Introduction to Arithmetic
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Book 1

1

The ancients defined philosophy as the love of wisdom, which means any knowledge or skill, even of a craft. Pythagoras was the first to restrict the meaning of wisdom to knowledge of truth or reality, and philosophy as the pursuit of wisdom. He further defined a thing to be 'real' when it continues to exist uniformly without change. Of course, a material object flows and changes, but its qualities and quantities, its size, and so on, do not.

2

Real qualities are immaterial, eternal, and unchanging. All change, such as birth, destruction, growth, etc., by definition, is not real in this sense; they come and pass, but never really are. Let us systematize these qualities of things. There are two forms. Firstly, some things are unified and continuous, such as extended objects, called "magnitudes". Secondly are discontinuous heaps, called "multitudes". Both are infinite, the former can be divided indefinitely, the latter never ceases to increase. In practice, one can only consider numbers/quantities (finite multitude) and sizes (finite magnitude).

3

Any quality can either be absolute, of itself, such as 'even', 'odd', etc., or relative to something else, such as 'smaller than', 'double of', etc. Thus, of quantity, there are two types: *arithmetic* is the study of absolute quantity, *harmony* is the study of relative quantity. Similarly, the study of size splits in two: *geometry* treats statics, *astronomy* treats that which moves.

Any attempt at philosophy must start with these mathematical subjects. They are the bridges that take us from the senses and opinions to the mind and understanding, from the concrete and familiar objects to immaterial and eternal abstractions, from matter to soul.

4–5

Of these four subjects, arithmetic comes first, for the others logically presuppose it. Just as the concept ‘animal’ is prior to ‘man’ and ‘man’ prior to ‘teacher’, so arithmetic is prior to geometry, for one cannot describe triangles, quadrilaterals, etc. without recourse to number, but numbers exist without shapes. Similarly harmony presupposes arithmetic, not only because the absolute precedes the relative, but also because musical ratios, such as 3:2, use numbers[1]. Finally, astronomy depends on geometry (rest precedes motion), on harmony (the music of the stars), and on arithmetic (risings, settings, etc. are numerical).

6–7

The universe seems to have been created using numbers. It is fitting, therefore, that numbers are harmoniously fitted together. All things are either equal or have some relation to each other. The most fundamental division of numbers is into odd and even, harmoniously interwoven together. They form the very essence of quantity, different from each other, yet both the same numbers.

An even number is that which can be divided into two equal parts; an odd number is that which cannot be so divided without leaving an extra unit. A Pythagorean would point out that two is the smallest number of possible parts.

Moreover, an even number can be divided into two equal and two unequal parts, except for 2, but an odd number admits only a division into unequal parts. In fact, a division of an even number always yields two even or two odd parts, while an odd number always gives even and odd parts. Note also that an odd is always flanked by two evens, and vice versa.

8

Every number is the average of its two neighbors, and of their neighbors, and so on as far as possible. Except for 1, which has only one neighbor and is half of it.

The evens can be classified into the *even-times even*, the *odd-times even* and the *even-times odd*.

The even-times even is that which itself and its parts can be divided equally as far as possible. For example, 64 is made of two 8s, which divide each into two 4s, these into two 2s each, and 2 into two 1s. They are pure in the sense that the parts of an even-times even are themselves even-times even. There is a way to generate all of them: starting from 1, double each term to get the next, thus

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots$$

Note that the product of two extreme terms remains the same down to the middle terms, e.g. 128 is equal to two 64s (i.e., 2×64) and sixty-four 2s, four

[1] The numbers and notation used in the original are Greek — the Indian numerals, especially 0, and the symbols $+, \times, :$ were not yet in use.

32s and thirty-two 4s, eight 16s and sixteen 8s. The odd term, such as 64, however, has a middle term, in this case eight 8s.

When one adds the terms one gets one less than the next in the series. For example, adding all the terms up to 64 gives 127.

9

The even-times odd is defined as an even number whose equal parts cannot be further divided, i.e., are odd. In fact, the number of parts are opposite in type to the parts themselves, odd and even. Thus 18 is 9×2 and 3×6. To generate these numbers, start with the odd numbers and multiply each by two,

$$6, 10, 14, 18, 22, 26, 30, \ldots$$

Thus each term is larger than the previous one by 4; the sum of two extreme terms remains the same down to the sum of their mean terms, e.g. $6 + 22 = 10 + 18 = 14 + 14$.

