Astrometrics

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1 Time

The time t measured by an inertial observer in the solar system's reference is called **Universal Time** (or GMT[1]) (assuming no relativistic effects).

The **Local Time** at a specific place on Earth is related to universal time by

$$\text{Local Time} = t + \text{“time zone”} + \text{“daylight saving”}$$

Each *time zone* is ideally a range of longitudes $15°(n \pm \frac{1}{2})$ but in practice they follow political borders. For the main regions and cities of the world:

−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
	<Hawaii <Poly>	←Ala	‘Pac’ Seat LA	‘Mnt’ Denv	‘Cntr’ Chic Mex Nic	‘East’ NY Col Peru	N.Sco Ven Bol ←Chl	<Grn> <Bra ←Arg	Atl	←Azor CVr	UK Gui Gha
1	2	3	4	5	6	7	8	9	10	11	12
Ger ←Fra ←Spa <Nig Ang	Fin Ukr Egypt S.Afr	Moscw <Turk Iraq ←Tanz Mdgr	Dub Mau	<Pak Ind+$\frac{1}{2}$	Bang	Thai Indo>	<China Phil Perth	Japan	Syd	Sol	Fiji <NZ

[1] Strictly speaking, physical time as measured by atomic clocks is called International Atomic Time. Universal Time counts the same seconds, but inserts a leap second every few years to keep in pace with the slowing day; so the difference between Atomic Time and Universal Time is of a few seconds over decades.

1.1 Day, Month, Year

Three unrelated time intervals are astronomically and historically important:

1. The average time it takes for the Earth to rotate on its axis so that the Sun appears in the same place in the sky is the

   $$\textbf{day} = 24\ \textbf{hours} = 24 \times 60\ \textbf{minutes} = 24 \times 60 \times 60\ \textbf{seconds}.$$

   It is a bit longer than the time it takes the Earth to rotate 360°, because it has to make up for the fact that the Earth has gone round the Sun an extra angle.

2. The average time it takes for the Earth to orbit once around the Sun (or as seen from Earth, for the Sun to go round the *ecliptic*) is called a **year**,

   $$1\ \textit{sidereal year} = 365.25636 \text{ days}$$

   This is more properly called the *sidereal* year because the angular speed is measured against the backdrop of 'fixed' stars. So the average angular speed of the Earth around the Sun is

   $$\omega_{\odot} = \frac{360^\circ}{\text{year}}$$

3. The average time it takes the Moon to orbit the Earth relative to the Sun ("synodic"), is called the **month**.

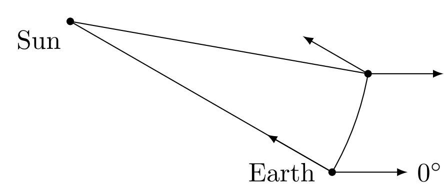
   $$1\ \textit{month} = 29.530588 \text{ days}$$

In general, the resultant of two angular speeds is $\omega_{res} = \omega_1 + \omega_2$, or equivalently by dividing with 360°,

$$\frac{1}{T_{res}} = \frac{1}{T_1} + \frac{1}{T_2}$$

The average angular speed of the Earth's rotation is

$$\omega_E = \frac{360^\circ}{\text{day}} + \frac{360^\circ}{\text{year}}$$



(since $\omega_E \times \text{day} = 360^\circ(1 + \frac{\text{day}}{\text{year}})$)

So, adding

$$\begin{aligned} 360^\circ/\text{day} &= 15^\circ/\text{hour} = 15'/\text{min} = 15''/\text{sec}, \\ \omega_{\odot} &= 360^\circ/\text{year} = 0.986^\circ/\text{day} = 2.464'/\text{hour} = 2.464''/\text{min}, \\ \omega_E &= 15^\circ 2' 28''/\text{hour} = 15.041''/\text{sec}. \end{aligned}$$

Three complications arise because the Earth's axis precesses (rotates) once every 25771 years, the orbital perihelion rotates every 113,000 years, and its daily rotational period is slowly and irregularly increasing due to tidal friction (2.7×10^{-8}). From an Earth perspective, the time it takes from one midsummer to the next, called the *tropical year*, is slightly less than a sidereal year as

$$\frac{1}{T_{trop.year}} = \frac{1}{T_{sid.year}} + \frac{1}{T_{prec.}} = \frac{1}{365.24219 \text{ days}}$$

For the Moon, the orbital period relative to the stars is given by

$$\frac{1}{T} = \frac{1}{\text{month}} + \frac{1}{\text{year}} = \frac{1}{27.321662 \text{ days}}$$

$$\therefore\ \omega_m = 0.549^\circ/\text{hour} = 0.549'/\text{min}$$

1.2 Calendar

The *Gregorian calendar* is set up to approximate the tropical year as closely as possible over millennia: $365.24219 \approx 365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400}$

A leap day is added to every year divisible by 4;
unless the leap year is the start of a century;
unless that century is divisible by 4.

This gives a *civil year* of 365.25 days in the short term, and 365.2425 days in the long term.

This calendar was introduced on 15 Oct 1582 in Catholic countries (and after 1700 in the rest of Protestant Europe). Before it, there was the *Julian calendar* which had a simpler structure with a leap day every four years: $365 + \frac{1}{4}$.

1.2.1 Changing from Date to Time

The number of days from start 1 January until the start of date $D/M/Y$ is

$$\text{days} = (D-1) + \begin{cases} 0 & \text{if January} \\ 31 & \text{if February} \\ \lfloor 30.6(M+1) \rfloor - 63 + \chi_{leap} & \text{if} \geqslant \text{March} \end{cases}$$

where χ_{leap} is 1 if the year is a leap year, and 0 otherwise

$$\chi_{leap} = 1 - \lceil Y/4 \rceil + \lfloor Y/4 \rfloor + \lceil Y/100 \rceil - \lfloor Y/100 \rfloor - \lceil Y/400 \rceil + \lfloor Y/400 \rfloor$$

The number of days from 2000.0 until the start of year $2000 + Y$ is, taking $y = Y - 1$,

$$\text{year-days} = 366 + 365y + \lfloor y/4 \rfloor - \lfloor y/100 \rfloor + \lfloor y/400 \rfloor$$

In general, the days from 2000.0 to a specific date number to

$$\text{year-days} + \text{days} + \text{time}$$

For example, the total time in days from 2000.0 to 2024 Sep 5 1022:43 (GMT) is given by: number of days 2000–2023 is 8766, Jan–Aug is 244, plus 4 and $(10 + (22 + 43/60)/60)/24$, in total 9009.43244 days.

To calculate the time between two dates, the **Julian Day** counts the number of whole days since 1 Jan 4713 BC (actually from noon, but more conveniently from midnight). To find the Julian day,

– From a Gregorian calendar date (post 15 Oct 1582):

$$JD = \lfloor 365.25(Y - 1 + 4712) \rfloor + 368 - \left\lfloor \frac{Y}{100} \right\rfloor + \left\lfloor \frac{Y}{400} \right\rfloor + \text{days} + \text{time}$$

– From a Julian calendar date (pre 4 Oct 1582):

$$JD = \lfloor 365.25(Y - 1 + 4712) \rfloor + 366 + \text{days} + \text{time}$$

(Note that there is no 0BC or 0AD, so Y BC $= -(Y - 1)$ AD, so to speak.)

To convert a Julian day to a date:

– To a Julian date:

$$\begin{aligned} Y &= \left\lfloor \frac{JD}{365.25} \right\rfloor - 4712 \\ \chi_{jleap} &= 1 - \left\lceil \frac{Y}{4} \right\rceil + \left\lfloor \frac{Y}{4} \right\rfloor \\ \text{days} &= JD - 1 + \chi_{jleap} - \lfloor 365.25(Y + 4712) \rfloor \end{aligned}$$

– To a Gregorian date:

$$\begin{aligned} &\texttt{a=}JD - 1721425;\ \texttt{a-=}\left\lfloor \frac{\texttt{a}}{146097} \right\rfloor;\ \texttt{a+=}\left\lfloor \frac{\texttt{a}}{36524} \right\rfloor;\ \texttt{a-=}\left\lfloor \frac{\texttt{a}}{1461} \right\rfloor \\ &Y = \left\lceil \frac{\texttt{a} - \frac{1}{2}}{365} \right\rceil \\ &\text{days} = JD - 1721426 - 365(Y - 1) - \left\lfloor \frac{Y-1}{4} \right\rfloor + \left\lfloor \frac{Y-1}{100} \right\rfloor - \left\lfloor \frac{Y-1}{400} \right\rfloor \end{aligned}$$

To convert days to D/M:

$$\text{days} < 59 + \chi_{leap} \ \Rightarrow \begin{cases} M = \lfloor \text{days}/31 \rfloor + 1 \\ D = \text{days} - 31(M - 1) + 1 \end{cases}$$

$$\text{days} \geqslant 59 + \chi_{leap} \ \Rightarrow \begin{cases} d := \text{days} + 64 - \chi_{leap} \\ M = \lfloor d/30.61 \rfloor - 1 \\ D = d - \lfloor 30.6(M + 1) \rfloor \end{cases}$$

1.2.2 Day of the Week

The *day of the week* is easily calculated from the Julian day by $JD \pmod 7$ where Mon $= 0$, ..., Sun $= 6$.

