stage, the layer of mass ω has some temperature α, and becomes $\frac{m\beta+\alpha\omega}{m+\omega}$ after touching; upon return the temperature becomes $\frac{\alpha m+\beta\omega}{m+\omega} = \alpha - (\alpha-\beta)\frac{\omega}{m}$, suppressing higher powers of ω. Thus $d\alpha = -(\alpha-\beta)\frac{\omega}{m} = -(\alpha-\beta)kdt$, proportional to the difference in temperature. The coefficient k thus represents the velocity of heat flow.

If we consider n cubes, each passing heat on to the next, we find

$$d\alpha_i = \frac{k}{m}dt((\alpha_{i+1}-\alpha_i) - (\alpha_i - \alpha_{i-1})).$$

According to the well-known method, substitute $\alpha_i = a_ie^{ht}$ to get $a_ih = \frac{k}{m}((a_{i+1}-a_i)-(a_i-a_{i-1}))$. These n equations lead to an equation of degree n in h; thus h could have n roots. To simplify, write $q = \frac{hm}{k}$, so $a_{i+1} = a_i(q+2) - a_{i-1}$, and in general $a_m = A\sin mu + B\sin(m-1)u$; at $m=0$ and $m=1$, a_m is known, and $a_{n+1} = a_n$. This last gives $\sin nu = 0$, i.e., $u = i\frac{\pi}{n}$, i an integer, hence $h+i = -2\frac{k}{m}\operatorname{versin} i\frac{\pi}{n}$ and[3]

$$\alpha_i = a_1\frac{\sin iu - \sin(i-1)u}{\sin u}e^{-\frac{2kt}{m}\operatorname{versin} u}$$

where u is 0 or $\frac{\pi}{n}$ because the others decrease quickly to 0. We see that when $u=0$, $\alpha_i = a_1$. Adding all the values gives $a_1\frac{\sin nu}{\sin u}e^{-\frac{2kt}{m}\operatorname{versin} u}$ which reduces to na_1 when $u=0$.

If we take u to be infinitely small, then $\frac{\sin iu - \sin(i-1)u}{\sin u}$ becomes $\cos iu$, which vanishes for the middle cube.

If the cubes are now taken to lie in a circle, the same analysis can be repeated, with the same equations but different conditions, $u_i = i\frac{2\pi}{n}$,

$$\begin{aligned}\alpha_{m+1} &= (A_1\sin mu_1 + B_1\cos mu_1)e^{-\frac{2kt}{m}\operatorname{versin} u_1} + \&c\\ &\quad + (A_n \sin mu_n + B_n\cos mu_n)e^{-\frac{2kt}{m}\operatorname{versin} u_n}\end{aligned}$$

[3] versin is $1-\cos$

The constants $A_1, \&c, A_n, B_1, \&c, B_n$ can be determined from the initial values of $a_1, \&c, a_n$. (There are only n unknowns by symmetry.)

The equations to be solved are

$$a_i = A_1 \sin(i-1)0\frac{2\pi}{n} + A_2 \sin(i-1)1\frac{2\pi}{n} + \&c + B_1 \cos(i-1)0\frac{2\pi}{n} + B_2 \cos(i-1)1\frac{2\pi}{n} + \&c$$

Noting that $\sum \sin i\frac{2\pi}{n} \sin j\frac{2\pi}{n}$ is nothing except when $i = j$ (and similarly for cosines), while $\sum \sin i\frac{2\pi}{n} \cos j\frac{2\pi}{n}$ is always nothing,

$$\begin{aligned} \frac{n}{2}A_j &= \sum a_i \sin(i-1)(j-1)\frac{2\pi}{n} \\ \frac{n}{2}B_j &= \sum a_i \cos(i-1)(j-1)\frac{2\pi}{n} \end{aligned}$$

Substituting gives the solution for the temperature

$$\begin{aligned} a_j = \frac{1}{n}\sum a_i + \sum \Big[& \frac{2}{n}\sin(j-1)\frac{2\pi}{n}\sum a_i \sin(i-1)p\frac{2\pi}{n} + \\ & \frac{2}{n}\cos(j-1)\frac{2\pi}{n}\sum a_i \cos(i-1)p\frac{2\pi}{n}\Big] e^{-\frac{2kt}{m}\operatorname{versin} p\frac{2\pi}{n}} \end{aligned}$$

where a_i are the initial temperatures. It is the sum of n solutions, each having nil initial temperature except for one mass. When the time is infinite, the temperature becomes $\frac{1}{n}\sum a_i$, the mean temperature.

In going to a continuous body, we must replace m by dx, n by $\frac{2\pi}{dx}$, k by $\frac{\pi g}{dx}$, i by $\frac{x}{dx}$, and a_i by some function $f(x)$. We then get the solution derived previously,

$$\begin{aligned} \phi(x,t) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx + \frac{1}{\pi}\left(\sin x\int f(x)\sin x\,dx + \cos x\int f(x)\cos x\,dx\right)e^{-g\pi t} + \&c \\ &= \frac{1}{2\pi}\int d\alpha f(\alpha)\textstyle\sum_{-\infty}^{+\infty}\cos i(\alpha - x)e^{-i^2kt} \end{aligned}$$

This shows that to solve the pde we can solve for a finite number of bodies, and then suppose that number to be infinite. Note that the formula contains an arbitrary function, corresponding to the initial condition. The solution must be unique, for given the initial state v_1, the next instant would give $v_2 = v_1 + k\frac{dv_1}{dt}dt$ which is determined ($\frac{dv}{dt} = k\frac{d^2f}{dx^2}$) and so on for the following instants.

5 Heat Flow in a Solid Sphere

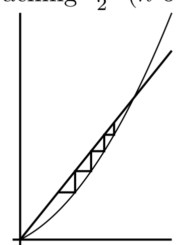
5.1 General Solution

$$\frac{dv}{dt} = k\left(\frac{d^2v}{dx^2} + \frac{2}{x}\frac{dv}{dx}\right), \quad \frac{dv}{dx} + hv = 0 \text{ at } x = X,$$

x is the radial variable, and v is initially $F(x)$. Putting $y = vx$ gives $\frac{dy}{dt} = k\frac{d^2y}{dx^2}$ which admits solutions $y = e^{mt}u$, where u is a function of x satisfying $mu = k\frac{d^2u}{dx^2}$. Since v cools to 0 as t becomes infinite, m can only take negative values $m = -kn^2$. Hence u is a circular function and if v is to remain finite at $x = 0$, it must equal

$$v = a\frac{e^{-kn^2t}}{x}\sin nx.$$

The condition at the surface then becomes $nX\cos nX + (hX - 1)\sin nX = 0$, i.e., $\frac{nX}{\tan nX} = 1 - hX$, an equation of the type $\frac{\epsilon}{\tan\epsilon} = \lambda$. There are an infinity of solutions obtained by intersecting the line $y = \frac{\epsilon}{\lambda}$ with the curve $y = \tan\epsilon$, approaching $\frac{n\pi}{2}$ (n odd) for n of an advanced order.



