Revisiting Kepler

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This project takes you to the time of Johannes Kepler just before the discovery of the telescope and the scientific revolution. How did he discover his three laws that proved so crucial to later developments? It is very hard to truly relive his experience, used as we are to calculators, Earth images from space, light pollution, and ... impatience. But here goes:

To accept Copernicus' heliocentric system required a leap of faith at the beginning of the 17th century. If Earth is rotating a whole circumference ($\approx$ 40000km) in 24 hours, then the ground under our feet is moving at 460m/s; that's four playing fields every single second (less with increasing latitude). Unlike his predecessor, Tycho Brahe, Kepler took this leap. Then it becomes natural that part of a planet's motion is due to Earth's orbital motion, which needs to be subtracted to obtain the planet's true orbital speed.

1. The first task is to find Earth's orbit. Choose a nearby planet and find its synodic period: for an outer planet, this is the average time interval between oppositions, when the planet is brightest; for an inner planet, it is the average time between greatest elongations (on the same side). Convert to its sidereal period by

   $$\frac{1}{T_{sid}} = \frac{1}{365.25} \pm \frac{1}{T_{syn}}$$

   $+$ for inner planets, $-$ for outer planets.
   (Alternatively, find these periods directly from Wikipedia).

   Example: Mars – synodic period 780 days, so sidereal period is 687 days,

   $$\frac{1}{T_{Mars}} = \frac{1}{365.25} - \frac{1}{780} = \frac{1}{686.9}.$$

2. Prepare a list of dates: start with an opposition of the planet (conjunction for an inner planet), then a sequence of dates separated by the sidereal period. Thus the planet will be fixed in one place of its orbit. Because of the round-up errors and approximations we'll be making, there needs to be at least twenty dates for the average orbit to appear properly. (So an accurate value for the sidereal period should be used.)

   [It can be tricky to calculate these dates on a computer. One can convert the date to a Julian number, add the periods, then convert each back to a date. See astrometrics.]

   Example: Mars – 3 Mar 2012 21:00 (opp.), 19 Jan 2014 17:00, 7 Dec 2015 14:00, 24 Oct 2017 14:00, ...

3. Take readings of the Sun and the planet on these dates, either with a telescope/phone app (but it would take years) or from NASA, or some other website.

   Example:

Date		3/3/12	19/1/14	7/12/15	24/10/17	...
Mars	RA	11 : 05 : 33	13 : 13 : 47	12 : 55 : 21	12 : 05 : 09	
	Decl.	$10°21'14''$	$-5°13'49''$	$-4°19'26''$	$0°41'44''$	

4. If the readings are in altazimuth or right-ascension/declination, then convert to an ecliptic longitude. (The websites above give direct readings).

   To convert from RA/Decl. to ecliptic longitude, use the following formula with $\epsilon = 23.4367°$:

$$Ec.long. = \operatorname{atan2}(\sin(RA)\cos\epsilon + \tan(Decl)\sin\epsilon, \cos(RA))$$

   [$\operatorname{atan2}(y, x) = \arctan(y/x)$, taking into account the correct quadrant.]

   Example: Ecliptic longitudes

Sun	343.691°	299.540°	255.132°	211.351°
Mars	163.492°	199.021°	194.422°	180.913°

   [Strictly speaking, the planet's orbital plane is not the same as Earth's (ecliptic), but the angular height above it, i.e., the ecliptic latitude, is at most about 1°, and will be ignored here. The readings are not accurate to 6 significant figures, but it is better not to round up any values.]

5. Make two lists of angles: (i) A, the relative angle of the Sun relative to the first angle; (ii) $B = \text{Mars} - \text{Sun}$. Confirm that B for the first date is 180° for an outer planet, and 0° for an inner planet.

   Example:

$A = \text{Sun} - 343.691°$	0°	315.849°	271.441°	227.660°
$B = \text{Mars} - \text{Sun}$	179.801°	259.481°	299.29°	329.562°

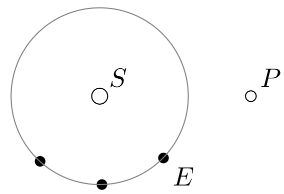
6. Mark a point O for the Sun. (This will be the origin.) Draw a horizontal line from it, representing 0°. On this line mark a second point P, representing the planet on the chosen dates. Then we want a point E such that $A = \angle POE$, $B = \angle OEP$.

   If $P = (1, 0)$, then the coordinates of E are

$$r = \frac{\sin(A+B)}{\sin B}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \cos A \\ \sin A \end{pmatrix}$$

   Because of the errors involved, one has to ignore points for which A or B are close to $0° \pm 20°$ or $180° \pm 20°$.

   Example: $(x, y) = (0.422, -0.410)$, $(0.015, -0.586)$, $(-0.394, -0.432)$



7. Fit an ellipse on these points. Verify that the Sun is at one focus. It's best to have enough points to fill the orbit twice and to ignore outliers. Note the relative distance $OP : OE$.

   Example: Radius is 0.59, so the inverse ratio gives Mars' orbital radius $R = 1.7$au. An 'eccentric' circle describes Earth's orbit adequately,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta - 0.0075 \\ \sin\theta - 0.0149 \end{pmatrix} \tag{1}$$

8. The second task is to find the orbits of the other planets. One can in theory repeat the above procedure for all planets and make a list of the relative distances, but in practice it becomes much harder due to errors.

   Choose a planet, find its sidereal period and make a list of pairs of dates separated by this period. It is not important now that you start with an opposition or conjunction, and each pair could be separated by any number of days (eg consecutive).