Traditionally, the *Littera Dominicalis* was used: If the letters $A \dots G$ are assigned to all dates of the year, starting 1 January, the *Dominical letter* is that letter which corresponds to the Sundays. (On leap years, there are two Dominical letters, the first for Jan, Feb, and the second for Mar onwards). Moreover, the weekday of 1 January is also given by the Dominical letter, if it is interpreted as $A =$ Sun, $B =$ Sat, $C =$ Fri, etc. For example, 2020 has two Dominical letters E and D, so the first Sunday is 5th January and 1 Jan 2020 is a Wednesday.

For a Gregorian date, the Dominical letter can be calculated by first dividing the year as $y = 100a + b$, then finding the number of leap years $b = 4c + d$ and the number of leap centuries $a = 4e + f$. As $365 = 1 \mod 7$, each normal year decreases the letter by one, four years by $-5 = 2 \mod 7$, a hundred years by $25 \times 2 + 1 = 2 \mod 7$, and four hundred years by $4 \times 2 - 1 = 0 \mod 7$. Combined in one formula, the (second) Dominical letter is then

$$\ell := -d + 2c + 2f \pmod 7 = 4y + a + 2d - e \pmod 7,$$

where $A = 0$, $B = 1, \dots, G = 6$. If the year is a leap year, the first Dominical letter is $\ell + 1 \pmod 7$. (For a Julian date, $\ell = 2 - d + 2c + a \pmod 7$.)

The weekday of 1 January is $7 - \ell$, where $0 =$ Sun, ..., $6 =$ Sat. (For leap years, use the first Dominical letter.) To calculate the day of the week of any given date, note that the first of a month is

March, November	$3 - \ell$	April, July	$6 - \ell$
May, January	$1 - \ell$	June, February	$4 - \ell$
August	$2 - \ell$	September, December	$5 - \ell$
October	$7 - \ell$		

(January and February refer to the following year, because a leap day may be added at the end of February.)

1.2.3 Full Moons

A simple way to calculate the age of the Moon (at midnight) is from the Julian day

$$\text{age} = JD - 2451550.6935 \pmod{29.530588}$$

where age $= 0$ for a new moon, and age $= 14.77$ for a full moon. If the dates of the New Moon, Full Moon, and Quarters are required, find the age u at the beginning of the month, then the dates are on $7.38 - u$, $14.77 - u$, $22.15 - u$, $29.53 - u$ days after. (Add 29.53 if negative.) The accuracy is of a few hours since each cycle differs slightly from the average lunar period of 29.53 days.

Traditionally, the age of the Moon was calculated using two numbers, the *Numerus Aureus* and the *Epactae*. The month and year are mutually irrational, but their ratio can be approximated by

$$\frac{29.530588}{365.25} \approx \cfrac{1}{12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{2}}}}} = \frac{19}{235},$$

i.e., 19 civil years are almost exactly 235 months (+1.5 hours). So there are only 19 different full-moon calendars (neglecting a possible leap day), one for each *golden number* $n = 1 + (Y \mod 19)$.

The age of the Moon on New Year's Day is called the *epact*. With each year a full moon occurs $13 \times 29.53 - 365 = 18.898$ days later, or equivalently 10.63 days earlier (11.63 when a leap year); on average, it is 10.88 days/year. The epact increases by 11 days, modulo 30, but one must take into effect the leap centuries; it would equal $11n - 3 + e - a \pmod{30}$. This agrees with increases of 10.88 modulo 29.53, but after 19 years there remains an error of 0.061839 days, i.e., 0.00325 days/year, or approximately 8 days every 25 centuries, so a correction of $8a/25$, rounded to the nearest integer, is applied

$$\text{epact } \epsilon = 8 + 11n - a + e + \text{round}(8a/25) \pmod{30}$$

From the epact one can then calculate the age on any specific date by adding the number of days to ϵ and subtracting 30 and 29 alternately.

1.2.4 Easter Date

Easter occurs on the first Sunday after the first full moon when spring starts (for computational purposes taken to be 21st March). There is an official algorithm to calculate it, the 'computus'. As decimal representations were not yet invented when it was devised, its workings may appear as numerical wizardry based on three numbers, the *Littera Dominicalis*, *Numerus Aureus*, and *Epactae*. Since Easter occurs after any leap day, only the second Dominical letter matters in the case of a leap year. There are only 19 different Easter full-moon dates, one for each *golden number*.

Consider 21 March:

$$\begin{aligned} \text{day of the week } &= 31 + 28 + 21 - \ell \pmod 7 \\ &= 3 - \ell \pmod 7, \end{aligned}$$

$$\begin{aligned} \text{age of Moon } &= \epsilon + 31 + 28 + 21 - 30 - 29 \pmod{30} \\ &= \epsilon + 21 \pmod{30} \end{aligned}$$

The number of days left after 21 March until the Easter Full Moon is then

$$\begin{aligned} h &:= 14 - (\epsilon + 21) \pmod{30} \\ &= 23 - \epsilon \pmod{30} \end{aligned}$$

The day of the week of Easter Full Moon is then $h + 3 - \ell \pmod 7$, and the number of days left until the following Sunday is

$$\begin{aligned} \lambda &:= 7 - (h + 3 - \ell) \pmod 7 \\ &= \ell - h + 4 \pmod 7 \end{aligned}$$

Easter should then be $h + \lambda$ days after 22 March. However, the way the traditional computus works, the period of the moon is increased alternately by 30 and 29 days, with the 'missing' day in the 'short' months taken as the 6th day; after 11 years, it can happen, when the epact is 24 or 25 (so the New Moon is on lunar day 5/6), that two years have the same full moon date; so when $\epsilon = 25$ or 26 and $n > 11$, there is a correction of -1; equivalently

$$\mu := \begin{cases} 1 & 11h + n \geqslant 320 \\ 0 & \text{o/w} \end{cases}$$

This will only have an effect on the Easter date if $\lambda = 6$, in which case we need to take the previous Sunday, i.e.,

$$\text{days from 22 March to Easter, } e := h + (\lambda + \mu \mod 7) - \mu$$

The last step is to convert this to a date

$$\text{month} = \begin{cases} 3 & e \leqslant 9 \\ 4 & e > 9 \end{cases} \qquad \text{day} = \begin{cases} 22 + e & e \leqslant 9 \\ e - 9 & e > 9 \end{cases}$$

or equivalently, $e + 114 = 31\,\text{month} + (\text{day} - 1)$.

There is a short algorithm, originally by Gauss and amended anonymously, that inputs the year `Y` and outputs the Easter date:

			Quotient	Remainder	
`Y`	$\div$	`19`	–	`m`	Golden number−1
`Y`	$\div$	`100`	`a`	`b`	
`b`	$\div$	`4`	`c`	`d`	
`a`	$\div$	`4`	`e`	`f`	
`8a+13`	$\div$	`25`	`g`	–	
`19m+a-e-g+15`	$\div$	`30`	–	`h`	$(19 = -11 \mod 30)$
`11h+m`	$\div$	`319`	μ	–	
`2c+2f-d`	$\div$	`7`	–	ℓ	Dominical letter
ℓ`-h+`μ`+32`	$\div$	`7`	–	λ	
`h+`λ`-`μ`+90`	$\div$	`25`	`j`	–	
`h+`λ`-`μ`+j+19`	$\div$	`32`	–	`i`	

The Easter date is then `i/j/Y`. For the Easter full moon take $\lambda = -1$.

1.3 Sidereal 'Time'

Sidereal Time is the *angle* that Earth's 0° longitude (Greenwich) is pointing at on the celestial equator. It is not really a time, but it is traditional to measure celestial angles using hours, minutes, and seconds; a great circle on the celestial sphere is divided into 24 angular 'hours', so one angular hour equals 15°.

$$\text{Sidereal Time GST} = \theta_0 + \omega_E(t - t_0)$$

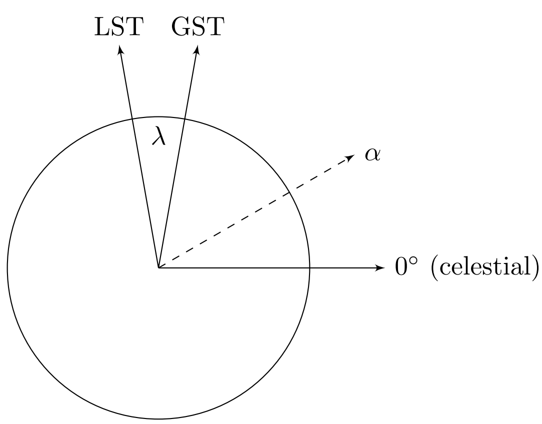
where θ_0 is the direction of Greenwich at t_0; for example,

t_0	θ_0
1 Jan 2000.0	99.967795°
1 Jan 2010.0	100.537624°
1 Jan 2020.0	100.116718°

Local Sidereal Time is the direction that a particular longitude λ on Earth is pointing at, i.e., if you imagine a line going from North to South passing directly overhead, LST is the angle between the celestial 0° and this imaginary line,

$$\text{LST} = \text{GST} + \text{longitude}$$

An angle α on the celestial equator is at LST $- \alpha$ away from this imaginary overhead line.