One can obtain these solutions by using the equations $\epsilon = \arctan u$, $u = \frac{\epsilon}{\lambda}$ and substituting a value for u into the first to get ϵ then in the second to get u, and repeating. We should not reverse the operations, using $u = \tan\epsilon$, $\epsilon = \lambda u$, for then we depart from the roots.

We now suppose these roots are known and call them n_iX. The elementary solutions are then $v = ae^{-kn^2t}\frac{\sin nx}{nx}$. The general state is formed from an infinite number of them $vx = \sum a_ie^{-kn_i^2t}\sin n_ix$. At $t = 0$, $F(x)x = \sum a_i\sin n_ix$. To determine the coefficients, multiply by $\sin nx\,dx$ and integrate.

$$\begin{aligned}\int_0^X \sin mx\sin nx\,dx &= \frac{1}{m^2-n^2}(-m\sin nX\cos mX + n\sin mX\cos nX)\\ &= \frac{\cos nX\cos mX}{m^2-n^2}(-m\tan nX + n\tan mX) = 0\end{aligned}$$

except when $m = n$ in which case the integral works out to $\frac{1}{2}X - \frac{1}{4n}\sin 2nX$, i.e., $a_i = \frac{2\int x\sin n_ixf(x)dx}{X - \frac{1}{2n_i}\sin 2n_iX}$.

5.2 Remarks

If h/K (or X) is very small, the equation $\frac{\epsilon}{\tan\epsilon} = 1 - \frac{h}{K}X$ becomes $\epsilon^2 = \frac{3hX}{K}$, omitting higher powers of ϵ, and the general solution becomes $v = e^{-\frac{3h}{CDX}t} + \&c.$, which is approximately independent of x, the radius of a spherical shell. The temperature reduces to a fraction $\frac{1}{m}$ when $t = \frac{X}{3h}CD\log m$, which is proportional to the diameter.

If the solid sphere has been cooling for a long time, then v is proportional to $\frac{\sin\frac{\epsilon_1x}{X}}{\frac{\epsilon_1x}{X}}$, which is practically 1 when ϵ is small.

The result $v = e^{-\frac{3h}{CDX}t}$ has been known to physicists, for if the solid is losing heat at the rate $hSvdt$ then its temperature decreases by $dv = -\frac{hSvdt}{CDV}$ (S is the surface area and V its volume), an equation whose solution is $e^{-\frac{hS}{CDV}t}$.

The constant $\frac{hS}{CDV}$ can thus be obtained by measuring the temperature at two different times. If h is the same, as when two liquids are in the same thin vessel, then the ratio of their specific heats can be obtained.

If a liquid is cooling at the rate $du = -Hudt$, and a thermometer is used to measure its temperature, the temperature of the thermometer changes as $dv = -h(v-u)dt$. These equations can be solved easily, and the error of the thermometer is $v-u = \frac{H}{h-H}u$ (since h is much larger than H). We tested this by dipping a thermometer into water at 8.5° (octagesimal scale) — its temperature dropped from 40° to 20° in six seconds. Thus e^{-h} is 0.000042. Then water at 60° was cooled in air at 12° — e^{-H} was found to be 0.98514. Thus the ratio of h to H is more than 673, and the error in the thermometer is less than the 600th part of the temperature it is meant to measure. We remark here that our new thermometers were calibrated, degree by degree, by putting them together with a number of calibrated thermometers in a vessel filled with fluid as it cools slowly.

Finally consider the case when X is large; then ϵ_i are very nearly $\pi, 2\pi, 3\pi, \&c.$ and the mean temperature is

$$z = \int \frac{vd(\frac{4\pi x^3}{3})}{\frac{4\pi X^3}{3}} = \frac{3}{X^3}\int x^2 v dx = \frac{6}{\pi^3} e^{-\frac{K\pi^2 t}{CDX^2}} + \&c.$$

This time, for a sphere to cool to a fraction $\frac{1}{m}$, a time equal to $\frac{CDX^2}{K\pi^2}\log m$ is needed, proportional to X^2, i.e., the cooling is very slow for large spheres.

6 Heat Flow in a Solid Cylinder

6.1 The General Solution

$$\frac{dv}{dt} = k\left(\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx}\right), \qquad hV + \frac{dV}{dx} = 0$$

Giving v the simple form $v = ue^{-mt}$ we find $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + gu = 0$ ($g = \frac{m}{k}$). The solution of this equation is

$$u = 1 - \frac{gx^2}{2^2} + \frac{g^2x^4}{2^2.4^2} - \frac{g^3x^6}{2^2.4^2.6^2} + \&c.$$

The condition at the surface $x = X$ becomes

$$\frac{hX}{2}(1 - \theta + \frac{\theta^2}{2^2} - \&c.) = \theta - \frac{2\theta^2}{2^2} + \frac{3\theta^3}{2^2.3^2} - \&c$$

where $\theta = g\frac{X^2}{2^2}$. The number g cannot be arbitrary, and we shall show that there are an infinity of roots g_i. The solution of the heat equation then consists of a sum of terms $a_i e^{-g_i kt} u_i(x)$, where the coefficients a_i are determined by the initial temperature.

Let $y = 1 - \theta + \frac{\theta^2}{2^2} - \frac{\theta^3}{2^2.3^2} + \&c. = f(\theta)$ be the value of u at $x = X$. The condition on g becomes $\frac{hX}{2} + \theta\frac{f'(\theta)}{f(\theta)} = 0$. Differentiating y twice we find $y + \frac{dy}{d\theta} + \theta\frac{d^2y}{d\theta^2} = 0$, and in general, $\frac{d^iy}{d\theta^i} + (i+1)\frac{d^{i+1}y}{d\theta^{i+1}} + \theta\frac{d^{i+2}y}{d\theta^{i+2}} = 0$. (Of course, these equations determine y if we know in addition that $y = 1$ when $\theta = 0$.)

Now from the theory of algebraic equations, if any real root of $\frac{d^{i+1}X}{dx^{i+1}} = 0$, when substituted into $\frac{d^iX}{dx^i}$ and $\frac{d^{i+2}X}{dx^{i+2}}$ gives two values of opposite sign, then the equation $X = 0$ has all roots real. This is so in our case, for if $\frac{d^{i+1}y}{d\theta^{i+1}} = 0$ it follows from the last equation in the previous paragraph that $\frac{d^iy}{d\theta^i}$ and $\frac{d^{i+2}y}{d\theta^{i+2}}$ are opposite in sign (and it is clear that there cannot be negative roots from the series expansion of y), hence all roots of $y = 0$ (and $y' = 0$) are real. As every root of y' lies between consecutive roots of $y = 0$, then $\theta\frac{y'}{y}$ alternates between nothing and infinity. Hence the equation in g, $\frac{hX}{2} + \theta\frac{f'(\theta)}{f(\theta)} = 0$ has all its roots θ_i (and thus m_i) real and positive.