10

The odd-times even is an even number whose equal parts can also be divided equally, but not all the way to 1 as in the even-times even, e.g. 24. It is intermediate between the other two types of evens. They can be split as the product of an even times even with an even times odd.

It is more complicated to generate these numbers: start with the odd series and the even-times even series, and multiply their numbers with each other, one from each series.

Odd numbers	3	5	7	9	11	13	15
Even-times evens	4	8	16	32	64	128	256
3×	12	24	48	96	192	384	768
5×	20	40	80	160	320	640	1280
7×	28	56	112	224	448	896	1792

Along each column, the property of the odd-times even prevails, namely the sum of the extremes is the sum of the means; while in each row, the property of the even-times even prevails, namely the product of the extremes is the product of the means.

11–13

The odd numbers are also classified into three types. The *primes* are those odd numbers which have no factors except themselves, e.g. 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. They are elementary in that they cannot be resolved by any number except 1.

A *secondary* number is an odd number that is not elementary but composite, i.e., it can be resolved into products, apart from 1s, e.g. 15 is 3×5.

The third type of odd is a composite number that has no common measure with respect to a given number. For example, relative to 25, there are 9, 21, 27, etc..

Eratosthenes' sieve is a way of separating out the three types of odds. Start with the odd numbers, and mark out the multiples of the first odd number 3 (skipping by a term each time); then take the second number, 5, and mark its multiples; continue like this. The marks distinguish the secondaries from the primes. A third number is that whose marks do not share with the given one, i.e., have no common measure.

Suppose we're given two odd numbers and we wish to determine if they have a common measure. Subtract the smaller from the larger as many times as possible; subtract the result from the first one, as many times as possible. When the subtractions result in unity, they are relatively prime, but otherwise they result in their common measure.

For example, given 23 and 45, $45-23=22$, $23-22=1$, so they are relatively prime. Given 21 and 49, $49-21-21=7$, $21-7-7=7$, so 7 is their common measure.

14–16

In a different direction, the even numbers split into three types. The *super-abundant* numbers are those whose parts are more than the whole. This is as excessive as an animal with ten tongues or three rows of teeth! One such is 12, for its parts are 1, 2, 3, 4, and 6, which sum up to 16.

A *deficient* number has less parts than the whole, like a man with one eye. Thus, 8 is deficient, for its parts 1, 2, and 4, sum to 7.

A *perfect* number is one whose parts are equal to the whole, neither more nor less. Only a few numbers are perfect; only $6=3+2+1$ is perfect from the units, only $28=14+7+4+2+1$ from the tens, only 496 is perfect from the hundreds, and only 8128 from the thousands. They are always even and end in 6 or 8.

To generate them, start with even-times even numbers from 1, and add the terms together; when the result is a prime, multiply by the current term to get a perfect number. Hence, $1+2=3$ is prime, so $3\times 2=6$ is perfect; $1+2+4=7$ is prime, so $7\times 4=28$ is perfect. But $1+2+4+8=15$ is not prime. The next one is prime $15+16=31$, so $31\times 16=496$ is perfect.

The number 1 is potentially perfect, for it is the first to be generated in this way (being prime). But it is not actually perfect, since it has no parts.

17–18

We now turn to relations between quantities. The highest division is between equality and inequality. Equality, as in 10 to 10 and 2 to 2, is a unique and elementary relation, for anything equal to an equal is equal, like 'friends'.

The unequal, however, is necessarily between a lesser and a greater, as in 'father' and 'son'. There are five possible such comparisons between unequals.

(Each comparison between greater and lesser has an opposite comparison between lesser and greater.)

The *multiple* is when the greater contains the lesser a whole number of times. For example every number from 2 onwards is a multiple of 1; the multiples of 2 are the even numbers; those of 3 are the triples (which are alternately odd and even).

The *super-particular* relation is when the greater contains the lesser and a part of it. In particular, the sesqui-alter contains the whole and a half of the lesser; e.g. 3:2, 6:4, ... The sesqui-tertian contains a whole and a third, etc., being 4:3, 8:6, ...

In the following table of multiples,

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
...									
10	20	30	40	50	60	70	80	90	100

notice how the second row (or column) is the double of the first row (or column), the third row is the triple of it, the fourth row the quadruple of it, etc. The third row is the sesqui-alter of the second row; in general, any row is a super-particular of the row above it. Clearly and objectively, the multiples precede as a notion the super-particulars as they involve the first row.