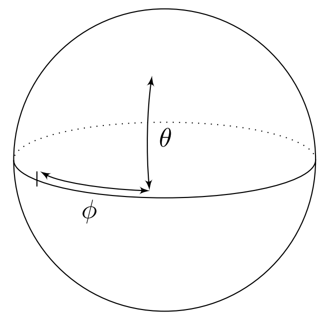
To convert from LST to local time,

$$\text{Local Time} = \frac{\text{LST} - \text{long.} - \theta_0}{\omega_E} + t_0 + \text{"time zone"} + \text{"daylight saving"}$$

2 Spherical Coordinates

The standard spherical coordinates are akin to the longitude and latitude that are used for positions on the Earth's surface; such coordinates are fully characterized by a great circle (the 'equator') and a reference point on it (the 0 longitude).

$$\boldsymbol{r} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix}$$



There are various spherical coordinates projected to the celestial sphere:

Name	*Coordinate*	*names*	*Reference circle and point*
Altazimuth	(A, a)	(azimuth, altitude)	horizon north
Equatorial	(α, δ)	(ascension, declination)	celestial equator 'first point of Aries'
Ecliptic	(λ, β)	(ecliptic longitude, latitude)	ecliptic 'first point of Aries'
Galactic	(l, b)	(galactic longitude, latitude)	Milky Way its center

The equatorial system is the most used. The 'first point of Aries' is, by definition, the point where the equator intersects the ecliptic in the constellation of Aries. Altazimuth coordinates of points on the celestial sphere are local and continuously changing. The rest are relatively fixed but have slight problems: Earth's rotational axis and equator move slowly (precession, 25ky); the ecliptic plane also precesses very slowly (430My) due to perturbations from the other planets; the galactic plane is hard to define precisely.

2.0.1 Angle between two Directions

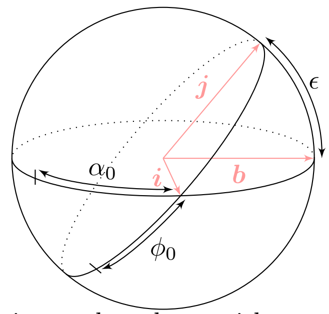
The *elongation* γ, or angular difference, between two coordinates (ϕ_1, θ_1) and (ϕ_2, θ_2) is given by

$$\cos\gamma = \begin{pmatrix} \cos\theta_1\cos\phi_1 \\ \cos\theta_1\sin\phi_1 \\ \sin\theta_1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_2\cos\phi_2 \\ \cos\theta_2\sin\phi_2 \\ \sin\theta_2 \end{pmatrix} = \cos(\phi_1 - \phi_2)\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$

2.1 Changing Coordinates

To change from one spherical coordinate system (ϕ, θ) to another (α, δ), consider their reference great circles and let $\boldsymbol{i}$ be a vector where they intersect. Suppose the reference points of the two systems have angles ϕ_0 and α_0 from $\boldsymbol{i}$. Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be a right-handed orthonormal vector basis with $\boldsymbol{j}$ in the plane of the (ϕ, θ) reference circle, and $\boldsymbol{i}, \boldsymbol{b}, \boldsymbol{c}$ a right-handed orthonormal basis with $\boldsymbol{b}$ in the plane of the (α, δ) reference circle. If the two planes are at an inclination of ϵ to each other, then the relation between the two bases is

$$\begin{aligned}\boldsymbol{b} &= \cos\epsilon\,\boldsymbol{j} - \sin\epsilon\,\boldsymbol{k}\\ \boldsymbol{c} &= \sin\epsilon\,\boldsymbol{j} + \cos\epsilon\,\boldsymbol{k}\end{aligned}$$



So the relation between the coordinates of a point on the sphere, with respect to the two bases $(\boldsymbol{i}, \boldsymbol{b}, \boldsymbol{c})$, $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, is

$$\begin{pmatrix}\cos\delta\cos(\alpha-\alpha_0)\\ \cos\delta\sin(\alpha-\alpha_0)\\ \sin\delta\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\epsilon & -\sin\epsilon\\ 0 & \sin\epsilon & \cos\epsilon\end{pmatrix}\begin{pmatrix}\cos\theta\cos(\phi-\phi_0)\\ \cos\theta\sin(\phi-\phi_0)\\ \sin\theta\end{pmatrix}$$

Dotting with the various unit vectors, one obtains the following identities:

$$\begin{aligned}
&(\boldsymbol{i}) & \cos(\alpha-\alpha_0)\cos\delta &= \cos(\phi-\phi_0)\cos\theta\\
&(\boldsymbol{b}) & \sin(\alpha-\alpha_0)\cos\delta &= \sin(\phi-\phi_0)\cos\theta\cos\epsilon - \sin\theta\sin\epsilon\\
&(\boldsymbol{c}) & \sin\delta &= \sin(\phi-\phi_0)\cos\theta\sin\epsilon + \sin\theta\cos\epsilon\\
&(\boldsymbol{j}) & \sin(\phi-\phi_0)\cos\theta &= \sin(\alpha-\alpha_0)\cos\delta\cos\epsilon + \sin\delta\sin\epsilon\\
&(\boldsymbol{k}) & \sin\theta &= -\sin(\alpha-\alpha_0)\cos\delta\sin\epsilon + \sin\delta\cos\epsilon
\end{aligned}$$

Dividing the first two equations, one gets

$$\boxed{\begin{aligned}\sin\delta &= \sin(\phi-\phi_0)\cos\theta\sin\epsilon + \sin\theta\cos\epsilon & (1)\\ \tan(\alpha-\alpha_0) &= \tan(\phi-\phi_0)\cos\epsilon - \sec(\phi-\phi_0)\tan\theta\sin\epsilon & (2)\end{aligned}}$$

Other useful identities are

$$\sin(\alpha-\alpha_0) = \frac{\sin\delta\cos\epsilon - \sin\theta}{\cos\delta\sin\epsilon}$$

$$\tan(\alpha-\alpha_0) = \frac{\sin\delta\cos\epsilon - \sin\theta}{\cos\theta\cos(\phi-\phi_0)\sin\epsilon}$$

To convert from (α, δ) to (ϕ, θ), use the same identities with ϵ replaced by $-\epsilon$.

Some useful reference data:

Altazimuth $\to$ Equatorial	$\epsilon = \phi - 90°$	$\alpha_0 = \mathrm{LST} - 90°$	$A_0 = 90°$
Ecliptic $\to$ Equatorial	$\epsilon = 23°26'16''$	$\alpha_0 = 0°$	$\lambda_0 = 0°$
Galactic $\to$ Equatorial	$\epsilon = 62°52'18''$	$\alpha_0 = 282°51'34''$	$l_0 = 32°55'55''$

where ϕ is the place's latitude.

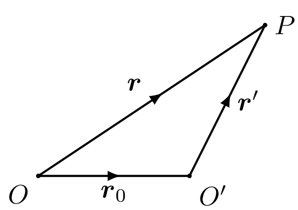
Earth's *obliquity* (angle between equator and ecliptic) changes (due to the other planets) as

$$\epsilon \approx 23.439291° - 0.8325° \sin(T/6523) - 0.00675° \sin(T/2778) \pm 0.0002°$$

where T is the number of years since 2000.

2.2 Changing the Origin of a Coordinate System

Another transformation is required when the origin of a coordinate system is changed, for example, from sun-centered to earth-centered.



$$\boldsymbol{r}_0 = r_0 \begin{pmatrix} \cos\theta_0 \cos\phi_0 \\ \cos\theta_0 \sin\phi_0 \\ \sin\theta_0 \end{pmatrix}$$

$$\boldsymbol{r} = r \begin{pmatrix} \cos\theta \cos\phi \\ \cos\theta \sin\phi \\ \sin\theta \end{pmatrix}$$

$$\boldsymbol{r}' = r' \begin{pmatrix} \cos\theta' \cos\phi' \\ \cos\theta' \sin\phi' \\ \sin\theta' \end{pmatrix}$$

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}_0$$

$$r' \begin{pmatrix} \cos\theta' \cos\phi' \\ \cos\theta' \sin\phi' \\ \sin\theta' \end{pmatrix} = r \begin{pmatrix} \cos\theta \cos\phi \\ \cos\theta \sin\phi \\ \sin\theta \end{pmatrix} - r_0 \begin{pmatrix} \cos\theta_0 \cos\phi_0 \\ \cos\theta_0 \sin\phi_0 \\ \sin\theta_0 \end{pmatrix}$$

The unknowns r', θ', ϕ' can be found from the three equations. In particular, distance $O'P = r'$ and the angle $\psi := OPO'$ can be obtained from:

$$(r')^2 = |\boldsymbol{r} - \boldsymbol{r}_0|^2 = r^2 + r_0^2 - 2\boldsymbol{r} \cdot \boldsymbol{r}_0$$

$$rr' \cos\psi = \boldsymbol{r} \cdot \boldsymbol{r}' = r^2 - \boldsymbol{r} \cdot \boldsymbol{r}_0$$

2.2.1 Parallax

For nearby objects P such as the Moon, the observed equatorial coordinates (α', δ') from a point O' on the surface of the Earth differ from the (calculated or published) geocentric equatorial coordinates (α, δ) relative to Earth's center O. If Earth's axes are $a = 6378.137$km and $b = 6356.752$km, and the observer has latitude ϕ, then the observed coordinates and distance r' are given by

$$r' \begin{pmatrix} \cos\delta' \cos\alpha' \\ \cos\delta' \sin\alpha' \\ \sin\delta' \end{pmatrix} = r \begin{pmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{pmatrix} - \begin{pmatrix} a\cos\phi \cos \mathrm{LST} \\ a\cos\phi \sin \mathrm{LST} \\ b \sin\phi \end{pmatrix}$$

Note: For more precise work, the angle ϕ is not exactly the published latitude $\tilde{\phi}$, which is the angle between the zenith and the equatorial plane. Neglecting the height of the point above the mean Earth, the two are related by $\frac{a\sin\phi}{\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}} = \sin\tilde{\phi}$, so

$$\tan\phi = \frac{b}{a}\tan\tilde{\phi}$$

2.3 Rising and Setting Times for (α, δ)

These are the times when the point (α, δ) has an altitude of 0. Substituting into (1) for a latitude ϕ,

$$\begin{aligned} 0 &= \sin a = \sin(\alpha - \mathrm{LST} + 90^\circ)\cos\delta\sin(90^\circ - \phi) + \sin\delta\cos(90^\circ - \phi) \\ &= \cos(\alpha - \mathrm{LST})\cos\delta\cos\phi + \sin\delta\sin\phi \\ \therefore\ \mathrm{LST} &= \alpha \pm \cos^{-1}(-\tan\delta\tan\phi) \end{aligned}$$

The local sidereal time can then be converted to local time.