Write

$$\begin{aligned}
2\cos(\alpha\sin r) &= e^{\frac{\alpha\omega}{2}}e^{-\frac{\alpha\omega^{-1}}{2}} + e^{-\frac{\alpha\omega}{2}}e^{\frac{\alpha\omega^{-1}}{2}} \\
&= 2\left(1 - \frac{\alpha^2}{2^2} + \frac{\alpha^4}{2^2.4^2} - \&c.\right) + \\
&\qquad 2\left(\frac{\alpha^4}{2.4.6.8} - \&c.\right)(\omega^2 + \omega^{-2}) + \&c. \\
&= 2A + B.2\cos 2r + C.2\cos 4r + \&c.
\end{aligned}$$

where $\omega = e^{r\sqrt{-1}}$. A is clearly u with $\alpha = x\sqrt{g}$; but recall how to find A in this series,

$$u = A = \frac{1}{\pi}\int_0^\pi \cos(\alpha\sin r)dr = \frac{1}{\pi}\int_0^\pi \cos(x\sqrt{g}\sin r)dr \qquad (6)$$

This is just one particular integral of the differential equation; the other can be found by putting uS in the equation and finding $S = a + b\int\frac{dx}{xu^2}$, where a and b are arbitrary constants. One can verify the fact (6) in another way

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi \cos(\alpha\sin r)dr &= \int_0^\pi dr\left(1 - \frac{\alpha^2\sin^2 r}{\underline{|2}} + \frac{\alpha^4\sin^4 r}{\underline{|4}} - \&c.\right) \\
&= \pi - \frac{\alpha^2}{\underline{|2}}S_2 + \frac{\alpha^4}{\underline{|4}}S_4 - \&c.
\end{aligned}$$

where $S_n = \int_0^\pi \sin^n r dr = \int_0^\pi A_n + B_n\cos 2r + \&c. = A_n\pi$. It can be verified that $A_2 = \frac{1}{2^2}.\frac{2}{1}$, $A_4 = \frac{1}{2^4}\cdot\frac{3.4}{1.2}$, &c.; substituting these values gives the desired formula.

One can also express $\frac{y'}{y}$ as the continued fraction $\frac{-1}{1-}\frac{\theta}{2-}\frac{\theta}{3-\&c.}$ by iterating $y + y' + \theta y'' = 0$, i.e., $\frac{y'}{y} = \frac{-1}{1+\theta\frac{y''}{y'}} = \&c.$ The condition $\frac{hX}{2}f(\theta) + \theta f'(\theta) = 0$ then gives an infinity of values θ_i.

Each elementary state is therefore

$$\pi v_i = e^{-\frac{2^2 k+\theta_i}{X^2}} \int_0^\pi \cos(2\frac{x}{X}\sqrt{\theta_i}\sin r)dr$$

and the general solution is a sum of such elementary states $v = a_1v_1 + a_2v_2 + \&c.$ Initially $\phi(x) = V = a_1u_1 + a_2u_2 + \&c. = a_1\psi(x\sqrt{g_1}) + a_2\psi(x\sqrt{g_2}) + \&c.$

In order to be able to find the coefficients we need to find functions σ_i such that $\int_0^X \sigma_i u_j dx = 0$ unless $i = j$. Indeed, since $gu + \frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = 0$, we find by integration by parts,

$$\begin{aligned}
0 &= -g\int_0^X \sigma u\,dx = \int_0^X \left(\frac{\sigma}{x}\frac{du}{dx} + \sigma\frac{d^2u}{dx^2}\right) dx \\
&= C + u\frac{\sigma}{x} - \int u\,d\left(\frac{\sigma}{x}\right) + D + \frac{du}{dx}\sigma - u\frac{d\sigma}{dx} + \int u\frac{d^2\sigma}{dx^2}dx \\
&= \int_0^X \left\{u\frac{d^2\sigma}{dx^2} - u\frac{d(\frac{\sigma}{x})}{dx}\right\} dx + \left(\frac{du}{dx}\sigma - u\frac{d\sigma}{dx} + u\frac{\sigma}{x}\right)_\omega - \left(\frac{du}{dx}\sigma - u\frac{d\sigma}{dx} + u\frac{\sigma}{x}\right)_\alpha
\end{aligned}$$

where the suffix α denotes $x = 0$ and ω denotes $x = X$. This would be possible if $\frac{d^2\sigma}{dx^2} - \frac{d(\frac{\sigma}{x})}{dx} = -\frac{n}{k}\sigma$ and also $\frac{\sigma}{x}$ and $\frac{d\sigma}{dx}$ were to vanish at $x = 0$ and $x = X$. But substituting $\sigma = xs$ in this equation gives $\frac{n}{k}s + \frac{d^2s}{dx^2} + \frac{1}{x}\frac{ds}{dx} = 0$, i.e., $s = \psi(x\sqrt{\frac{n}{k}})$. Thus it is enough to take $\sigma = x\psi(x\sqrt{\frac{n}{k}})$ for the method of finding the coefficients to work.

It remains to work out $\int \sigma u\,dx$ when $m = n$. Letting $\sqrt{\frac{m}{k}} = \mu$, $\sqrt{\frac{n}{k}} = \nu$, we have

$$\begin{aligned}
\int x\psi(\mu x)\psi(\nu x)\,dx &= \frac{\mu X\psi'(\mu X)\psi(\nu X) - \nu X\psi'(\nu X)\psi(\mu X)}{\nu^2 - \mu^2} \\
&= \frac{\mu X^2\psi'^2 - X\psi\psi' - \mu X^2\psi\psi''}{2\mu}
\end{aligned}$$

when $\nu = \mu$. But $h\psi + \mu\psi' = 0$ and $(\mu^2 - \frac{h}{x})\psi + \mu^2\psi'' = 0$, hence the above becomes equal to

$$\frac{1}{2}X^2\psi^2\left(\frac{\mu^2 + h^2}{\mu^2}\right) = \frac{X^2U^2}{2}\left(1 + \frac{kh^2}{m}\right)$$

where U is the value of u at $x = X$, i.e.,

$$\int_0^X xu_iu_j\,dx = 0 \text{ and } \int_0^X xu_i^2\,dx = \left\{1 + \left(\frac{hX}{2\sqrt{\theta_i}}\right)^2\right\}\frac{X^2U_i^2}{2}$$

7 Heat Flow in a Rectangular Prism

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0$$

with one end at a constant temperature and initially the temperature is nil. Trying a solution of the type $ae^{-mx}\cos ny\cos pz$ we find $m^2-n^2-p^2=0$. To satisfy also $\frac{h}{k}v+\frac{dv}{dy}=0$ at $y=\pm l$ as well as $\frac{h}{k}v+\frac{dv}{dz}=0$ at $z=\pm l$ we find that we need $\frac{hl}{k}=nl\tan nl=pl\tan pl$, i.e., n and p are equal to $\frac{\epsilon}{l}$ where ϵ solves $\epsilon\tan\epsilon=\frac{hl}{k}$, of which there are an infinite number.