Moreover, going along the first row or column and then down the last column or row, we progress from unity to ten to hundred; the product of two opposite corners are equal. The diagonal consists of squares. On either side of it are the heteromecic numbers 2, 6, 12, ...; adding two successive squares and both heteromecic numbers between them gives another square; adding two successive heteromecic numbers and twice the square between them also gives a square. The interested person can find many other such pleasing rules.

20–21

A number is *super-partient* when it contains a lesser number and multiple parts of it (the parts must necessarily be thirds or higher order). For example, a super-bi-partient contains a whole number and two parts of it.

The roots of this relation can be obtained by comparing the odds, starting with 5, with all the numbers, starting with 3,

$$5:3,\quad 7:4,\quad 9:5,\quad 11:6,\quad 13:7$$

Each root form gives rise to others by doubling, tripling, etc., both terms; thus 5:3 gives 10:6, 15:9, ...

22

The *multiple super-particular* is when the greater contains the lesser a multiple of times as well as a part of it. Thus there are a great variety of such relations:

a double sesqui-tertian contains two wholes and a third of the lesser, such as 7:3, a quadruple sesqui-alter contains four and a half of the lesser, such as 9:2.

The double super-particulars can be obtained by comparing the odd numbers from 5 with all the numbers from 2 onwards,

$$5 : 2, \quad 7 : 3, \quad 9 : 4, \quad 11 : 5$$

In general one can take any row of multiples in the table above and compare with a previous row of multiples.

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The *multiple super-partient* is a number that contains a lesser number a multiple times as well as several parts (as before, at least thirds or higher). For example, 8 is the double super-bi-partient of 3. This concludes the ten arithmetic relations.

There is a method that demonstrates that equality is naturally prior to inequality, so that the beautiful and limited precedes the ugly and infinite, and the rational soul is the agent which puts in order the irrational appetites. Given any number, copied three times, create a new triple according to the following rule: the first equals the first, the second equals the sum of the first and second, the third is the first added to twice the second and to the third; the result are doubles. Now repeat the same rule to get triples, and furthermore all the multiples in order. Now reverse the order of one of these triples and re-apply the rule; the result is a triple of super-particulars. Reverse this again and apply the rule to get super-partients. If the rule is applied to a triple of super-particulars directly, the result is multiple super-particulars, and from the super-partients come the multiple super-partients. As this rule is not applied by chance, it follows that the natural order of the relations is: equality, multiples, super-particulars, super-partients, multiple super-particulars, and multiple super-partients.

For example, from 1:1:1, one obtains 1:2:4, then reversing, 4:6:9, reversing again, 9:15:25, and finally 4:14:25. Note that the first and third terms are always squares.

Book 2

1–2

An *element* is the smallest thing which constitutes an object, and which is indecomposable. For example, letters are the elements of speech, notes are the elements of music, the so-called four elements are the elements of the universe.

The elements of numbers are 1 and 2. The elementary principle of the relative relations is equality, as will be shown next. Let three terms be given in any increasing ratio, whether multiple, super-particular, etc., provided it is the same ratio of the first to the second, and the second to the third. Subtract the first from the second; add the first term to twice the new second term, and subtract the answer from the third. Then the new triple will be in a more primitive ratio. Re-applying this rule will simplify the ratio until an equality is reached.

3–5

The multiples can generate the super-particulars. For example, the doubles produce the sesqui-alters, e.g. from 1, 2, 4, 8, …, we obtain first 3 (from 2), then 6 (from 4) and 9 (from 6), etc.

1	2	4	8	16	32
	3	6	12	24	48
		9	18	36	72
			27	54	108
				81	162
					243

Note how the diagonal double ratios consists of the triple ratios. A similar table can be drawn starting from the triples 1, 3, 9, …, to obtain the sesqui-tertian ratios. (The diagonal is then the quadruple ratio series 1, 4, 16, 64, …)

Every double ratio is the combination of a sesqui-alter and a sesqui-tertian; e.g. 2:4 is 2:3 combined with 3:4. A triple ratio consists of a double and a sesqui-alter, such as 6:12:18 or 6:9:18. In general any multiple is the combination of a lesser multiple and a super-particular.

6

The digits that we use to indicate numbers, such as 5 for five, are used by convention, not by nature. In fact, the simplest notation would be to use • for 1, •• for 2, ••• for 3, etc. Unity is thus the beginning of an interval, as a point is the start of a line. Just as a point added to a point gives nothing new, and nothing added to nothing makes nothing, so unity multiplied with unity produces unity. Unity is therefore an element of the numbers, without dimension.