The corresponding azimuth at rising and setting is given by

$$\begin{aligned} \sin(A - 90^\circ) &= \frac{\sin 0 \cos(90^\circ - \phi) - \sin\delta}{\cos 0 \sin(90 - \phi)} \\ \Rightarrow \qquad \cos A &= \frac{\sin\delta}{\cos\phi} \end{aligned}$$

In practice, one needs to correct for atmospheric refraction, which increases the altitude. Thus a point (star) rises/sets when its altitude is -0.575°.

Sunrise/set occurs when the Sun has an altitude of about $a = -0.83^\circ$ $(= -0.575^\circ - 0.25^\circ)$, twilight until it is about -6°, quite dark at -12°, and completely dark below -18°. Similarly, moonrise/set occurs at an altitude of -0.83°. For these cases, use the more general formula that gives the time when (α, δ) is at an altitude a:

$$\mathrm{LST} = \alpha \pm \cos^{-1}(\sin a \sec\delta\sec\phi - \tan\delta\tan\phi)$$

If the point is moving (e.g. Moon) then calculate (α, δ) at two times (separated by Δt), and find their corresponding rising/setting times T_1, T_2. The correct rising/setting time is then the point of intersection $t = T_1 + (T_2 - T_1)t/\Delta t$. If the object is near (e.g. Moon), use the observer-corrected coordinates (α, δ) (see below, parallax).

2.3.1 Refraction

The altitude angle a is refracted to a' by Snell's law

$$\sin z' = \frac{\sin z}{1.000282} \qquad \Rightarrow \ \Delta a = 0.016^\circ \cot a$$

where $z = 90^\circ - a$ is the *zenith* angle (the refractive index depends on the light frequency). This is only accurate for z small.

For lower altitudes, an empirical fit is $\Delta a = 0.017^\circ \cot(a + \frac{10.3^\circ}{a+5.11^\circ})$.

2.3.2 Stellar Aberration and Parallax

Suppose a star has position $d\boldsymbol{e}$ relative to the Sun; the Earth has position $a\binom{\cos\theta}{\sin\theta} = a\boldsymbol{\theta}$ and velocity $\boldsymbol{u} = \frac{2\pi a}{T}\binom{-\sin\theta}{\cos\theta} = 2\pi c\alpha\boldsymbol{\theta}^\perp$, where $\alpha = \frac{a}{Tc} = 0.0000158$ is the Earth's orbital radius in light years.

The star has position $\boldsymbol{r} = d\boldsymbol{e} - a\boldsymbol{\theta} = d(\boldsymbol{e} - \frac{\alpha}{d_{ly}}\boldsymbol{\theta})$ relative to Earth, so light coming from the star appears to come from a direction $-(-c\hat{\boldsymbol{r}} + \boldsymbol{u}) = c(\hat{\boldsymbol{r}} - \frac{1}{c}\boldsymbol{u}) = c(\hat{\boldsymbol{r}} - 2\pi\alpha\boldsymbol{\theta}^\perp)$.

Note that if $\boldsymbol{x}$ is much smaller than $\boldsymbol{e}$, then the unit vector of $\boldsymbol{e} + \boldsymbol{x}$ is $\boldsymbol{e} + (1 - \boldsymbol{e}\boldsymbol{e}^*)\boldsymbol{x}$ to first order (since $|\boldsymbol{e}+\boldsymbol{x}|^2 \approx 1 + 2\boldsymbol{e}\cdot\boldsymbol{x}$, so $|\boldsymbol{e}+\boldsymbol{x}|^{-1} \approx 1 - \boldsymbol{e}\cdot\boldsymbol{x}$.) In particular, the apparent direction of the star can be found from

$$\hat{\boldsymbol{r}} = \boldsymbol{e} - \frac{\alpha}{d_{ly}}(1 - \boldsymbol{e}\boldsymbol{e}^*)\boldsymbol{\theta}$$

$$\tilde{\boldsymbol{e}} = \boldsymbol{e} - \alpha(1 - \boldsymbol{e}\boldsymbol{e}^*)(2\pi\boldsymbol{\theta}^\perp + d_{ly}^{-1}\boldsymbol{\theta})$$

Thus the star appears to form ellipses around the central direction $\boldsymbol{e}$. If $\boldsymbol{e} = \begin{pmatrix}\cos\beta\cos\lambda\\ \cos\beta\sin\lambda\\ \sin\beta\end{pmatrix}$ are the ecliptic coordinates of the star, then the tangent vectors are $\begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}$ and $\begin{pmatrix}-\sin\beta\cos\lambda\\ -\sin\beta\sin\lambda\\ \cos\beta\end{pmatrix}$; by dotting $\tilde{\boldsymbol{e}} - \boldsymbol{e}$ with these vectors, we get that the apparent position of the star over the course of a year is

$$\tilde{\boldsymbol{e}} = 2\pi\alpha\begin{pmatrix}\cos(\theta-\lambda)\\ \sin\beta\sin(\theta-\lambda)\end{pmatrix} + \frac{\alpha}{d_{ly}}\begin{pmatrix}\sin(\theta-\lambda)\\ -\cos(\theta-\lambda))\end{pmatrix}$$

The dominant effect is *spherical aberration* (especially at the ecliptic poles) with radius of $2\pi\alpha = 20.48''$; the second effect is *parallax*, visible by comparing a group of nearby stars, of order $\alpha/d_{ly} = 3.26''/d_{ly}$; its reciprocal is the distance in *parsecs*.

2.4 Precession

Like any symmetric top with moments of inertia (A, A, C), the Earth precesses at a rate ω_{pre} where

$$\frac{3G}{2}\left(\frac{M_\odot}{r_\odot^3}+\frac{M_m}{r_m^3}\right)(C-A)\cos\epsilon\sin\epsilon = \text{torque} = \omega_{pre}\omega_E C\sin\epsilon$$

For the oblate Earth, $A/C = 0.9967$, so $\omega_{pre} \approx 0.0142°/\text{year} = 360°/25400\,\text{year}$. In fact, the equatorial plane is rotating about the ecliptic north at a rate of $360°/25771\,\text{years} = 50.23''/\text{year}$, e.g. after more than two millennia, the first point of Aries is now in Pisces. So the ecliptic coordinates of a star are changing at the rate of $\dot\lambda = 50.23''/\text{year}$, $\dot\beta = 0$. Differentiating (1) and (2) gives

$$\dot\delta\cos\delta = \cos\beta\sin\epsilon\;\dot\lambda$$

$$-\sin\alpha\cos\delta\;\dot\alpha - \cos\alpha\sin\delta\;\dot\delta = -\sin\lambda\cos\beta\;\dot\lambda$$

so

$$\begin{aligned}\dot\delta &= \cos\alpha\sin\epsilon\;\dot\lambda = 19.98''\cos\alpha/\text{year},\\ \dot\alpha &= (\cos\epsilon+\sin\alpha\tan\delta\sin\epsilon)\dot\lambda\\ &= (46.09''+19.98''\sin\alpha\tan\delta)/\text{year}\end{aligned}$$

Note that there is a separate tiny precession (wobble) of a few metres every 1.18 years, due to the Earth not being exactly axially symmetric.