The elementary states are therefore $e^{-\sqrt{n_i^2+n_j^2}x}\cos n_iy\cos n_jz$ where $n_i=\frac{\epsilon_i}{l}$. To satisfy $v=1$ at $x=0$ we need

$$1=(a_1\cos n_1y+a_2\cos n_2y+\&\mathrm{c})(b_1\cos n_1z+b_2\cos n_2z+\&\mathrm{c}).$$

It is sufficient to take each series equal to 1 separately. The coefficients a_i are found in the same manner as before even if n_i are not now the odd integers. However, unless $n=\nu$,

$$\begin{aligned}\int_0^l\cos ny\cos\nu y\,dy &= \frac{1}{2}\int_0^l\cos(n-\nu)y\,dy+\frac{1}{2}\int_0^l\cos(n+\nu)y\,dy\\ &= \frac{1}{2}\frac{(n+\nu)\sin(n-\nu)l+(n-\nu)\sin(n+\nu)l}{n^2-\nu^2}\\ &= \frac{n\sin nl\cos\nu l-\nu\cos nl\sin\nu l}{n^2-\nu^2}\\ &= 0\end{aligned}$$

on account of $n\tan nl=\frac{h}{k}=\nu\tan\nu l$. The dominant elementary state is $e^{-x\sqrt{2n_1^2}}\cos n_1y\cos n_2z$, especially if l is small, in which case $\epsilon_1\tan\epsilon_1=\frac{hl}{k}$ becomes $\epsilon_1^2=\frac{hl}{k}$ and $n_1=\sqrt{\frac{hl}{k}}$. Conversely if l is large, then $\epsilon_i=m\frac{\pi}{2}$ (m odd) and at $y=0$, $z=0$, writing α for $e^{-\frac{x}{l}\frac{\pi}{2}}$,

$$v\left(\frac{\pi}{4}\right)^2=1(\alpha^{\sqrt{1^2+1^2}}-\frac{1}{3}\alpha^{\sqrt{1^2+3^2}}+\&\mathrm{c})-\frac{1}{3}(\alpha^{\sqrt{3^2+1^2}}-\frac{1}{3}\alpha^{\sqrt{3^2+3^2}}+\&\mathrm{c})+\&\mathrm{c}$$

8 Heat Flow in a Solid Cube

$$\frac{dv}{dt}=k\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)\qquad k=\frac{K}{CD}$$

At the surface $K\frac{dv}{dx}+hv=0$ at $x=\pm a$ and similarly $K\frac{dv}{dy}+hv=0$ at $y=\pm a$, etc. Taking the simplest solution $e^{-mt}\cos nx\cos py\cos qz$, we find $m-k(n^2+p^2+q^2)$. By the same working as the previous section, $na\tan na=\frac{h}{K}a$ and similarly for p and q. So the solutions are

$$e^{-kt(n_1^2+n_2^2+n_3^2)}\cos n_1x\cos n_2y\cos n_3z$$

the principal one being $e^{-3kn_1^2t}\cos n_1x\cos n_2y\cos n_3z$. If we suppose $v=XYZ$ where X depends on x and t, Y on y and t, Z on z and t, then

$$\frac{1}{X}\frac{dX}{dt}+\frac{1}{Y}\frac{dY}{dt}+\frac{1}{Z}\frac{dZ}{dt}=k\left(\frac{1}{X}\frac{d^2X}{dt^2}+\frac{1}{Y}\frac{d^2Y}{dt^2}+\frac{1}{Z}\frac{d^2Z}{dt^2}\right)$$

which implies $\frac{dX}{dt} = k\frac{d^2X}{dx^2}$, $\frac{dY}{dt} = k\frac{d^2Y}{dy^2}$, $\frac{dZ}{dt} = k\frac{d^2Z}{dz^2}$ as well as $\frac{dX}{dx} + \frac{h}{K}X = 0$, $\frac{dY}{dy} + \frac{h}{K}Y = 0$, $\frac{dZ}{dz} + \frac{h}{K}Z = 0$. These equations have been solved earlier.

We can compare the cooling of a cube with that of a sphere. In the sphere, the temperature diminishes as e^{-kn^2t} where $\frac{na}{\tan na} = 1 - \frac{h}{K}a$, while in the cube it diminishes as $e^{-3\frac{\epsilon^2}{a^2}kt}$ where $\epsilon \tan\epsilon = \frac{h}{K}a$. When a is small, these ratios become both $e^{-\frac{3h}{CDa}t}$. When a is large, however, ϵ becomes $\frac{\pi}{2}$, and the two ratios become $e^{-\frac{3k\pi^2}{4a^2}t}$ and $e^{-\frac{k\pi^2}{a^2}t}$. Note that the time it takes for the temperature to halve is proportional in both cases but in the ratio 4 to 3 (cube to sphere).

9 Heat Diffusion

9.1 Heat Flow in an Infinite Line

$$\frac{dv}{dt} = k\frac{d^2v}{dx^2} \qquad \left(k = \frac{K}{CD}\right), \quad \phi(x,0) = F(x).$$

We treat this first in one dimension to illustrate the concepts. Take $F(x)$ to be a symmetric function on the portion ab and 0 otherwise. Trying the solution $a\cos qx\, e^{-kq^2t}$, q and a can take arbitrary values, so we can take q_i close to each other, in which case a_i becomes a function $f(q)$ and

$$v = \int_0^\infty dq\, f(q)\cos qx\, e^{-kq^2t}.$$

$$\text{At } t = 0, \qquad F(x) = \int_0^\infty dq\, f(q)\cos qx. \tag{7}$$

The question is whether one can find a suitable function $f(q)$ given $F(x)$, i.e., $F(x) = dq\, f(q_1)\cos q_1x + dq\, f(q_2)\cos q_2x + \&c$ $(q_i = i\,dq)$. Using the method established before, multiply by $dx \cos rx$ and integrate from $x = 0$ to $x = n\pi$ with $n = \frac{1}{dq}$ infinite

$$\int_0^{n\pi} F(x)\cos q_jx\, dx = dq\, f(q_j)\cdot\frac{1}{2}n\pi$$

or, at $n = \infty$,

$$f(q) = \frac{2}{\pi}\int_0^\infty F(x)\cos qx\, dx, \text{[4]}$$

The solution v is thus found. For example, taking $F(x)$ to be 1 between $x = 0$ and $x = 1$ and 0 otherwise, we find $f(q) = \frac{2}{\pi}\frac{\sin q}{q}$. Note that, by (7), even discontinuous functions can be represented by integrals. Similarly if $F(x) = e^{-x}$ for x positive, then $f(q) = \frac{2}{\pi}\frac{1}{1+q^2}$.

