Banach Spaces

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1 Banach Spaces

Definition A **normed vector space** X is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a function called the norm $\|.\| : X \to \mathbb{R}$ such that,

$$\|x + y\| \leqslant \|x\| + \|y\|,$$

$$\|\lambda x\| = |\lambda| \|x\|,$$

$$\|x\| \geqslant 0, \|x\| = 0 \Leftrightarrow x = 0.$$

1.0.1 Easy Consequences

$$\|x - y\| \geqslant \|x\| - \|y\|, \quad \| - x\| = \|x\|,$$

$$\|x_1 + \ldots + x_n\| \leqslant \|x_1\| + \ldots + \|x_n\|.$$

1.0.2 Examples

1. The set of real numbers with the norm taken to be the absolute value.

2. $\mathbb{R}^N$ or $\mathbb{C}^N$ with the norm defined by $\|\mathbf{x}\|_2 = \left(\sum_i |x_i|^2\right)^{1/2}$. There are other possibilities however, $\|\mathbf{x}\|_1 = \sum_i |x_i|$ or $\|\mathbf{x}\|_\infty = \max_i |x_i|$. Thus $\left\|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right\|_1 = 3 + 4 = 7$, $\left\|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right\|_2 = \sqrt{9 + 16} = 5$, $\left\|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right\|_\infty = \max(3, 4) = 4$.

3. Sequences: sequences can be added and multiplied by scalars, and form a vector space. But there are different ways of taking the length of a sequence:

   The space $\ell^1 = \{ (a_n) : \sum_n |a_n| \leqslant \infty \}$ with norm defined by $\|(a_n)\|_{\ell^1} = \sum_n |a_n|$.

$\ell^2 = \{\, (a_n) : \sum_n |a_n|^2 \leqslant \infty \,\}$ with norm defined by $\|(a_n)\|_{\ell^2} = \sqrt{\sum_n |a_n|^2}$.
$\ell^\infty = \{\, (a_n) : |a_n| \leqslant C \,\}$ with norm defined by $\|(a_n)\|_{\ell^\infty} = \sup_n |a_n|$.

4. Functions: functions also form a vector space, and different norms can be defined for them as with sequences

   The space $L^1 = \{\, f : A \to \mathbb{R} : \int_A |f(x)|\, dx < \infty \,\}$, with norm defined by $\|f\|_{L^1} = \int_A |f(x)|$.

   The space $L^2 = \{\, f : A \to \mathbb{R} : \int_A |f(x)|^2\, dx < \infty \,\}$, with norm defined by $\|f\|_{L^2} = \sqrt{\int_A |f(x)|^2\, dx}$.

   The space $L^\infty = \{\, f : A \to \mathbb{R} : f \text{ is measurable, and } |f(x)| \leqslant C \; a.e. \,\}$, with norm defined by $\|f\|_{L^\infty} = \sup_{x \; a.e.} |f(x)|$ (i.e. the smallest c such that $|f(x)| \leqslant c \; a.e.$

   The space $C = \{\, f : [a,b] \to \mathbb{R} : f \text{ is continuous} \,\}$ with $\|f\|_C = \|f\|_{L^\infty} = \max_{x \in [a,b]} |f(x)|$.

Proposition 1.1 **A vector space is normed if and only if it has a translation invariant and scalar-homogeneous distance,**

$$d(x, y) = \|x - y\|.$$

Proof: Given a norm, the function $d(x,y) = \|x - y\|$ is translation invariant in the sense that $d(x + a, y + a) = \|x + a - y - a\| = d(x, y)$, scalar homogeneous in the sense that $d(\lambda x, \lambda y) = |\lambda| d(x, y)$. Conversely if $d(x, y)$ is translation invariant, then $d(x, y) = d(x - y, y - y) = d(x - y, 0)$ and we can define $\|x\| = d(x, 0)$ so that $d(x, y) = \|x - y\|$. We will now show that the axioms for the norm correspond precisely to the axioms for a distance.

d satisfies the triangle inequality $\Leftrightarrow$ the norm $\|.\|$ does.

$$d(x, y) \leqslant d(x, z) + d(z, y)$$

corresponds to $\|x - y\| \leqslant \|x - z\| + \|z - y\|$. Similarly $d(x, y) = d(y, x)$ corresponds to $\|x - y\| = \|y - x\|$; $d(x, y) \geqslant 0$ is $\|x - y\| \geqslant 0$, while $d(x, y) = 0 \Leftrightarrow x = y$ becomes $\|x - y\| = 0 \Leftrightarrow x - y = 0$.

The scale-homogeneity of the metric supplies the final axiom for the norm.