Dimensions 1, 2, or 3, are called, in geometry, a line, a surface, and a solid for the last one has width, depth, and height. Before them, the point is non-dimensional.

7

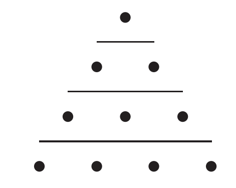
The point has no dimension but is the beginning of dimension, i.e., it is not a line but the start of one. Similarly, a line is the beginning of a surface, but not itself a surface. A surface is the start of a body, but not itself solid.

With numbers, equally, unity is the beginning of numbers, linear number is the beginning of a planar number, which is the start of a solid number. The linear numbers are 2, 3, 4, 5, etc.

The most elementary form of planar number is the triangle. Indeed, every rectilinear figure can be decomposed into triangles, but the triangle itself cannot be resolved any further.

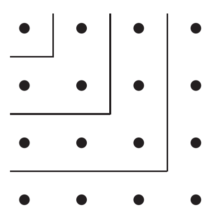
8

A *triangular* number is one that can be laid out as an equilateral triangle of units, e.g. 3, 6, 10, 15,... In this series, 1 is potentially a triangle but not actually. The sides of the triangle increase as with the numbers, so that the difference between triangle numbers increase as the successive numbers. Conversely, starting with the numbers 1, 2, 3, 4, 5,... and adding them up gives the triangle numbers.

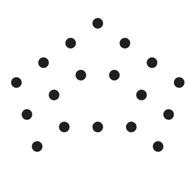


9–11

A *square* number is one that can be laid as a square, e.g. 1, 4, 9, 16, ... corresponding to squares with sides 1, 2, 3, 4, ... respectively. They can be obtained by adding the odd numbers, as 1, $1+3$, $1+3+5$, $1+3+5+7$, etc.



The *pentagonal* numbers are those that form regular pentagons, i.e., 1, 5, 12, 22, ..., corresponding to sides 1, 2, 3, 4, ... The differences between successive terms start with 4 and increase by 3 each time.



The hexagonal, heptagonal, and succeeding numbers are those that form hexagons, heptagons, etc. Just as the previous series are constructed by adding the numbers, then the odds (i.e., skipping one), then the numbers skipping 2, these higher type numbers can be produced by adding the numbers that are formed by skipping 3, 4, etc. (These 'skipping' numbers are just three less than the number of sides of the polygon.)

	skip by	add							
Triangular		1	2	3	4	1	3	6	10
Square	1	1	3	5	7	1	4	9	16
Pentagonal	2	1	4	7	10	1	5	12	22
Hexagonal	3	1	5	9	13	1	6	15	28
Heptagonal	4	1	6	11	16	1	7	18	34

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Just as every square can be divided into two triangles along the diagonal, so every square number is the sum of consecutive triangular numbers, and conversely. Similarly, adding a triangle to a square makes it a pentagon, so adding the row of triangular numbers above to the row of squares, but displaced by one place, gives the row of pentagonal numbers, e.g. $1 + 4 = 5$, $3 + 9 = 12$.

By the same way, adding the triangle numbers to the displaced pentagonals gives the hexagonals, and in general, converts one polygon type to a higher one. This confirms further that the triangle numbers are the elements of the other polygons.

13–14

A *solid* number is that whose units can form a solid shape. The simplest are the *pyramids* which start with a polygonal base and taper to an apex. There is one pyramidal series for each polygon type. The triangular pyramids are built by adding each successive triangle number, 1, $4 = 1 + 3$, $10 = 1 + 3 + 6$, $20 = 1 + 3 + 6 + 10, \ldots$

The square pyramids are formed in the same way by adding squares to the base, 1, $5 = 1 + 4$, $14 = 1 + 4 + 9$, $30 = 1 + 4 + 9 + 16, \ldots$

One can also speak of 'truncated' pyramids, when the apex is cut off at some height from the base.

15–17

The *cubes* are, as their name implies, numbers formed by multiplying an equal length, width and height, thus 1, 8, 27, etc. Each square is bounded by 6 squares, 12 edges, and 8 solid angles.

Opposed to the cubes are the scalene *wedges*, consisting of differing lengths, widths and heights, e.g. $3 \times 5 \times 12$.