[4] became known as Fourier's theorem

In an entirely analogous manner, one can write $F(x) = \int_0^\infty dq\, f(q)\sin qx$ where $f(q) = \frac{2}{\pi}\int_0^\infty F(x)\sin qx\, dx$ when $F(x)$ satisfies $F(-x) = -F(x)$. In general, an arbitrary function $\phi(x)$ can be expressed as the sum $F(x) + f(x)$ where F is symmetric and f alternate. We would then obtain

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} d\alpha\, \phi(\alpha)\cos(q-\alpha)x$$

since $\int_{-\infty}^{+\infty} d\alpha\, f(\alpha)\cos q\alpha = 0$ and $\int_{-\infty}^{+\infty} d\alpha\, F(\alpha)\sin q\alpha = 0$. These are the same results we obtained for series. The main difference between solids of definite form and infinite ones is primarily this, that the first have series solutions and the latter integral ones.

One can see this another way. Starting from

$$\frac{\pi}{2}\phi(u) = \sin u\int_0^\pi du\, \phi(u)\sin u + \sin 2u\int_0^\pi du\, \phi(u)\sin 2u + \&\text{c}$$

and taking $u = \frac{x}{n}$, $f(x) = \phi(\frac{x}{n})$, $n = \frac{1}{dq}$, $q = i\, dq$, we find a sum of

$$\sin\frac{ix}{n}\int_0^{n\pi}\frac{dx}{n}\phi\left(\frac{x}{n}\right)\sin\frac{ix}{n} = dq\ \sin qx\int_0^\infty dx\, f(x)\sin qx$$

Thus

$$\frac{\pi}{2}f(x) = \int_0^\infty dq\ \sin qx\int_0^\infty dx\, f(x)\sin qx.$$

Equivalently, (5) can be rewritten as

$$f(x) = \frac{1}{2\pi}\sum_i\int_a^b d\alpha\, f(\alpha)\cos\frac{2i\pi}{X}(x-\alpha)$$

as X is made larger, we find

$$f(x) = \frac{1}{2\pi}\int_a^b d\alpha\, f(\alpha)\int_{-\infty}^{+\infty} dp\ \cos p(x-\alpha),$$

$$\text{and}\quad \frac{d^i}{dx^i}f(x) = \frac{1}{2\pi}\int d\alpha\, f(\alpha)\int dp\, p^i\cos(px - p\alpha + i\frac{\pi}{2}).$$

Thus arbitrary functions can be differentiated, integrated or summed. In fact this transformation gives an easy way of changing derivatives into algebraic terms.

As an example, take $f(x) = x^r$, then

$$\int_0^\infty dx\ \sin qx\, x^r = \frac{1}{q^{r+1}}\int_0^\infty du\ \sin u\, u^r = \frac{\mu}{q^{r+1}}$$

and

$$\int_0^\infty dq\ \sin qx\frac{\mu}{q^{r+1}} = x^r\int_0^\infty du\,\frac{\sin u}{u^{r+1}}\mu = \mu\nu x^r$$

so $\mu\nu = \frac{\pi}{2}$. For example, when $r = -\frac{1}{2}$, $\mu = \nu$ and we find $\int_0^\infty du\, \frac{\sin u}{\sqrt{u}} = \sqrt{\frac{\pi}{2}}$. Similarly, $\int_0^\infty du\, \frac{\cos u}{\sqrt{u}} = \sqrt{\frac{\pi}{2}}$ and from these two conclude[5] that $\int_0^\infty dq\, e^{-q^2} = \frac{1}{2}\sqrt{\pi}$, a well-known result. Similarly, the integrals $\int dq\, \cos q^2$ can be obtained by substituting $x = y\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)$ in $\sqrt{\pi} = \int_{-\infty}^{+\infty} dx\, e^{-x^2}$.

The identity can also be used to find the form of the arbitrary function $f(x)$ when x is replaced by $\mu + \nu\sqrt{-1}$,

$$\begin{aligned} f(\mu + \nu\sqrt{-1}) &= \frac{1}{2\pi}\int d\alpha\, f(\alpha) \int dp\, \cos(p\mu - p\alpha)(e^{p\nu} + e^{-p\nu}) \\ &\quad + \sqrt{-1}\frac{1}{2\pi}\int d\alpha\, f(\alpha)\int dp\, \sin(p\mu - p\alpha)(e^{p\nu} - e^{-p\nu}) \end{aligned}$$

It is worthwhile to investigate the nature of these transforms

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} d\alpha\, f(\alpha)\frac{\sin(p\alpha - px)}{\alpha - x}$$

Firstly, it is clear that $\int_0^\infty dx \frac{\sin px}{x} = \int_0^\infty dx \frac{\sin x}{x} = \frac{\pi}{2}$. Secondly, as p becomes greater, $\frac{\sin px}{x}$ becomes infinitely oscillatory, so for x not small, $\int_\omega^\infty \frac{\sin px}{x} dx$ becomes zero. On each oscillation, $f(\alpha)\frac{\sin(p\alpha - px)}{\alpha - x}$ has a zero area because $\frac{f(\alpha)}{\alpha - x}$ is practically constant. So the only part of the integral which contributes is that part near to x, from $x - \omega$ to $x + \omega$, with ω infinitely small. Over this interval

$$\int_{x-\omega}^{x+\omega} d\alpha\, f(\alpha)\frac{\sin p(\alpha - x)}{\alpha - x} = f(x)\int_{x-\omega}^{x+\omega} d\alpha \frac{\sin(p\alpha - px)}{\alpha - x} = \pi f(x)$$

(In fact this analysis is true for many other functions apart from $\frac{\sin px}{x}$).

Let us solve $\frac{du}{dt} = k\frac{d^2u}{dx^2}$ in a different way using the above. From $\sqrt{\pi} = \int_{-\infty}^{+\infty} dq\, e^{-q^2}$ we get

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} dq\, e^{-(q+b)^2} = e^{-b^2}\int_{-\infty}^{+\infty} dq\, e^{-q^2} e^{-2qb}$$

so taking $b^2 = kt$,

$$\sqrt{\pi} e^{kt} = \int_{-\infty}^{+\infty} dq\, e^{-q^2} e^{-2q\sqrt{kt}}.$$

The elementary solutions are $u = e^{-nx}e^{n^2kt} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} dq\, e^{-q^2}e^{-n(x+2q\sqrt{kt})}$ and in general

$$u = \int_{-\infty}^{+\infty} dq\, e^{-q^2}(a_1 e^{-n_1(x+2q\sqrt{kt})} + \&\mathrm{c}) = \int_{-\infty}^{+\infty} dq\, e^{-q^2}\phi(x + 2q\sqrt{kt})$$

since a_i are arbitrary. This was first derived by M. Laplace.

[5] by taking $e^{-q^2} = \cos iq^2 + i\sin iq^2$ and changing variable $u = iq^2$.