3 Orbits

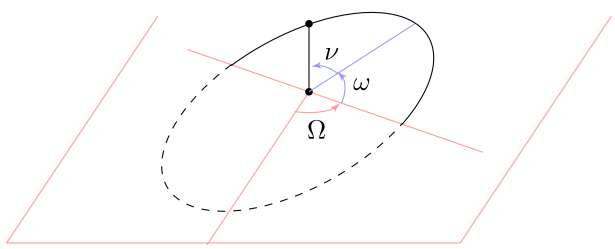
Consider the path of an object orbiting the Sun. Relative to the center of mass of the system, $m\boldsymbol{r} + M_\odot \boldsymbol{R}_\odot = \boldsymbol{0}$, hence $\boldsymbol{r} - \boldsymbol{R}_\odot = (1 + \frac{m}{M})\boldsymbol{r}$. The force acting on the object is

$$m\ddot{\boldsymbol{r}} = -\frac{GM_\odot m}{|\boldsymbol{r} - \boldsymbol{R}_\odot|^2}\hat{\boldsymbol{r}} = -\kappa m \frac{\hat{\boldsymbol{r}}}{|\boldsymbol{r}|^2}$$

where $\kappa = GM_\odot(1 + m/M_\odot)^{-2}$. By conservation of angular momentum (the force is independent of angle), the motion is in a plane, so polar coordinates (r, θ) suffice, and the equations reduce to $r^2\dot{\theta} = h$ (constant) and $\ddot{r} - r\dot{\theta}^2 = -\frac{\kappa}{r^2}$. Combining the two and substituting $u := 1/r$ gives $u'' + u = \frac{\kappa}{h^2} = \alpha$ whose solution

$$r = \frac{\alpha}{1 + e\cos(\theta - \theta_0)} \tag{3}$$

is a conic.



It is an ellipse when the *eccentricity* $e < 1$, with $\alpha = a(1 - e^2)$, a the semi-major axis, and θ_0 the direction of the point closest to the focus, called the *perihelion*. Substituting into $r^2\dot{\theta} = h$, a differential equation is obtained for the "true anomaly" $\nu := \theta - \theta_0$ measured from the perihelion

$$\dot{\nu} = \beta(1 + e\cos\nu)^2 \tag{4}$$

where $\beta = \frac{\kappa^2}{h^3} = \frac{2\pi}{T}(1 - e^2)^{-3/2}$. The period of the orbit is

$$T = \int_0^{2\pi} \frac{\mathrm{d}\nu}{\dot{\nu}} = \frac{2\pi}{\sqrt{\kappa}}a^{3/2}$$

known as Kepler's third law. Equation (4) is of separable type,

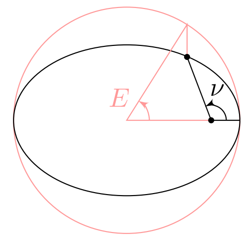
$$\int \frac{\mathrm{d}\nu}{(1 + e\cos\nu)^2} = \beta t$$

To solve, it is best to change variables from ν to the "eccentric anomaly" E defined by

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2} \tag{5}$$

(Equivalently, $\tan\nu = \sqrt{1 - e^2}\frac{\sin E}{\cos E - e}$.) After some calculations, the equation becomes

$$\dot{E} = \frac{2\pi}{T}\frac{1}{1 - e\cos E}$$



Changing coordinates again from E to the "mean anomaly" (Kepler)

$$M := E - e\sin E \tag{6}$$

gives the simplified equation $\dot{M} = 2\pi/T$, i.e.,

$$M(t) = \frac{2\pi}{T}(t - t_0) + M(t_0) \tag{7}$$

Sometimes, the reference time t_0 is that of perihelion, in which case $M(t_0) = 0$. More often, angles are measured from a reference direction; either this is the node where the orbital plane cuts the ecliptic, $M(t_0) = \varepsilon - \omega$, where ε is the mean anomaly and ω is the perihelion angle; or the reference direction is the first point of Aries and $M(t_0) = L - \omega'$ where $L = \varepsilon + \Omega$, $\omega' = \omega + \Omega$.

Orbital characteristics: given an object of mass m orbiting another M in an ellipse of eccentricity e and semi-major axis a,

period	$T = \sqrt{\frac{4\pi^2}{\kappa}}a^{3/2}$
angular momentum mh	$h^2 = \kappa a(1 - e^2)$
total energy	$\mathcal{E} = -\frac{\kappa m}{2a}$
max/min speeds	$v_\pm = \sqrt{\frac{\kappa}{a}\frac{1\pm e}{1\mp e}}$
perihelion/aphelion distances	$a_\pm = (1 + \frac{m}{M})(1 \pm e)a$

Note that these formulæ apply only to a two-body orbit; the Sun has several orbiting planets, so to a good approximation it can be taken to be at the origin, and the factor $(1 + m/M)$ can be ignored ($\kappa = GM_\odot$).

3.0.1 Parabolic and Hyperbolic Orbits

For hyperbolic orbits $e > 1$ (e.g. near Earth asteroids/meteoroids), the equation $\dot{\theta} = \beta(1 + e\cos\theta)^2$ can be solved by applying the change of variable,

$$\tanh\frac{E}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}$$

to get

$$\dot{E} = \frac{\beta(e^2 - 1)^{3/2}}{e\cosh E - 1}$$

A second change of variables, $M = e\sinh E - E$ leads to $\dot{M} = \beta(e^2-1)^{3/2}$, so

$$M(t) = \beta(e^2-1)^{3/2}t$$

where $\beta^2 = \frac{\kappa}{q^3(1+e)^3}$, since the perigee distance is $q = h^2/\kappa(1+e)$ and $\beta = \kappa^2/h^3$. The deflection angle (between asymptotes) is $2\arccos(1/e)$.

For parabolic orbits $e = 1$ (e.g. some comets), the equation $\dot{\theta} = \beta(1+\cos\theta)^2$ can be solved directly,

$$\beta t = M = \int \frac{\mathrm{d}\theta}{(1+\cos\theta)^2} = \frac{\sin\theta(2+\cos\theta)}{3(1+\cos\theta)^2} = \tfrac{1}{6}\tan^3\tfrac{\theta}{2} + \tfrac{1}{2}\tan\tfrac{\theta}{2}$$

hence $x := \tan\theta/2$ satisfies a cubic equation, so

$$\tan\frac{\theta}{2} = (3M + \sqrt{9M^2+1})^{1/3} - (3M + \sqrt{9M^2+1})^{-1/3}$$

Escape velocity: In both cases, $0 \leqslant \mathcal{E} = \frac{1}{2}mv^2 - \frac{\kappa m}{r}$, so the escape velocity is $v \geqslant \sqrt{\frac{2\kappa}{r}}$. For example, on Earth, $v = \sqrt{\frac{2GM_E}{R_E}} = 11.2$km/s. An object in orbit, however, already has some kinetic energy, so it needs a boost of $\frac{\kappa m}{2a} = \frac{1}{2}mv^2$, or $v = \sqrt{\frac{\kappa}{a}}$ to escape; e.g. , solar escape velocity from Earth is $\sqrt{\frac{GM_\odot}{1au}} = 29.9$km/s.

Orbit transfer: To go from one orbit to another, a probe needs an orbit with perihelion a and aphelion b, so semi-major axis of $\frac{a+b}{2}$ and eccentricity $e = \frac{b-a}{b+a}$; it needs a boost of kinetic energy of $v^2 = \kappa(\frac{1}{a} - \frac{2}{a+b})$, in addition to escaping the planet; but such an orbit would take much longer than to sling a proble via another planet.

3.1 Calculating the position

To find the position (α, δ) of an orbiting object (planet), as seen from Earth:

1. Find the mean anomaly $M(t)$, using (7) and t, t_0, T.

2. Find the eccentric anomaly $E(t)$, using (6) and $M(t)$, e; since the equation has no closed-formula solution, a Newton-Raphson or other iterative method is used.

3. Find the true anomaly $\nu(t)$, using (5) and $E(t)$, e. The heliocentric longitude in the orbit's plane is then $l' = \nu + \omega = \nu + \omega' - \Omega$ measured from the node, and latitude $= 0$.

4. Find the distance $r(t)$, using (3) and $\nu(t)$, e, a.

5. Find the heliocentric coordinates of Earth: $l_E = \nu_E + \omega_E$, $\beta_E = 0$. If the Sun's position is sought, then go straight to step 8, using $\lambda = l_E + 180°$, $\beta = 0$. (This is not extremely accurate, because it is the Moon-Earth system that moves in an elliptical orbit.) Find also r_E.

6. The heliocentric coordinates with respect to the ecliptic plane can then be found by changing coordinates from $(l', 0)$ to (l, τ); the angle between the object's orbit and the ecliptic is the *inclination* i of the orbit,while the longitude reference points are $l'_0 = 0$, $l_0 = \Omega$; so using (1) and (2), ($\epsilon = i$, $\phi_0 = 0$, $\alpha_0 = \Omega$)

$$\begin{aligned}\sin\tau &= \sin l' \sin i \\ \tan(l - \Omega) &= \tan l' \cos i\end{aligned}$$

7. The geocentric ecliptic coordinates (λ, β) are obtained from (l, τ) using the next section.

8. The geocentric equatorial coordinates (α, θ) are found from (λ, β).

9. If necessary, find the apparent equatorial coordinates by calculating the parallax; then the altazimuth coordinates if required.

3.1.1 Changing from Heliocentric to Geocentric Coordinates

The connection between heliocentric and geocentric ecliptic coordinates, (l, τ) and (λ, β), is:

$$\rho \begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix} = r \begin{pmatrix} \cos\tau\cos l \\ \cos\tau\sin l \\ \sin\tau \end{pmatrix} - r_E \begin{pmatrix} \cos l_E \\ \sin l_E \\ 0 \end{pmatrix}$$

If $\boldsymbol{x} = (x, y, z)$ is the vector on the right, then $\rho = |\boldsymbol{x}| = \sqrt{x^2 + y^2 + z^2}$, $\beta = \arctan \frac{z}{\sqrt{x^2+y^2}}$, $\lambda = \arctan \frac{y}{x}$. For specific formulas,

$$\begin{aligned}\rho^2 &= r^2(1 + \alpha^2 - 2\alpha\cos\tau\cos(l - l_E)) \\ \tan\beta &= \frac{\sin\tau}{\sqrt{\alpha^2 + \cos^2\tau - 2\alpha\cos\tau\cos(l - l_E)}} \\ \tan\lambda &= \frac{\alpha\sin l_E - \cos\tau\sin l}{\alpha\cos l_E - \cos\tau\cos l}\end{aligned}$$

where $\alpha = r_E/r$.