$\square$

This invariance under translations and scaling has the following easy consequences

Proposition 1.2

$$B_r(x) = rB_1(x/r) = x + B_r(0) = x + rB_1(0)$$

$$\overline{B_r(x)} = \{\, y : d(x,y) \leqslant r \,\}$$

Proof: The first statement is the set equivalent of

$$d(x,y) < r \Leftrightarrow d(x/r, y/r) < 1 \Leftrightarrow d(0, y-x) < r \Leftrightarrow d(0, (y-x)/r) < 1$$

Let a be a limit point of $B_r(x)$, so that we can find $y \in B_r(x)$ such that $\|y-a\| < \epsilon$, which implies that $\|a-x\| \leqslant \|a-y\| + \|y-x\| < \epsilon + r$. Hence $a \in \{\, y : d(x,y) \leqslant r \,\}$ as required.

Conversely, suppose $d(x,y) \leqslant r$; let $a = x + \lambda(y-x)$ with $1 - \epsilon/r < |lambda| < 1$. Then $\|a-x\| = |\lambda|\|y-x\| < r$, and $\|a-y\| = |1-\lambda|\|x-y\| < \epsilon$.

□

1.0.3 Note

Normed vector spaces are therefore metric spaces, as well as vector spaces. So we can apply ideas related to both, in particular open/closed sets, limit points, convergence of sequences, completeness, continuity, compactness, as well as linear subspaces, linear independence and spanning sets, linear transformations, kernels etc.

Since, at the end of the day, we are interested in issues like convergence and so on, we say that two norms are *equivalent* when they induce the same open sets.

For example, the norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ are all equivalent on $\mathbb{R}^N$. (But note that they are not equivalent for sequences or functions! In particular, a sequence of functions may converge in L^1 but not in L^∞ or vice-versa.)

1.0.4 Banach Spaces

When the induced metric is complete, the normed vector space is called a **Banach space**.

So, a closed linear subspace of a Banach space is itself a Banach space.

1.0.5 Example

Not every norm is complete of course. For example, suppose we take the vector space of continuous functions $f : [0,1] \to \mathbb{R}$ with the norm $\|f\| = \int_0^1 |f(x)|$. This is indeed a norm but it is not complete, for consider the sequence of continuous functions $f_n(x) = n(x - \frac{1}{2})\chi_{[\frac{1}{2},\frac{1}{2}+1/n]} + \chi_{(\frac{1}{2}+1/n,1]}$; it is Cauchy since $\|f_n - f_m\| = \int_0^1 |f_n - f_m| = \frac{1}{2}|1/n - 1/m| \to 0$ as $n, m \to \infty$, but it does not converge in $C[0,1]$, because suppose it does i.e. $\|f_n - f\| \to 0$ as $n \to \infty$. This means that $\int_0^1 |f_n(x) - f(x)|\,dx \to 0$, which implies that $\int_0^1 |f(x)|\,dx \to 0$, and $\int_{\frac{1}{2}+1/n}^1 |1 - f(x)|\,dx \to 0$, so that $f(x) = 0$ on $[0, \frac{1}{2})$ and $f(x) = 1$ on $(\frac{1}{2}, 1]$, implying it is discontinuous.

1.1 Metric and Vector Properties

Proposition 1.3 **The maps $x \mapsto x + y$, $x \mapsto \lambda x$ and $x \mapsto \|x\|$ are continuous.**

Proof: Addition by a vector is continuous,

$$\|x_1 - x_2\| < \epsilon \;\Rightarrow\; \|(x_1 + y) - (x_2 + y)\| = \|x_1 - x_2\| < \epsilon.$$

Scalar multiplication is continuous,

$$\|x_1 - x_2\| < \epsilon \;\Rightarrow\; \|\lambda x_1 - \lambda x_2\| = |\lambda|\|x_1 - x_2\| < |\lambda|\epsilon.$$

The norm is continuous,

$$\|x_1 - x_2\| < \epsilon \;\Rightarrow\; |\|x_1\| - \|x_2\|| \leqslant \|x_1 - x_2\| < \epsilon.$$

Corollary

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$
$$\lim_{n\to\infty} \lambda x_n = \lambda \lim_{n\to\infty} x_n.$$

Proposition 1.4 **If M is a linear subspace of X, then so is $\bar{M}$.**

Proof: Let x, y be limit points of M. Then we can find a sequence $x_n \to x$ and $y_n \to y$; but $x_n + y_n \in M$, so $x + y = \lim_{n\to\infty}(x_n + y_n) \in \bar{M}$. Similarly $\lambda x = \lim_{n\to\infty}(\lambda x_n) \in \bar{M}$.

$\square$

For example, the set of polynomials forms a linear subspace of $C[0,1]$; its closure in this space is a closed linear subspace (and hence complete).