To solve $\frac{dv}{dt} = k\frac{d^2v}{dx^2} - hv$ with constant temperature 1 at $x = 0$, let $v = e^{-x\sqrt{\frac{h}{k}}} + e^{-ht}u$ then $\frac{du}{dt} = \frac{d^2u}{dx^2}$ so u is given by the above integral. If the initial temperature v is 0 (except at $x = 0$), then initially $u = -e^{-x\sqrt{\frac{h}{k}}}$ for x larger than 0 and $e^{x\sqrt{\frac{h}{k}}}$ when x is less than 0. The integral from $x + 2q\sqrt{kt} = 0$ to $x + 2q\sqrt{kt} = +\infty$ becomes

$$-\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{kt}}}^{\infty} dq\, e^{-q^2}e^{-(x+2q\sqrt{kt})\sqrt{\frac{h}{k}}} = -\frac{e^{-x\sqrt{\frac{h}{k}}}}{\sqrt{\pi}}e^{ht}\int dq\, e^{-(q+\sqrt{ht})^2}$$
$$= -\frac{e^{-x\sqrt{\frac{h}{k}}}}{\sqrt{\pi}}e^{ht}\int_{\sqrt{ht}-\frac{x}{2\sqrt{kt}}}^{\infty} dr\, e^{-r^2}$$

The second integral from $x + 2q\sqrt{kt} = -\infty$ to $x + 2q\sqrt{kt} = 0$ has to be worked out separately. The integrals of the type $\psi(R) = \frac{1}{\sqrt{\pi}}\int_R^\infty dr\, e^{-r^2}$ are now quite well-known and can be calculated using series. The final solution is then

$$v = e^{-x\sqrt{\frac{h}{k}}} - e^{-x\sqrt{h}}\psi(\sqrt{ht} - \frac{x}{2\sqrt{kt}}) + e^{x\sqrt{h}}\psi(\sqrt{ht} + \frac{x}{2\sqrt{kt}})$$

If heat cannot escape ($h = 0$) the solution becomes

$$v = 1 - \left(\psi\left(-\frac{x}{2\sqrt{kt}}\right) - \psi\left(\frac{x}{2\sqrt{kt}}\right)\right) = 1 - 2\phi\left(\frac{x}{2\sqrt{kt}}\right)$$

where $\phi(R) = \frac{1}{\sqrt{\pi}}\int_0^R dr\, e^{-r^2} = \frac{1}{\sqrt{\pi}}(R - \frac{1}{1}\frac{1}{3}R^3 + \frac{1}{\underline{|2}}\frac{1}{5}R^5 - \&\text{c})$.

Note in passing that

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} dq\, e^{-q^2} = \int_{-\infty}^{+\infty} dq\, e^{-(q+a)^2} = \int_{-\infty}^{+\infty} dq\, e^{-q^2}e^{-2aq}e^{-a^2}$$

$$\text{so}\quad \sqrt{\pi}(1 + a^2 + \frac{a^4}{\underline{|2}} + \&\text{c}) = \int_{-\infty}^{+\infty} dq\, e^{-q^2}(1 - 2aq + \frac{2^2a^2q^2}{\underline{|2}} - \&\text{c})$$

from which follows the known result

$$\int_{-\infty}^{+\infty} dq\, e^{-q^2}q^{2m} = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 2\cdot 2\cdots 2}\sqrt{\pi}.$$

9.2 Heat Flow in an Infinite Solid

It should be clear that the foregoing analysis can be repeated for 3 orthogonal space variables to get a solution of $\frac{dv}{dt} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}$, with initial temperature $f(x, y, z)$,

$$u = \pi^{-\frac{3}{2}}\int_{-\infty}^{+\infty} dn\int_{-\infty}^{+\infty} dp\int_{-\infty}^{+\infty} dq\, e^{-(n^2+p^2+q^2)}f(x+2n\sqrt{t}, y+2p\sqrt{t}, z+2q\sqrt{t}) \tag{8}$$

Now $e^{-n^2 t}\cos nx$ is a solution of this equation, hence

$$\begin{aligned}
\int_{-\infty}^{+\infty} dn\, e^{-n^2 t}\cos nx &= \frac{1}{\sqrt{t}}\int_{-\infty}^{+\infty} dp\, e^{-p^2}\cos 2pu \qquad \text{where } p^2 = n^2 t,\ 2pu = nx \\
&= \frac{1}{2}\frac{1}{\sqrt{t}}e^{-u^2}\int dp\, e^{-(p+u\sqrt{-1})^2} + \frac{1}{2}\frac{1}{\sqrt{t}}e^{-u^2}\int dp\, e^{-(p+u\sqrt{-1})^2} \\
&= \frac{1}{\sqrt{t}}e^{-u^2}\sqrt{\pi} \qquad \text{since } \int_{-\infty}^{+\infty} dq\, e^{-(q+b)^2} = \sqrt{\pi} \text{ whatever } b \text{ is} \\
&= \frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{x^2}{4t}}
\end{aligned}$$

must also be a solution (it being a sum of solutions). Thus $\int d\alpha\, f(\alpha)\frac{e^{-\frac{(x-\alpha)^2}{4t}}}{\sqrt{t}}$ is also a solution. This integral is equal to $\int dq\, f(x+2q\sqrt{t})e^{-q^2}$ with the substitution $\frac{(x-\alpha)}{2\sqrt{t}} = q$. It follows that in three dimensions we have also the solution

$$v = \int_{-\infty}^{+\infty} d\alpha \int_{-\infty}^{+\infty} d\beta \int_{-\infty}^{+\infty} d\gamma\, f(\alpha,\beta,\gamma)t^{-\frac{3}{2}}e^{-\frac{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2}{4t}}$$

which is essentially the same as (8).

Note that as a general principle, if two functions satisfy the same differential equation and start with the same values then they must necessarily be equal. In particular, two integrals, such as the above, which are equal at $t=0$, must be identical.

Going back to $v = \int d\alpha\, f(\alpha)\frac{e^{-\frac{(\alpha-x)^2}{4kt}}}{2\sqrt{\pi kt}}$, if the initial temperature is restricted to a line segment from $x=-h$ to $x=+g$ and we consider the temperature after a very long time, we get $\frac{e^{-\frac{x^2}{4kt}}}{2\sqrt{\pi kt}}\int_{-h}^{g} d\alpha\, f(\alpha)$ when, say, $(1 - e^{-\frac{\alpha^2+2\alpha x}{4kt}}) < \frac{1}{100}$, i.e., $t > \frac{100gx}{2k}$. Taking typical values of k, this gives t of about three days and a half when $g = 0.1$ and $x = 1$. This approximation is independent of the initial distribution of heat, but depends only on its total quantity. In particular if the heat is initially concentrated in a small region ω, then $v = \frac{e^{-\frac{x^2}{4kt}}}{2\sqrt{\pi t}}\omega f$. The same is true in three variables except now $v = \frac{e^{-\frac{x^2+y^2+z^2}{4kt}}}{2^3\sqrt{\pi^3k^3t^3}}\omega^3 f$.