Orbital data of planets at 2000.0 (with change every 10 years)

	a (AU)	e	i (°)	L (°)	ω_p (°)	t_0	Ω (°)
Mercury	0.38709927	0.20563593	7.00497902	252.25032350	77.45779628	2000.1280	48.33076593
	0.00000004	0.00000191	-0.00059475	1494.72674112/yr	0.01604769		-0.01253408
Venus	0.72333566	0.00677672	3.39467605	181.97909950	131.60246718	2000.5332	76.67984255
	0.00000039	-0.00000411	-0.00007889	585.17815387/yr	0.000268329		-0.027769418
Earth-Moon	1.00000261	0.01671123	-0.00001531	100.46457166	102.93768193	2000.0110	0.0
	0.00000056	-0.00000439	-0.00129467	359.99372450/yr	0.03232736		0.0
Mars	1.52371034	0.09339410	1.84969142	-4.55343205	-23.94362959	2001.7836	49.55953891
	0.00000185	0.00000788	-0.00081313	191.40302685/yr	0.04444109		-0.02925734
Jupiter	5.20288700	0.04838624	1.30439695	34.39644051	14.72847983	2011.2108	100.47390909
	-0.00001161	-0.00001325	-0.00018371	30.34746128/yr	0.02125267		0.02046911
Saturn	9.53667594	0.05386179	2.48599187	49.95424423	92.59887831	2003.5304	113.66242448
	-0.00012506	-0.00005099	0.00019361	12.22493622/yr	-0.04189722		-0.02886779
Uranus	19.18916464	0.04725744	0.77263783	313.23810451	170.95427630	2050.7349	74.01692503
	-0.00019618	-0.00000440	-0.00024294	4.28482028/yr	0.04080528		0.00424059
Neptune	30.06992276	0.00859048	1.77004347	-55.12002969	44.96476227	2045.6635	131.78422574
	0.00002629	0.00000510	0.00003537	2.18459453/yr	-0.03224146		-0.00050866

1 AU=149,597,871km; the period of a planet is equal to $a^{3/2}$ if a is measured in AU; t_0 is the perihelion time. Earth's perihelion rotates 360° in 112000years. Sun's diameter is 1.392×10^6km.

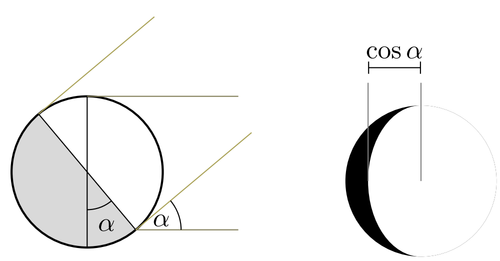
Data from ssd.jpl.nasa.gov. For asteroids etc., see www.minorplanetcenter.org/iau/MPCORB.html, ssd.jpl.nasa.gov/sbdb.cgi, or asteroid.lowell.edu. It is hard to track satellites' orbits because they are easily perturbed, but see www.amsat.org/amsat-new/tools. Check calculations with www.jb.man.ac.uk/almanac

3.1.2 Angular Size and Phase

The *angular size* of the object is

$$\text{angular size (in radians)} = \frac{\text{diameter of object}}{\rho}$$

The *phase* of the object is the fraction of the lit up area of the object to the whole apparent area.



For a unit spherical object with incident light at an angle α, the terminator is a great circle which appears as an ellipse with semi-minor axis of $\cos\alpha$. The apparent lit up area is $\frac{\pi}{2} + (\cos\alpha)\frac{\pi}{2}$, so the phase is

$$F = \frac{1+\cos\alpha}{2}$$

For an astronomical object, α is the angle Sun-Object-Earth, i.e., between the spherical coordinates (l, τ) and (λ, β):

$$\cos\alpha = \cos\tau\cos\beta\cos(l-\lambda) + \sin\tau\sin\beta$$

The inclination of the planets is fixed (up to precession). The direction vector of their 'north pole' is, in equatorial coordinates (α, δ): Mercury (281.01°, 61.45°), Venus (272.76°, 67.16°), Earth (0°, 90°), Mars (317.68°, 52.89°), Jupiter (268.06°, 64.50°), Saturn (40.59°, 83.54°) (including rings), Uranus (257.31°, −15.18°), Neptune ($299.36 \pm 0.7°$, $43.46 \pm 0.51°$) (precesses every 6.9y).

3.2 Analemma

The *analemma* is that curve that describes the Sun's position on the celestial sphere at 24 hour intervals, starting from the winter solstice. The (negative of) right ascension for each day, converted to time, is called the *equation of time*.

The analemma can be calculated by finding the geocentric position $\tilde{\lambda}$ of the Sun on the ecliptic, relative to the winter solstice, converted to equatorial coordinates, then retraced to the winter solstice by rotating Earth back for an equal time period.

Let t be the day of the year, expressed as an angle from the winter solstice, $t = M+\pi+\omega+\frac{\pi}{2} \approx M+12.93°$. It is related to λ by (5), (6), and $\lambda = \nu+\omega+\pi$, $\tilde{\lambda} = \lambda + \frac{\pi}{2}$. Then the analemma position expressed in equatorial coordinates is $(\alpha(t), \delta(t))$ (where $\alpha = 0$ refers to the winter solstice)

$$\begin{pmatrix}\cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\epsilon & 0 & \sin\epsilon\\ 0 & 1 & 0\\ -\sin\epsilon & 0 & \cos\epsilon\end{pmatrix}\begin{pmatrix}\cos\tilde{\lambda} & \sin\tilde{\lambda} & 0\\ -\sin\tilde{\lambda} & \cos\tilde{\lambda} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\delta\cos\alpha\\ \cos\delta\sin\alpha\\ \sin\delta\end{pmatrix}$$

The equation of time is $(t, -\tilde{t})$, where $\tilde{t} = \alpha/\omega_E = \alpha \times \frac{\text{day}}{360°}(1+\frac{\text{day}}{\text{year}})^{-1}$ (negative since it is meant as a correction for sundial time).

At sunrise and sunset, the analemma needs to be rotated by the latitude to show its orientation relative to the horizon. The Sun's varying position explains why sunrise is latest after the winter solstice and earliest before the summer solstice (in the northern hemisphere), while sunsets are opposite.

3.3 Finding an Orbit from Observations

An orbit requires 6 parameters to be determined in space: $a, e, i, \omega, \Omega, M(t_0)$, so three observations $(\alpha(t_i), \delta(t_i))$ should in principle be enough.

There are errors in the calculation: light takes time to arrive to Earth, and the observations are from a point on the surface of the Earth, not its center.

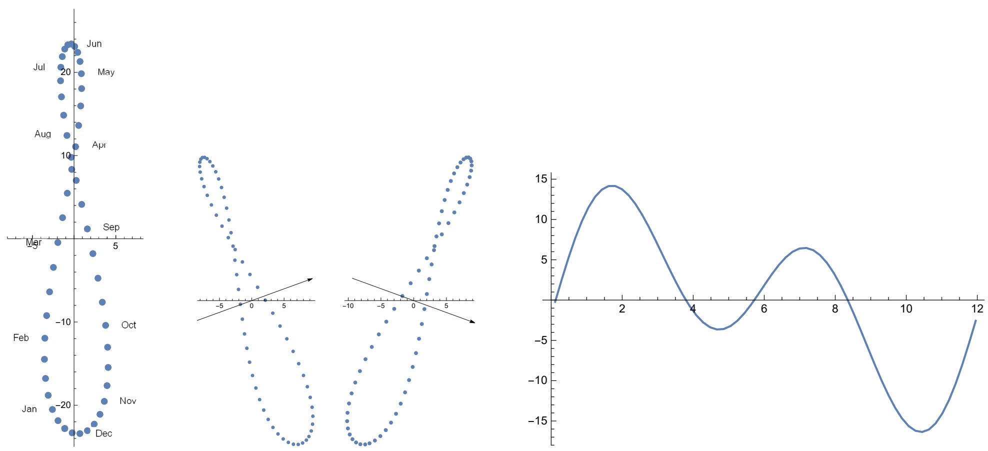


Figure 1: Analemma, at surise and sunset, Equation of Time

3.4 Lunar Orbit

The Moon's motion is much more complicated because of the gravitational effect of the Earth and the Sun:

$$\ddot{\boldsymbol{r}}_E = -\frac{GM_\odot \boldsymbol{r}_E}{|\boldsymbol{r}_E|^3} + \frac{GM_m \boldsymbol{r}}{|\boldsymbol{r}|^3}$$
$$\ddot{\boldsymbol{r}}_m = -\frac{GM_\odot \boldsymbol{r}_m}{|\boldsymbol{r}_m|^3} - \frac{GM_E \boldsymbol{r}}{|\boldsymbol{r}|^3}$$

where $\boldsymbol{r} = \boldsymbol{r}_m - \boldsymbol{r}_E$. The Earth's motion is dominated by the Sun's attraction (99.4%), so can be taken to be an elliptic orbit, even a circular orbit to a good enough approximation $\boldsymbol{r}_E = a(\cos\theta_E, \sin\theta_E, 0)$; but the Moon's is dominated by both the Sun (69%) and Earth (31%).