A *parallelipipedon* is a number with heteromecic sides. Recall that a heteromecic number is, in geometric terms, a rectangle whose length is one more than the width, i.e., $2 = 1 \times 2$, $6 = 2 \times 3$, $12 = 3 \times 4$, $20 = 4 \times 5$, ... Now, the Pythagoreans assign the meaning of sameness to 1, and of otherness to 2, or more generally to odd and even. Accordingly the squares also have the nature of sameness because they are the sum of odd numbers; while the heteromecic numbers are the sum of the evens, hence belong to the 'other' class.

To build on this, note that multiplying odd numbers together gives another odd number, just as sameness results from equality. But an even number multiplied with other numbers changes them all to even.

A *brick* is the case where the length and width are the same but the height is smaller than them. A *beam* is similar but the height is larger. A *spherical* number is a cube which, after the three multiplications, ends in the same number where it began, e.g. $1 = 1 \times 1 \times 1$, $125 = 5 \times 5 \times 5$, $216 = 6 \times 6 \times 6$. In fact, any number ending in 1, 5, or 6 gives rise to spherical numbers. (One can also talk of 'circular' numbers as squares ending in 1, 5, or 6.)

18–20

We wish to elaborate further on how the principles of same and other, equality and inequality, odd and even, square and heteromecic, give rise to all the peculiar properties of numbers. Plato adds to these the indivisible and the divisible, and Philolaus adds the bounded and boundless. Objects are made of these opposing principles, resulting in *harmony*, the reconciliation of the diverse..

Take two rows, one of the squares, the other of the heteromecic:

1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
2	6	12	20	30	42	56	72	90	110	132	156	182	210	240

Their ratios, 2:1, 6:4, 12:9, ..., are the double, the sesqui-alter, etc. Their differences are the successive numbers 1, 2, 3, ... If the squares are displaced by one place, then their ratios are again 4:2, 9:6, ..., the same as before; but their differences are now 2, 3, 4, ... Thus the same differences but different ratios, as in 2:4 and 4:6, or equal ratios and unequal differences, as in 1:2 and 2:4; same in quantity and opposing in quality, or vice versa.

If we now compare two successive square numbers with a heteromecic as their mean, such as 9:12:16, then the product of the extremes equals the square of the mean; and the extremes plus twice the mean gives another square. Best of all, adding a square number with the next heteromecic, and a heteromecic with the next square, give all the triangle numbers.

Furthermore, adding or subtracting a side from its square makes it heteromecic. Thus the 'other' is both larger and smaller than the 'same'. That the odd is sameness is confirmed by starting with any series of multiples, such as triples, 1, 3, 9, 27, 81, ..., and noting that every odd position is a square, but not the even ones. Finally, the cubes, 1, 8, 27, ..., are obtained by starting with the odd numbers and adding first one, then the next two, the next three etc.

21

A *proportion* is the combination of two or more ratios, or more generally, of relations (such as differences). For example, 1:2:4 is a continued proportion of doubling, while 1:2:3 is a continued proportion of quantity. With more terms than three, one can compare totally different terms, and then one speaks of 'disjunct' proportions, as in 1:2:4:8.

22–23

The three most ancient proportions are the *arithmetic*, *geometric*, and *harmonic*. There are ten proportions in all, if one includes their opposite proportions and four others discovered by the moderns.

An arithmetic proportion is that which preserves the differences (but not ratios) between terms. It is the simplest one in that it is the proportion of the natural numbers 1, 2, 3, 4, ... If one takes equally separated terms in this series, one still obtains an arithmetic proportion. The mean term of an odd numbered arithmetic proportion is equal to half the sum of the extreme terms; the two mean terms of an even numbered proportion add up to this sum. Moreover, the ratio of corresponding terms of two arithmetic proportions is the same as the ratio of their differences. Also, and this is original, the product of the extremes plus the square of the common difference equals the square of the mean. Fourthly, the ratio of consecutive terms decreases as more terms are taken.

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A geometric proportion is one in which the ratio (but not the difference) between consecutive terms remains the same. For example, 1, 2, 4, 8, 16, ... A peculiar property is that the differences between the terms themselves form a geometric proportion with the same ratio. For the double ratio, the differences form the same proportion; for the triple ratio, they form twice the same proportion; for the quadruple ratio, thrice; etc.

A geometric proportion may also be super-particular, or super-partient, or mixed. In all these cases, the product of the extremes is the square of the mean; while for even numbered proportions, the product equals the product of the two means. If the square numbers are interlaced with the heteromecic, then the ratio between terms start with the double, then sesqui-alter, the sesqui-tertian, and so on.