9.2.1 The Highest Temperatures in an Infinite Solid

From $v = f\int_{-g}^{+g} d\alpha\, \frac{e^{-\frac{(\alpha-x)^2}{4kt}}}{2\sqrt{\pi kt}}$ we get $\frac{dv}{dx} = \frac{f}{2\sqrt{\pi kt}}\left(e^{-\frac{(x+g)^2}{4kt}} - e^{-\frac{(x-g)^2}{4kt}}\right)$

$$\frac{dv}{dt} = \frac{d^2v}{dx^2} = \frac{f}{2\sqrt{\pi kt}}\left(-\frac{2(x+g)}{4kt}e^{-\frac{(x+g)^2}{4kt}} + \frac{2(x-g)}{4kt}e^{-\frac{(x-g)^2}{4kt}}\right)$$

So the maximum temperature occurs when $\frac{dv}{dt} = 0$, i.e., $t = \frac{gx}{h\log(\frac{x+g}{x-g})}$. If g is infinitely small, then we obtain $t = \frac{x^2}{2k}$.

If one repeats this for $v = \frac{bf}{\sqrt{\pi}} \frac{e^{-\frac{x^2}{4kt}}}{2\sqrt{kt}} e^{-ht}$ we would get $kt = \frac{1}{\frac{1}{x^2}+\sqrt{\frac{1}{x^4}+\frac{4h}{kx^2}}}$, which is approximately $t = \frac{1}{2}\sqrt{\frac{1}{hk}}x$ for large x. Thus the heat moves outward in a wave-like manner increasingly with a constant speed. Similarly in an infinite solid, we obtain in like fashion $t = \frac{r^2}{6k}$ for large r.

9.3 Comparison of the Integrals

We have derived three formulas for the solution of $\frac{dv}{dt} = \frac{d^2v}{dx^2}$, initially with value $F(\alpha)$, namely

$$v = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\alpha\, F(\alpha) \int_{-\infty}^{+\infty} dq\, e^{-q^2 t} \cos(qx - q\alpha)$$

$$v = \int_{-\infty}^{+\infty} \frac{d\alpha\, F(\alpha)}{2\sqrt{\pi}\sqrt{t}} e^{-(\frac{\alpha-x}{2\sqrt{t}})^2}$$

$$v = \frac{1}{\sqrt{\pi}} \int d\beta\, e^{-\beta^2} F(x + 2\beta\sqrt{t})$$

The relation between the three is that $\int_{-\infty}^{+\infty} dq\, e^{-q^2 t} \cos(qx - q\alpha) = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-(\frac{\alpha-x}{2\sqrt{t}})^2}$ and $\beta = \frac{\alpha-x}{2\sqrt{t}}$.

It is easy to develop these solutions in series form (as per M. Poisson). Denoting $\phi' = \frac{d\phi}{dx}$, $\phi'' = \frac{d^2\phi}{dx^2}$, &c, and $\phi_{\prime} = \frac{d\phi}{dt}$, &c, we get from $\frac{dv}{dt} = \frac{d^2v}{dx^2}$,

$$v = c + \int dt\, v'' = c + \int dt(c'' + \int dt\, v^{\mathrm{iv}}) = c + tc'' + \frac{t^2}{\underline{|2}} c^{\mathrm{iv}} + \&\mathrm{c}$$

where c is an arbitrary function of x; and

$$\begin{aligned} v &= a + bx + \int dx \int dx\, v_{\prime} = a + bx + \int dx \int dx (a_{\prime} + b_{\prime} x + \int dx \int dx\, v_{\prime\prime}) \\ &= a + \frac{x^2}{\underline{|2}} a_{\prime} + \frac{x^4}{\underline{|4}} a_{\prime\prime} + \&\mathrm{c} + xb + \frac{x^3}{\underline{|3}} b_{\prime} + \frac{x^5}{\underline{|5}} b_{\prime\prime} + \&\mathrm{c} \end{aligned}$$

where a and b are arbitrary functions of t. The first series is nothing else but $v = e^{tD^2}c(x)$ and the first part of the second is $v = \cos(x\sqrt{-D})a(t)$.

In like manner, the solutions of the following equations are

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0, \qquad v = \cos(yD)\phi(x) + \sin(yD)\psi(x),$$

$$\frac{d^2v}{dt^2} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}, \qquad v = \cos(t\sqrt{-D})\phi(x,y) + \int dt\, \cos(t\sqrt{-D})\psi(x,y)$$

where $D = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$. In fact this furnishes a general method for solving partial differential equations. For example, to solve $\frac{dv}{dt} = a\frac{d^2v}{dx^2} + b\frac{d^4v}{dx^4} + \&\mathrm{c}$, let D be

$a\frac{d^2}{dx^2} + b\frac{d^4}{dx^4} + \&c$, then $v = e^{tD}\phi(x)$ is a solution, for $\frac{dv}{dt} = De^{tD}\phi = Dv$ as required.

The general argument used in this book can also be used for such equations. Trying a solution of the type $v = e^{-mt}\cos px$ we find $m = ap^2 + bp^4 + \&c$. So the infinite sum of such solutions is also a solution

$$v = \int d\alpha\, \phi(\alpha) e^{-t(ap^2+bp^4+\&c)} \cos(px - p\alpha).$$

At $t = 0$ this gives $f(x) = \int d\alpha\, \phi(\alpha)\cos(px - p\alpha)$ so that obtaining $\phi(\alpha)$ from the initial function $f(x)$ gives the solution v. But recall that $f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\alpha\, f(\alpha)\int_{-\infty}^{+\infty} dp\, \cos(px - \alpha x)$ so $\phi(\alpha) = \frac{1}{2\pi}f(\alpha)$.

As another example, take $\frac{d^2v}{dt^2} + \frac{d^4v}{dx^4} = 0$ which is the equation of an elastic string. A simple solution is $\cos q^2 t \cos qx$, so a more general solution is

$$u = \int d\alpha\, F(\alpha) \int dq\, \cos q^2 t \cos(qx - q\alpha) = \frac{1}{2\pi}\int d\alpha\, \phi(\alpha)\frac{\sqrt{\pi}}{\sqrt{t}}\sin\left(\frac{\pi}{4} + \frac{(x-\alpha)^2}{4t}\right)$$

At $t = 0$ we get $\phi(x) = 2\pi F(\alpha)$. But there is another solution, namely

$$w = \frac{1}{2\pi}\int d\alpha\, \psi(\alpha) \int dq \frac{1}{q^2}\sin q^2 t \cos(qx - q\alpha) = \frac{1}{2\pi}\int d\alpha\, \psi(\alpha)\frac{\sqrt{\pi}}{\sqrt{t}}\sin(\frac{\pi}{4} - \frac{(x-\alpha)^2}{4t})$$

which gives, at $t = 0$, $\frac{dw}{dt} = \psi(\alpha)$. So the complete integral of the equation is the sum $v = u + w$.