In general, the three-body problem has no simple solution. If only the leading terms in $\boldsymbol{r}/|\boldsymbol{r}_E|$ are kept, assuming $|\boldsymbol{r}_E| = a$, rescaling $\boldsymbol{R} := \boldsymbol{r}/a_m$, $T := \omega_m t$, and letting $G(M_E + M_m) = \omega_m^2 a_m^3$, $GM_\odot = \omega_E^2 a^3$, $\beta := \omega_E/\omega_m$, simplifies the second equation to (using a Taylor series $\frac{\boldsymbol{r}_m}{|\boldsymbol{r}_m|^3} \approx \frac{\boldsymbol{r}_E}{|\boldsymbol{r}_E|^3} + \frac{\boldsymbol{r}}{|\boldsymbol{r}_E|^3} - 3\frac{(\boldsymbol{r}\cdot\boldsymbol{r}_E)\boldsymbol{r}_E}{|\boldsymbol{r}_E|^5}$)

$$\ddot{\boldsymbol{R}} = -\frac{\boldsymbol{R}}{|\boldsymbol{R}|^3} + \beta^2(3(\boldsymbol{R}\cdot\widehat{\boldsymbol{r}}_E)\widehat{\boldsymbol{r}}_E - \boldsymbol{R})$$

As $\beta^2 \approx 0.006$, the dominant part is $\ddot{\boldsymbol{R}} = -\boldsymbol{R}/|\boldsymbol{R}|^3$, i.e., an elliptical orbit around the Earth in a plane inclined at an angle i to the ecliptic, eccentricity e, and with mean angular speed $\omega_m = 360°/27.321662$ days (=27.321582 days with respect to the precessing ecliptic coordinates). But this behavior is modified by the remaining term; the real orbit is not planar: both the ellipse and the plane are slowly precessing or rotating with mean periods of $T_P = 5.99685$ years and $T_N = 18.5996$ years respectively. The (ascending) *node* is the direction where this plane intersects the ecliptic, and it moves as

$$N(t) = N_0 - \frac{360°}{T_N}(t - t_0) - A_n(t)$$

where $N_0 = N(t_0)$ is the node longitude at the reference date and $A_n = 0.16° \sin M_\odot(t)$ ($M_\odot(t)$ is the solar longitude). Note that the Moon crosses the Sun with a period t_s, the nodes with a period t_n, and the perigee with period t_p, where

$$\begin{array}{rrl}
\text{“synodic” month} & \frac{1}{t_s} & = \frac{1}{27.321662\text{days}} - \frac{1}{365.25636\text{days}} \\
 & & = \frac{1}{27.321582\text{days}} - \frac{1}{365.24219\text{days}} = \frac{1}{29.53059\text{days}} \\
\text{“draconic” month} & \frac{1}{t_n} & = \frac{1}{27.321662\text{days}} + \frac{1}{T_N} = \frac{1}{27.21222\text{days}} \\
\text{“anomalistic” month} & \frac{1}{t_p} & = \frac{1}{27.321662\text{days}} - \frac{1}{T_P} + \frac{1}{T_N} = \frac{1}{27.55455\text{days}}
\end{array}$$

Reference data for Moon 2000.0 (from ssd.jpl.nasa.gov/?sat_elem):

a	e	ω_p	t_0	i	N_0	diameter
384399km	0.0549006	83.1862°	2000.05177	5.1454°	125.1240°	3474km
		+40.6901°/year			−19.341°/year	

(It's best to look up t_0 in a table of lunar perigees, e.g. www.fourmilab.ch/earthview/pacalc.html.)

One can only find the main periodic terms of this motion. A fair approximation is elliptical motion with various corrections:

$$\begin{array}{lrll}
\text{Mean anomaly} & M(t) & = 360°(t - t_0)/27.321582\text{days} & \\
\text{‘True’ anomaly} & \theta & = \nu & \text{(elliptic angle)} \\
 & & +1.2739° \sin(2D - M) & \text{“evection”} \\
 & & +0.658° \sin 2D & \text{“variation”} \\
 & & -0.1858° \sin M_\odot & \text{“annual equation”}
\end{array}$$

where $D = M + \omega_p - \lambda_\odot$.

Then the geocentric ecliptic longitude and latitude are given by

$$\lambda = \theta + \omega',$$
$$\beta = i \sin(\lambda - N) + 0.173° \sin(2D - \lambda + N),$$

The angle Earth-Moon-Sun is then $\theta = 180° - \lambda_\odot + \lambda$, so its phase is $F = \frac{1}{2}(1 + \cos\theta)$. The distance from the Earth is approximately

$$\frac{\rho}{a} = \frac{1 - e^2}{1 + e\cos\theta}.$$

If one wants to track the Moon's motion over short periods of time, one can calculate the rates of change of λ_m and β_m, by differentiating:

$$\begin{array}{ll}
\dot\lambda_m \approx \dot\nu = \beta(1 + e\cos\nu)^2 \approx \frac{2\pi}{T}(1 - e^2)^{-3/2}(1 + 2e\cos\nu) & \approx (0.55° + 0.06° \cos M)/\text{hour} \\
\dot\beta_m \approx i\cos(\lambda - N)(\dot\lambda - \dot N) & \approx 0.05° \cos(\lambda - N)/\text{hour.}
\end{array}$$

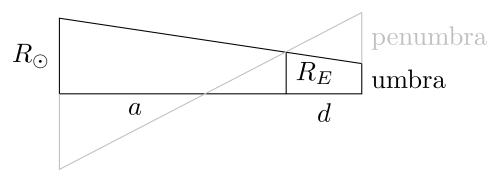
See www.solexorb.it for an example of a direct numerical orbit calculator.

3.4.1 Eclipses

Eclipses occur when the Moon and Sun are in exactly the same or opposite directions; since the lunar orbital plane is different from the ecliptic, these directions must be the nodes.

A **solar eclipse** occurs when both the Sun and the Moon are at the same node, $\lambda_m = \lambda_\odot$ and $|\lambda - N| < 18°31'$ or $|\lambda - N - 180°| < 18°31'$. The longest duration of a total eclipse is $7^m 31^s$, that of an annular eclipse is $12^m 24^s$.

A **lunar eclipse** occurs when the Sun is at one node, and the Moon at the other node; the angular size of the umbra and penumbra are, on average, 1.37° and 2.44°; more precisely, when the Earth is at distances a, d from the Sun and Moon, they are (in radians):



$$\text{umbra} = \frac{2R_E}{d}\left(1 - (\alpha - 1)\frac{d}{a}\right), \qquad \text{penumbra} = \frac{2R_E}{d}\left(1 + (\alpha + 1)\frac{d}{a}\right),$$

where $\alpha = R_\odot / R_E = 695842/6371 = 109.22$. The longest duration of a lunar eclipse is $1^h 47^m$ ($3^h 45^m$ from tip to tip).

Both the Sun and the Moon move along approximately uniformly during the eclipse,

$$\begin{aligned} \lambda_\odot(t) &= \lambda_\odot(t_0) + \dot{\lambda}_\odot(t - t_0), & \beta_\odot(t) &= 0 \\ \lambda_m(t) &= \lambda_m(t_0) + \dot{\lambda}_m(t - t_0), & \beta_m(t) &= \beta_m(t_0) + \dot{\beta}_m(t - t_0) \end{aligned}$$

where $\dot{\lambda}_\odot \approx 360°/\text{year} = 0.04°/\text{hour}$. The Moon's position relative to the Sun is $(\lambda_m - \lambda_\odot, \beta_m)$, a straight line with a slope of $\dot{\beta}_m/(\dot{\lambda}_m - \dot{\lambda}_\odot)$ coming in from right to left. It can be calculated at two times to determine the straight line. The Sun's angular diameter can be calculated from its true diameter divided by the distance Earth-Sun; similarly the Moon's. In either case, the time of eclipse maximum is given at the point when this relative line is closest to the origin,

$$\frac{\beta_m}{\dot{\beta}_m} + \frac{\lambda_m - \lambda_\odot}{\dot{\lambda}_m - \dot{\lambda}_\odot} = 0 \;\Rightarrow\; t = t_0 - \frac{1}{2}\left(\frac{\beta_m(t_0)}{\dot{\beta}_m} + \frac{\lambda_m(t_0) - \lambda_\odot(t_0)}{\dot{\lambda}_m - \dot{\lambda}_\odot}\right)$$

Saros Cycle: The lunar eclipses are fairly predictable (unlike total solar eclipses) because of the large size of the Earth's shadow relative to the Moon. Two numbers are important: the month relative to the sun (29.53059 days) and the time taken for the Moon to cross the orbital nodes (27.21222 days). Using continued fractions, their ratio can be approximated by

m/n	$27.2n - 29.5m$	$29.5m/27.5$	$29.5m/365.25$ (yr)
12/13	-0.608	12.86	1 year -11 days
35/38	0.494	37.51	3 years -62 days
47/51	-0.115	50.37	4 years -73 days
223/242	0.0357	239.0	18 years $+11$ days