1.1.1 Series

A **series** is a sequence of vectors obtained by adding, i.e. $\sum_{n=0}^{N} x_n$. A series therefore converges when $\|\sum_{n=0}^{N} x_n - x\| \to 0$ as $N \to \infty$. It is said to converge **absolutely** when $\sum_n \|x_n\|$ converges in $\mathbb{R}$.

We can show that a series is absolutely convergent by using the usual tests for real series, for example, the comparison, ratio or root tests.

Proposition 1.5 **For a Banach space, absolutely convergent series converge.**

Proof: Suppose that $\sum_n \|x_n\|$ converges. Let $y_N = \sum_{n=0}^{N} x_n$, so that $\|y_M - y_N\| = \|\sum_{n=N+1}^{M} x_n\| \leqslant \sum_{n=N+1}^{M} \|x_n\| \to 0$. Hence y_N is a Cauchy sequence in X, and so converges to y, say. This means that $\sum_n x_n = y$.

It is also true (see the exercises) that if a normed vector space is such that all its absolutely convergent series converge, then the space is also complete, i.e. a Banach space.

1.1.2 Quotient Spaces

If M is a closed linear subspace, then we can factor it out and form the space X/M with addition, scalar multiplication and norm defined by

$$\begin{gathered}(x+M)+(y+M)=(x+y)+M,\\ \lambda(x+M)=\lambda x+M,\\ \|x+M\|=d(x,M)=\inf_{v\in M}\|x-v\|.\end{gathered}$$

That the addition and scalar multiplication satisfy the axioms of a vector space is a triviality; let us show that we do indeed get a norm:

$$\begin{aligned}\|(x+M)+(y+M)\| &= \|x+y+M\| = \inf_{v\in M}\|x+y-v\|\\ &= \inf_{v_1,v_2\in M}\|x+y-v_1-v_2\|\\ &\leqslant \inf_{v_1\in M}\|x-v_1\| + \inf_{v_2\in M}\|y-v_2\|\\ &= \|x+M\|+\|y+M\|\end{aligned}$$

$$\begin{aligned}\|\lambda(x+M)\| &= \|\lambda x+M\|\\ &= \inf_{v\in M}\|\lambda x-v\|\\ &= \inf_{u\in M}\|\lambda x-\lambda u\|\\ &= \inf_{u\in M}|\lambda|\|x-u\|\\ &= |\lambda|\|x+M\|\end{aligned}$$

$$\|x + M\| = \inf_{v \in M} \|x - v\| \geqslant 0.$$

$$\|x + M\| = 0 \Leftrightarrow \inf_{v \in M} \|x - v\| = 0 \Leftrightarrow x \in \bar{M} = M \Leftrightarrow x + M = 0 + M.$$

Proposition 1.6 **If X is complete, and M is a closed linear subspace, then X/M is also complete.**

Proof: let $x_n + M$ be an absolutely convergent series i.e. $\sum_n \|x_n + M\|$ converges. Now $\forall n > 0, \exists v_n \in M, \|x_n - v_n\| \leqslant \|x_n + M\| + 1/2^n$, so the left hand side converges by comparison with the right. So $\sum_n (x_n - v_n) = x$ since X is complete, and thus

$$\|\sum_{n=1}^{N} (x_n + M) - (x + M)\| = \|\sum_{n=1}^{N} x_n - x + M\| \leqslant \|\sum_{n=1}^{N} (x_n - v_n) - x\| \to 0$$

1.1.3 Exercises

1. Show that the norms defined for $\mathbb{R}^N$ and $\mathbb{C}^N$ are indeed norms.

2. Show that the norms defined for ℓ^1 and ℓ^∞ are indeed norms.

3. Show that the norm defined for L^1 is indeed a norm, except that it may happen that $\int_A |f(x)|\,dx = 0$ without $f = 0$; however from the theory of Lebesgue integration, in this case we get $f = 0 a.e.$, so that the failure of this axiom is not drastic. In fact we can identify those functions that are equal almost everywhere into equivalence classes and work with these, but the notation would be too pedantic to be useful.

4. Suppose that $\|.\|_1$ and $\|.\|_2$ are two norms on a vector space X, such that there are positive constants c, d,

   $$\|x\|_1 \leqslant c\|x\|_2, \quad \|x\|_2 \leqslant d\|x\|_1.$$

   Show that every ball of norm 1 has a smaller ball of norm 2 with the same centre, and vice-versa. Deduce that the two norms have the same open sets. Show that the converse is also true.

5. Show that in $\mathbb{R}^N$, the three norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ defined previously are all equivalent. What do the unit balls in each norm look like?

