1 Quasi-Linear Partial Differential Equations

Definition 1.1 *An n'th order **partial differential equation** is an equation involving the first n partial derivatives of u,*

$$F(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^n u}{\partial x^n}, \ldots) = 0.$$

*A **linear first-order p.d.e.** on two variables x, y is an equation of type*

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} = c(x, y)u(x, y).$$

We will be able to solve equations of this form; in fact of a slightly more general form, so called **quasi-linear**:

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u).$$

2 Solution

Define a curve in the x, y, u space as follows

$$\mathbf{r}(t) = (x(t), y(t), z(t)) \text{ satisfying } \frac{d\mathbf{r}}{dt} = (a, b, c)$$

with initial condition $\mathbf{r}(0)$ on the surface $z = u(x, y)$.

From Picard's theorem, assuming that a, b and c are well-behaved functions of x, y and u, we know that there is a unique solution.

Then we can compare this curve with the curve $u = u(x(t), y(t))$.

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\ &= a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} \\ &= c \\ &= \frac{dz}{dt} \end{aligned}$$

and since both curves start at the same point, we must have $u = z$.

Of course, this only gives a single curve on the surface, but if we are specified a whole line of initial points of the type $u(x_s, y_s) = u_s$ then we can solve the equation $\dot{\mathbf{r}} = (a, b, c)$ with initial conditions $x(0) = x_s$, $y(0) = y_s$ and $u(0) = u_s$ to give a collection of lines all lying on the surface i.e. we get $u = u(t, s)$.

If we can now eliminate the variables t and s using the equations $x = x(t, s)$ and $y = y(t, s)$, then we get our required solution $u = u(x, y)$.

2.1 Example

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = x - y$$

given that $u(0, s) = 0$.

Solution: Define the following curves

$$\begin{aligned} \dot{x} &= x & x(0) &= 0 \\ \dot{y} &= -y & y(0) &= s \\ \dot{u} &= x - y & u(0) &= 0 \end{aligned}$$

Solving the first two equations is straightforward to give

$$x(t) = e^t \qquad y(t) = se^{-t}.$$

We can now substitute into the third equation to get

$$\dot{u} = e^t - se^{-t} \qquad u(0) = 0.$$

Integrating,

$$\begin{aligned} u(t, s) &= e^t + se^{-t} - (1 + s) \\ &= x + y - 1 - xy. \end{aligned}$$

The required solution is therefore $u(x, y) = x + y - 1 - xy$.

3 Second Method

The method described above is straightforward, but it may prove difficult in practice to change variables from s, t back to x, y. A second method makes this easier by automatically suppressing the t variable to start with.

The object is to find two independent integrable identities that, when integrated, do not involve t (but may, and usually do, involve the initial condition variable s). Once these two relations are found, it is then usually straightforward to eliminate s from the two to leave a relation involving u, x and y.

3.1 Example

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + 2xy = 0$$

with $u(s, 2s) = 2s^2$.

Solution: Set up the differential equations on $x(t)$, $y(t)$ and $u(t)$:

$$\begin{aligned} \dot{x} &= y & x(0) &= s \\ \dot{y} &= x & y(0) &= 2s \\ \dot{u} &= -2xy & u(0) &= 2s^2. \end{aligned}$$

Two integrable relations are the following:

$$x\dot{x} - y\dot{y} = 0$$

$$\dot{u} + 2x\dot{x} = 0,$$

which integrate to $x^2/2 - y^2/2 = A$ and $u + x^2 = B$, where the constants A and B are determined from the initial conditions to get

$$x^2 - y^2 = -3s^2$$

$$u + x^2 = 3s^2.$$

We can now eliminate s from the two relations to get

$$u + x^2 = -(x^2 - y^2)$$

i.e.

$$u(x, y) = y^2 - 2x^2.$$

4 General Solution

The second method is particularly useful in finding the general solution to a quasi-linear p.d.e., that is, one for which the initial conditions are not specified.

If we follow the same steps as before, we again end up with two integrated relations that have two undetermined constants $A(s)$ and $B(s)$. However s can still be eliminated from the two equations in the sense that if the relations are given as $R(x, y, u) = A(s)$ and $T(x, y, u) = B(s)$ then

$$R(x, y, u) = A(B^{-1}(T(x, y, u))) = F(T(x, y, u))$$

for a general function F.