Since the Moon moves at a rate of $360°/29.53059\text{days} = 1.37°/0.11\text{days}$, the required accuracy is almost achieved by the ratio 47/51 and better by 223/242. The former is a period of 3.8 years of recurring eclipses, typically the 0,6,12,18,24,30,36,41,42,(47) 'moons'; but this pattern changes slightly for the subsequent 3.8 years. The second fraction leads to the *Saros* cycle of 18.03 years = 18 years 11 days, in which recurrent eclipses are much more similar to each other. Coincidentally, the Moon's cycle of distances from the Earth is of 27.55455 days (third column), which almost exactly divides 223 synodic months, so corresponding eclipses in consequent Saros cycles are very similar to each other. The current Saros cycle has the following eclipses:

Year	Lunar Eclipses				Solar Eclipses	
2006	14 Mar		7 Sep		29 Mar	22 Sep
2007	3 Mar		28 Aug		(19 Mar)	(11 Sep)
2008	21 Feb		16 Aug		(7 Feb)	(1 Aug)
2009	9 Feb	7 Jul	6 Aug	31 Dec	26 Jan	22 Jul
2010		26 Jun		21 Dec	15 Jan	11 Jul
2011		15 Jun		10 Dec	(4 Jan)	(25 Nov)
2012		4 Jun		28 Nov	20 May	13 Nov
2013	25 Apr	25 May		18 Oct	10 May	3 Nov
2014	15 Apr			8 Oct	(29 Apr)	(23 Oct)
2015	4 Apr			28 Sep	(20 Mar)	(13 Sep)
2016	23 Mar		18 Aug	16 Sep	9 Mar	1 Sep
2017	11 Feb		7 Aug		26 Feb	21 Aug
2018	31 Jan		27 Jul		(15 Feb)	(11 Aug)
2019	21 Jan		16 Jul		2 Jul	26 Dec
2020	10 Jan	5 Jun	5 Jul	30 Nov	21 Jun	14 Dec
2021		26 May		19 Nov	(10 Jun)	(4 Dec)
2022		16 May		8 Nov	(30 Apr)	(25 Oct)
2023		5 May		28 Oct	20 Apr	14 Oct
2024		25 Mar		18 Sep	8 Apr	2 Oct

3.4.2 Lagrangian points

Consider an object M_J orbiting another of larger mass $M_\odot$ in a circle about their common center of mass,

$$\boldsymbol{x}_J = \frac{M_\odot a}{M_\odot + M_J}\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}, \qquad \boldsymbol{x}_\odot = -\frac{M_J a}{M_\odot + M_J}\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$$

where $\omega^2 a^3 = G(M_\odot + M_J)$.

Now consider a third, much smaller, mass m moving around the pair, $\boldsymbol{x}(t)$. Using a rotating frame of reference $a\boldsymbol{X} := R_{-\omega t}\boldsymbol{x}$, $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, and noting $\dot{R} = \omega PR$, with $P^2 = -I$, $PR = RP$, $\alpha := \frac{M_J}{M_\odot + M_J}$, then

$$\begin{aligned}
\boldsymbol{X}_J &= (1-\alpha)\boldsymbol{i},\\
\boldsymbol{X}_\odot &= -\alpha\boldsymbol{i},\\
\dot{\boldsymbol{x}} &= aR_{\omega t}(\dot{\boldsymbol{X}} + \omega P\boldsymbol{X})\\
\ddot{\boldsymbol{x}} &= aR_{\omega t}(\ddot{\boldsymbol{X}} + 2\omega P\dot{\boldsymbol{X}} - \omega^2\boldsymbol{X})
\end{aligned}$$

Then the equation of motion of m

$$\ddot{\boldsymbol{x}} = -\frac{GM_\odot(\boldsymbol{x}-\boldsymbol{x}_\odot)}{|\boldsymbol{x}-\boldsymbol{x}_\odot|^3} - \frac{GM_J(\boldsymbol{x}-\boldsymbol{x}_J)}{|\boldsymbol{x}-\boldsymbol{x}_J|^3}$$

becomes

$$\ddot{\boldsymbol{X}} + 2\omega P\dot{\boldsymbol{X}} - \omega^2\boldsymbol{X} = -\frac{(1-\alpha)\omega^2(\boldsymbol{X}-\boldsymbol{X}_\odot)}{|\boldsymbol{X}-\boldsymbol{X}_\odot|^3} - \frac{\alpha\omega^2(\boldsymbol{X}-\boldsymbol{X}_J)}{|\boldsymbol{X}-\boldsymbol{X}_J|^3}$$

At equilibrium, $\dot{\boldsymbol{X}} = \ddot{\boldsymbol{X}} = \boldsymbol{0}$,

$$\begin{pmatrix}X\\Y\end{pmatrix} = \boldsymbol{X} = \frac{(1-\alpha)(\boldsymbol{X}-\boldsymbol{X}_\odot)}{d_\odot^3} + \frac{\alpha(\boldsymbol{X}-\boldsymbol{X}_J)}{d_J^3} = \begin{pmatrix}\frac{(1-\alpha)(X+\alpha)}{d_\odot^3} + \frac{\alpha(X-1+\alpha)}{d_J^3}\\ \frac{(1-\alpha)Y}{d_\odot^3} + \frac{\alpha Y}{d_J^3}\end{pmatrix}$$

There are thus five equilibrium points, $L1$–$L5$; for $Y = 0$, write $X = \pm 1 + \xi$ and, keeping only lowest order terms,

$$L1 \approx a\begin{pmatrix}1-(\alpha/3)^{1/3}\\0\end{pmatrix}, \quad L2 \approx a\begin{pmatrix}1+(\alpha/3)^{1/3}\\0\end{pmatrix}, \quad L3 \approx -a\begin{pmatrix}1+\frac{5\alpha}{12}\\0\end{pmatrix}$$

These are not stable. For $Y \neq 0$, it follows that $d_\odot = d_J$, so $M_\odot$, M_J, m form an equilateral triangle,

$$L4, L5 = a\begin{pmatrix}\frac{1}{2}-\alpha\\ \pm\frac{\sqrt{3}}{2}\end{pmatrix}$$

They are stable when $(1-2\alpha)^2 \geqslant \frac{23}{27}$, i.e., α small enough.

4 Luminosity

The **luminosity** L is the *emitted* (visual) power output of an object. The **absolute magnitude** is a logarithmic scale of the luminosity, with respect to a reference luminosity $L_0 := 3.31\ 10^{28}W$ (approx., the output of the star Vega),

$$M_v := -\frac{5}{2}\log_{10}\frac{L}{L_0}$$

The Sun has a luminosity of $L_\odot = 3.8\ 10^{26}W$, so its absolute magnitude is 4.85, and relative to it,

$$M_v = 4.85 - \frac{5}{2}\log_{10}\frac{L}{L_\odot}.$$

Note that a small hot star may give out as much light as a large 'cool' star.

The *apparent luminosity* L' is the *received* light *intensity* after it has traveled a distance r, i.e.,

$$L' = \frac{L}{4\pi r^2}$$

Since relative comparisons are much easier to do, it was originally estimated as a ratio L'/L'_{Vega} with respect to the star Vega's apparent luminosity, $L'_{Vega} = 2.75\ 10^{-8}W/\mathrm{m}^2$. The **apparent magnitude** is a logarithmic scale of the intensity

$$\begin{aligned} m_v &:= -\tfrac{5}{2}\log_{10}\tfrac{L'}{L'_{Vega}} \\ &= -18.9 - \tfrac{5}{2}\log_{10}\tfrac{L}{4\pi r^2} && \text{S.I. units} \\ &= -26.74 - \tfrac{5}{2}\log_{10}\tfrac{L/L_\odot}{r^2} && r \text{ measured in a.u.} \\ &= -7.57 + M_v + 5\log_{10} r && r \text{ measured in light-years} \\ &= M_v + 5\log_{10}(r/10) && r \text{ measured in parsecs} \end{aligned}$$

For example,

	Light intensity(W/m^2)	apparent magnitude
Sun	1380	−26.74
Full Moon	0.00331	−12.7
Venus	0.00000023	−4.8
Sirius	0.0000001	−1.46

For light reflected from the Sun by an object at a distance r from the Sun and ρ from Earth, the brightness is approximately

$$\frac{L_\odot}{4\pi r^2} \times \frac{\pi D^2}{4} \times F \times A \times \frac{1}{4\pi\rho^2},$$

where D is the diameter of the object, F its phase, and A its 'albedo' (the fraction of light reflected); so the apparent magnitude is

$$\begin{aligned} m_v &= -26.74 + \frac{5}{2}\log_{10}\frac{16r^2\rho^2}{D^2FA} \qquad r, \rho, D \text{ in a.u.} \\ &= -23.7 + 5\log_{10}\frac{r\rho}{D\sqrt{FA}} \\ &= 17.2 + 5\log_{10}\frac{r\rho}{D\sqrt{FA}} \qquad r, \rho \text{ in a.u., } D \text{ in km} \end{aligned}$$

For a comet, the size of the coma depends on the distance from the Sun, i.e., D is proportional to r^{-1} (or r^{-2}), so $m_v = 17.2 + c + 5\log_{10}\frac{r^2\rho}{\sqrt{FA}}$.

References:

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