For the equation $\frac{d^2v}{dt^2} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}$ we first obtain the elementary solution $\cos mt \cos px \cos qy$ where $m^2 = p^2 + q^2$ so then

$$\begin{aligned} v = {} & \frac{1}{(2\pi)^2}\int d\alpha\, d\beta\, \phi(\alpha,\beta) \int dp\, \cos(px - p\alpha) \int dq\, \cos(qy - q\beta) \cos t\sqrt{p^2+q^2} \\ & + \frac{1}{(2\pi)^2}\int d\alpha \int d\beta\, \psi(\alpha,\beta)\int dp\, \cos(px - p\alpha)\int dq \frac{\cos(qy - q\beta)\sin t\sqrt{p^2+q^2}}{\sqrt{p^2+q^2}} \end{aligned}$$

For $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0$, start with $\cos px \cos qy\, e^{mx}$ with $m^2 = p^2 + q^2$.

1827: MEMOIR
on the temperature of the Earth
and planetary spaces[6]

One of the most important modern questions is the determination of the internal temperature of the Earth. There are three sources of heat for the Earth:

1. The Sun, whose unequal heat radiation produces the diversity of climates;
2. The Stars in the universe;
3. The Earth's original heat when the planets formed.

Of these, the numerous stars produce a constant temperature that is lower than the polar regions. It is the other two that are most important for heating the Earth, and they have different effects: the Sun heats mostly the outer surface, while the Earth's heat is felt mostly in the interior. Observations show that as one proceeds to greater depths the temperature increases at 1 degree centigrade every 32 meters; the temperature at the center must be very high, and cannot be due to the Sun's radiation. This rate of heat flow will diminish with time and must have been greater in the past, but the surface temperature will remain mostly the same because of the Sun's heat.

Below about 30 to 60metres of the surface (or inwards in the case of mountains), the temperature is practically constant, although decreasing as one moves polewards. As one approaches the surface, we start seeing the familiar seasonal and daily variations (from 3metres). The air and water generally tend to make the temperature distribution more homogeneous by their movements. The Sun's light penetrates the upper rarified layers of the atmosphere, but is absorbed to greater extents by the lower denser regions and the ground until it is converted to 'heat radiation' when it loses its penetrating power. This explains why 'transparent' media (such as the sea) cause warming: light manages to pass through them but heat rays are trapped in. Also this is why it is very cold at high elevations: as the warm air close to the land rises and expands it cools. The trade winds are caused by differences in temperature not tidal forces, but both affect ocean currents.

The solid earth by contrast *conducts* heat, which is a much slower process. Earth's surface is sandwiched between a hot interior and a freezing sky. The sky's temperature is that of the coldest places on Earth, which is the same as the surface of the farthest planets.

We can treat the effect of each of these sources separately:

9.3.1 Solar Radiation

The Earth can be divided, for our purposes, into the surface region, going down to about 30m, and the interior. On this region, the temperature is constant.

[6] from http://www.wmconnolley.org.uk/sci/fourier_1827/fourier_1827.html#text

This invariance is due to a balance of heat inflow and outflow. In the surface region, the incoming solar radiation of the Sun would raise Earth's temperature indefinitely unless an equal amount escapes. The different parts of the globe have reached a balance, resulting in the different climates, effected by the elevation, proximity to the sea, state of the surface, and the wind direction. The average over a whole year is in fact this constant temperature. A temperature variation at the surface will only be felt to a depth proportional to the square root of that variational period. For half the year, heat flows inwards, but reverses for the rest of the year, with a time lag of about an eight of a year after the seasons.

9.3.2 Stellar Radiation

That deep space does have a temperature can be most easily seen with this argument: if one forgets for a moment the atmosphere, then the poles, receiving little or no solar heat, would cool down without limit, affecting the latitudes nearer the equator. If the Sun were the only heat source, any change in its distance (due to the orbital eccentricity) would be detectable, and the commencement of night would expose a place to extreme cold. It is the heat of deep space that acts as a moderating factor.[7] It is difficult to determine how much the movement of the sea and air affects the global temperature, but they cannot possibly compensate such enormous differences.

The cause of this radiation is clearly the innumerable multitude of stars and the inter-stellar rarified atmosphere. The whole solar system is equally bathed in a uniform sky temperature. Each planet's surface temperature would be equal to this with the addition of the solar radiation, depending on its distance, its inclination of the axis of rotation, and the nature of its surface and atmosphere. It is very probable, however, that the temperature of the planet's poles, and of the farthest planets such as Herschel's[Uranus], is only slightly higher than the sky temperature. It is therefore about -40 degrees[8].

An important experiment was done by M. Saussure who enclosed a vase, containing cork, inside a number of glass flasks, one inside another. At midday, the vase temperature rises to more than 110 degrees, while the temperature of the intermediate flasks is less. This is the greenhouse effect mentioned earlier. This effect is present on Earth, and would be more pronounced if the air and sea were motionless.

Extra: Questions arise immediately: how are the depths at which there is no variation in temperature dependent on the period of variation? Why is this depth so small compared to Earth's radius? How long does it take for the temperature to stabilize at the surface? The analysis of heat depends in general on three phenomena — the internal conduction of heat, the properties of the surface in losing heat, and the initial distribution of heat.

[7] Perhaps he is considering the atmosphere as part of the sky; it was probably not clear at the time whether space was entirely vacuum, and by how much the weather mixed the air.

[8] All references to degrees are in the Reaumur scale

9.3.3 Interior Heat Conduction

What I propose to understand is how a hot solid sphere cools when immersed in a constant temperature bath, and why below the depth of about 30m there is still an increase in temperature of about 1 degree every 30m or 40m. This increase cannot of course be due to sunlight, for otherwise the temperature would *decrease* with depth, not increase. There must therefore be an inner source of heat, due to elevated temperatures at the center. This is in accordance with the theories of the formation of the planet.

The mathematical equations that I deduced to tackle this problem indicate that the rate of heat outflow must have been much higher in the past, but now changes imperceptibly (it may take roughly thirty thousand years for it to reduce by half). It will certainly not affect the climate; it is more likely that human society may change the climate by changing the surface and air flow. They also show that the temperature increase is less at greater depths, nevertheless the increase, accumulating over thousands of kilometers, results in a temperature much higher than that of incandescent materials.

Although precise values of conduction of the materials in the interior are obviously unavailable, we can still say something in a general sense. The total amount of heat outflow needed to reproduce the observed temperature differences is not enormous: it is proportional to the square root of the heat capacity divided by the volume and permeability [i.e., $\sqrt{C/V\sigma}$]; even if the conductivity was as high as that of iron, this heat would not be higher than that needed to melt a cylinder of ice with diameter 1m and height 3.1m. Conduction is a very slow process: were the interior of the Earth below twenty leagues be replaced by a material at five hundred times the boiling point of water, we at the surface would observe a rise of one degree after two hundred thousand years!

One might expect this heat outflow to affect the surface temperature. In actual fact, the excess in the surface temperature is related to the rate of temperature decrease at the surface; using the values for iron, one would find that the excess would be an imperceptible quarter of a degree! Using more realistic values, one would probably find the excess to be less than a thirtieth of a degree. Also this excess cannot have changed since the time of the Greeks by more than 1/300th of a degree.

One may surmise that the planets, including Earth, formed from hot parts of the Sun.