4.1 Example

$$(3y - 2u)\frac{\partial u}{\partial x} + (u - 3x)\frac{\partial u}{\partial y} = 2x - y$$

Define the curves satisfying the equations

$$\begin{aligned} \dot{x} &= 3y - 2u \\ \dot{y} &= u - 3x \\ \dot{u} &= 2x - y. \end{aligned}$$

We can find two integrable relations,

$$\dot{x} + 2\dot{y} + 3\dot{u} = 0,$$

$$x\dot{x} + y\dot{y} + u\dot{u} = 0,$$

which integrate to

$$x + 2y + 3u = A(s),$$

$$x^2/2 + y^2/2 + u^2/2 = B(s).$$

Eliminating s, we get

$$x^2 + y^2 + u^2 = F(x + 2y + 3u)$$

which is the general solution of the differential equation.

If initial conditions are now specified, we can find what F is from them as follows. Suppose we specify that $u(s, s) = 0$. Therefore substituting $x = s$, $y = s$ and $u = 0$ in the general solution we get

$$2s^2 = F(3s).$$

Hence, if we write $3s = x + 2y + 3u$, we get the specific solution

$$x^2 + y^2 + u^2 = 2s^2 = 2\left(\frac{x + 2y + 3u}{3}\right)^2.$$

Rearranging the terms we finally get

$$9u^2 + 12(x + 2y)u - (7x^2 - 8xy + y^2) = 0$$

to give

$$u(x, y) = -\frac{2}{3}(x + 2y) \pm \frac{1}{3}\sqrt{11x^2 + 8xy + 17y^2}.$$

4.2 Exercises

Find the general solution, and then solve using the given data, for the following equations

1. $$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u/2 = 0 \qquad u(s, 0) = 1$$

2. $$(y + u)\frac{\partial u}{\partial x} - (x + u)\frac{\partial u}{\partial y} + (x - y) = 0 \qquad u(s, 2s) = s$$

3. $$x(y + u^2)\frac{\partial u}{\partial x} + y(x^2 - u^2)\frac{\partial u}{\partial y} + u(x^2 + y) = 0 \qquad u(0, s) = 0$$

4. $$\frac{3}{x - y}(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}) = 2 \qquad u(s, 0) = -s^2$$

5. $$(3x - u)\frac{\partial u}{\partial x} + (3y - u)\frac{\partial u}{\partial y} = x + y \qquad u(s, 2s) = 0$$

Tutorials P.D.E.s

Find the solution $u(t,x)$ of the following equations:

1. $$\partial_t u - \partial_x^2 u = 0 \qquad u(t,0) = 0 \quad u(t,1) = 0 \quad u(0,x) = 6\sin(9\pi x)$$

2. Same as above but with $u(0,x) = \begin{cases} 1 & 0 < x < 1/2 \\ 2 & 1/2 < x < 1 \end{cases}.$

3. $$\partial_t u - \partial_x^2 u = 0 \quad \partial_x u(t,0) = 0 \quad \partial_x u(t,1) = 0 \quad u(0,x) = f(x)$$

   Show that the solution is

   $$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} \cos(n\pi x)$$

   where

   $$f(x) = \sum_{n=1}^{\infty} A_n \cos(n\pi x).$$

   Find formulas for A_n in terms of $f(x)$.

4. $$\partial_t u - \partial_x^2 u + u = 0 \quad u(t,0) = 0 = u(t,1) \quad u(0,x) = 1$$

5. $$\partial_x^2 u + \partial_y^2 u = 0 \quad u(0,y) = y, u(x,0) = u(1,y) = u(x,2) = 0$$

6. $$\partial_t^2 u - \partial_x^2 u = 0 \quad u(t,0) = u(t,1) \quad \partial_x u(t,0) = \partial_x u(t,1) \quad \partial_t u(0,x) = 0$$

   $$u(0,x) = \begin{cases} 2x & 0 < x < 1/3 \\ 1 - x & 1/3 < x < 1 \end{cases}$$