   Example: Mercury. Synodic period is 115.9 days, so sidereal period is 87.97 days. List of dates: 6 Jan 2019 19:00, 4 Apr 2019 18:00; 16 Jan 2019 19:00, 14 Apr 2019 18:00; ...

9. Take/read observations of the Sun (A) and planet (B) on these pairs of dates, and convert to ecliptic longitude.

   Example:

Date	6/1/19,	4/4/19	16/1/19,	14/4/19
Sun A	286.162°	14.677°	296.352°	24.503°
Mercury B	272.464°	348.324°	287.935°	357.080°

10. Calculate the position of Earth using formula (1) with $\theta = A + 180°$.

    Example:

Date	6/1/19,	4/4/19	16/1/19,	14/4/19
Earth x	−0.286	−0.975	−0.451	−0.917
y	0.945	−0.268	0.881	−0.430

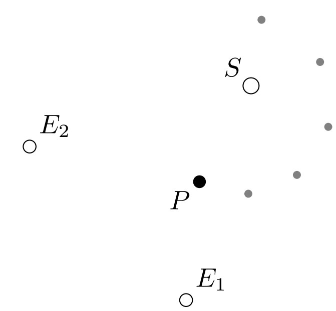
11. Draw xy axes, with the Sun at the origin. Place each pair of points of Earth, $E_1 = (x_1, y_1)$, $E_2 = (x_2, y_2)$. We want a point P such that $xE_1P =$

B_1 and $xE_2P = B_2$. Draw lines from each point at the observed angles of the planet (to the x-axis), or compute the point P using trigonometry:

$$P = \frac{1}{\sin(B_1 - B_2)} \left(\sin B_1 \cos B_2 \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} - \cos B_1 \sin B_2 \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} + \begin{pmatrix} (y_2 - y_1) \cos B_1 \cos B_2 \\ (x_1 - x_2) \sin B_1 \sin B_2 \end{pmatrix} \right)$$

Repeat for each pair of points.

Example: Take $(x_1, y_1) = (-0.286, 0.945)$, $(x_2, y_2) = (-0.975, -0.268)$, $\theta_1 = 272.464°$, $\theta_2 = 348.324°$, to get $(x, y) = (-0.227, -0.423)$.



12. By labeling each point P with its date, one can try to find a law that relates them. Kepler's first guess was that the distance between successive points (i.e., the planet's speed) is inversely proportional to its distance from the Sun. Then he settled on his second law: the area of the triangles SP_iP_{i+1} is constant.

    Example Average distance $R = 0.4$. Areas of triangles in 10-day periods.

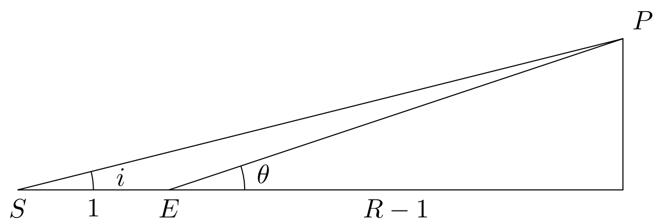
Date	6 Jan	16 Jan	26 Jan	5 Feb	15 Feb
Area	0.103	0.101	0.097	0.091	0.084

    [The triangles, especially the nearer ones, are underestimates of the area traced by the planet. For 1-day periods, the constancy should become evident.]

13. Repeat for other planets to get a list of relative distances. To find the inclination of the orbit of an inner planet, the easiest way is to mark the angle above the ecliptic (*ecliptic latitude*) at several instances of greatest elongations; the largest of these angles is the inclination. For an outer planet, mark the ecliptic latitude at opposition; if θ is the largest among these, then the inclination is given by

$$\tan i = \left(1 - \frac{1}{R}\right) \tan \theta$$

    where R is the relative distance of the planet.



14. Make a list of sidereal periods T (in years) and relative distances R, as well as T^2 and R^3 – Kepler's third law suggests itself.

Planet	Me	V	E	M	J	S
T_{sid} (years)	0.24	0.62	1.0	1.88	11.9	29.5
R (au)	0.39	0.72	1.0	1.52	5.2	9.6
T^2	0.058	0.38	1	3.53	142	870
R^3	0.059	0.37	1	3.51	141	885

At the time, it proved difficult to verify Kepler's workings because of the errors involved, e.g. angles were not accurate down to seconds of arc. Kepler's laws were not accepted as fact and hardly quoted. With the telescope and the discovery of Jupiter's moons it became possible to quickly and accurately determine their relative radii and orbital periods.

Moon	Io	Europa	Ganymede	Callisto
T (days)	1.77	3.55	7.16	16.69
T (relative)	1	2	4	9.4
R (relative)	1	1.59	2.54	4.46
T^2	1	4	16	88
R^3	1	4.1	16.4	89

Once Kepler's third law was seen to be true in the 1650s/60s, for both planets and Jupiter's moons, it suggested a 'universal law of attraction'. Several people (Hooke, Halley, Newton) realized, that when combined with Huygens' formula for circular acceleration, v^2/r, it implied an inverse square law

$$a = \frac{v^2}{r} = \frac{1}{r}\left(\frac{2\pi r}{T}\right)^2 = 4\pi^2 \frac{r^2}{r^3 r} \propto \frac{1}{r^2}$$

Newton, with his ability to solve fluxion equations, completed this project by proving that an inverse square law implies motion in an ellipse.