Any two consecutive squares have precisely one mean that makes the three a geometric proportion, e.g. 4:6:9; it is in fact the product of the sides of the two squares. Any two consecutive cubes have precisely two means that together form a geometric proportion, e.g. , 8:12:18:27; they are found by mixing the sides of the two cubes, $2 \times 2 \times 3$ and $2 \times 3 \times 3$.

In general, the product of two squares is another square, but that of a square with a heteromecic is not a square. Similarly, the product of two cubes is another cube. This is just as the product of two evens is even, of two odds is odd, but of an even and an odd is again even.

25

The harmonic proportion occurs when the ratio of the largest term to the smallest term is equal to the ratio of the difference of the largest from the mean to the difference of the mean from the smallest. For example, 3, 4, 6 is harmonic since 3:6 equals $(4-3):(6-4)$.

One property is that the ratio of consecutive terms increases; note how the arithmetic, the geometric, and the harmonic form a natural series in this regards because the ratios decrease, remain the same, or increase respectively. If the extremes are added and multiplied by the mean, the result is twice the product of the extremes.

26

The musical ratios 6:4:3:2 are harmonic, hence the name. The most elementary is the *diatessaron* which is the sesqui-tertian ratio 4:3; then comes the *diapente* which is the sesqui-alter ratio 3:2; then the *diapason*, which is the multiple 6:3, and the diapason and diapente combined, i.e., 6:2; finally the *di-diapason* 4:1.

Philolaus, however, maintains it is called harmonic because of geometry: a cube has the harmonic proportions 12 sides, 8 angles, and 6 faces.

27

Just as in a taut musical string, the bridge can be inserted in between to produce the arithmetic, the geometric, or the harmonic mean; and similarly, the fingerholes in a musical pipe; in an analogous way, between two odds or two evens, one can insert these three means. For example, between 10 and 40 one can fit 25 for the arithmetic mean, 20 for the geometric, and 16 for the harmonic mean. Or, given 5 and 45, one fits 25, 15, and 9 for the three means.

To find the arithmetic mean, add the extremes and divide by two; for the geometric mean, you need to take the square root of the product of the extremes, or observing the ratio of the extremes, find its square root. For the harmonic mean, multiply the difference of the extremes by the lesser term and divide the product by the sum of the extremes, then add the result to the lesser term.

28

The other means will be dealt with more briefly. The *subcontrary* proportion occurs when the ratio of the largest to the smallest is the same as the ratio of the difference between the two smaller terms and the two larger terms (but opposite to the harmonic), e.g. 3:5:6. It has the property that the product of the greater with the mean is twice the product of the mean with the smaller term, thus $5 \times 6 = 2 \times 5 \times 3$.

The fifth proportion occurs when the ratio of the mean to the lesser is the same as the ratio of the two differences, in reverse fashion to the geometric mean, e.g. 2:4:5. The product of the greatest with the mean is twice that of the extremes, e.g. $5 \times 4 = 2 \times 5 \times 2$.

The sixth proportion occurs when the ratio of the greatest to the mean is the same as the ratio of the two differences, again in reverse order of the geometric mean, e.g. 1:4:6.

There are yet four other minor proportions: the seventh one occurs when the ratio of extremes is the same as the ratio of their difference to the difference of the lesser terms, e.g. 6:8:9. The eighth proportion occurs when this same ratio equals that of the difference of the extremes to the difference of the greater terms, e.g. 6:7:9. The ninth proportion occurs when the ratio of the mean to the least is the same as that of the difference of the extremes to the difference of the smallest terms, e.g. 4:6:7. Finally, the tenth proportion occurs when this same ratio is the same as that of the difference of the extremes to the difference of the greatest terms, e.g. 3:5:8.

29

There is one proportion, most perfect, which embraces all the ten proportions. It is a three-dimensional proportion comprising four terms and two means. It occurs when the extreme terms are products of three numbers, and the two means are the harmonic and arithmetic means of the extremes. The geometric mean is also to be found here, because the ratio of the greatest to the second term is the same as the ratio of the third term to the least.

For example, given $6 = 1 \times 2 \times 3$ and $12 = 2 \times 2 \times 3$, their means are $8 = 1 \times 2 \times 4$ and $9 = 1 \times 3 \times 3$. Moreover, 8:6 and 12:9 are the diatessaron, 9:6 and 12:8 is the diapente, the ratio 12:6 is the diapason, while 9:8 is the interval of a tone.

Reference:
`https://ia700709.us.archive.org/27/items/NicomachusIntroToArithmetic/nicomachus_introduction_arithmetic.pdf`