6. Show that if $\lambda_n \to \lambda$ and $x_n \to x$ then $\lambda_n x_n \to \lambda x$. In Corollary 1.4, strictly speaking, we can only immediately deduce that $\lim_{n\to\infty}(x_n + y) = x + y$; give a proper proof of the corollary.

7. Show that if a series $\sum_n x_n$ converges, then $x_n \to 0$. (Hint: convergent sequences are Cauchy)

8. * Let X be a normed vector space for which every absolutely convergent series converges. Show that X is complete as follows: let x_n be a Cauchy sequence in X, and let $y_n := x_{n+1} - x_n$; show that there is a subsequence y_{n_r} such that $\|y_{n_r}\| \leqslant 1/2^r$ and so $\sum_r \|y_{n_r}\| \leqslant 1$. Deduce that $\sum_{r=0}^{N} y_{n_r}$ converges, and hence x_n.

2 Continuous Linear Operators

Definition An **operator** is a linear continuous transformation $T : X \to Y$ between normed vector spaces,

$$T(x+y) = Tx + Ty, \quad T(\lambda x) = \lambda Tx.$$

In particular, a **functional** is a continuous linear map $\phi : X \to \mathbb{C}$.

The set of operators is denoted by $B(X, Y)$. The set of functionals is called the **dual space**.

Examples of functionals on sequences are $\sum_n \alpha_n a_n$, on functions $\int k(x) f(x)\, dx$.

In general linear maps are not continuous. For example, let X be the vector space of finite real sequences with the ℓ^1 norm. Consider the functional $\phi(\mathbf{x}) = \sum_n n a_n$ (check linearity!), and the sequence of sequences $\mathbf{x}_1 = (1, 0, \ldots), \mathbf{x}_2 = (0, \frac{1}{2}, 0, \ldots), \mathbf{x}_3 = (0, 0, 1/3, \ldots), \ldots$. Then $\|\mathbf{x}_r\|_{\ell^1} = \sum_n |a_n| = 1/n \to 0$ as $n \to \infty$, so that $\mathbf{x}_n \to \mathbf{0}$, yet $\phi(\mathbf{x}_n) = n \cdot \frac{1}{n} = 1 \not\to 0$.

Another important example is differentiation: although the map is linear (say on the vector space of polynomials) it is not continuous eg $D\cos(nx) = -n\sin(nx)$ so $\|D\cos(nx)\|_C = n$ cannot be bounded by a multiple of $\|\cos(nx)\|_C = 1$.

A simple test for continuity of a linear transformation is the following "bounded" property,

Proposition 2.1 **A linear transformation is continuous if, and only if,**

$$\exists c > 0, \forall x \in X, \quad \|Tx\|_Y \leqslant c\|x\|_X$$

Proof: If T is linear and continuous, then

$$\exists \delta > 0, \|y\| < \delta \;\Rightarrow\; \|Ty\| < 1$$

In particular, putting $y = \frac{1}{2}\delta x/\|x\|$, we get $\|y\| = \frac{1}{2}\delta < \delta$, so that $\|Tx\| = |2\|x\|/\delta|\|Ty\| < \frac{2}{\delta}\|x\|$.

Conversely, suppose T is bounded, then

$$\|Tx - Ty\| = \|T(x-y)\| \leqslant c\|x - y\|.$$

So for any $\epsilon > 0$, let $\delta = \epsilon/c$ and

$$\|x - y\| < \delta \;\Rightarrow\; \|Tx - Ty\| < \epsilon$$

Proposition 2.2 **The kernel of an operator is a closed linear subspace.**

Proof: If $Tx = 0$ and $Ty = 0$ then $T(x+y) = Tx + Ty = 0$ and $T(\lambda x) = \lambda Tx = 0$, so that the kernel is a linear subspace. Let x be a limit point of the kernel i.e. there is a sequence $x_n \to x$ with $Tx_n = 0$. Then $Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = 0$, and x is in the kernel.

$\square$

Definition An **isomorphism** between normed vector spaces is a map $T : X \to Y$ such that both T and T^{-1} are linear and continuous (ie T is a linear homeomorphism).

An **isometry** between normed vector spaces is a bijective linear map $T : X \to Y$ such that $\|Tx\| = \|x\|$

It follows, of course, that isometries are isomorphisms (since continuity of T follows from the bounded property, and $\|T^{-1}x\| = \|TT^{-1}x\| = \|x\|$ as well).

Theorem 2.3 **Every N-dimensional normed vector space is isomorphic to $\mathbb{C}^N$, and so is complete.**

Proof: Let $T : \mathbb{C}^N \to X$ be the standard map $T \begin{pmatrix} x_1 \\ \vdots x_N \end{pmatrix} := x_1 e_1 + \ldots x_N e_N$ where $e_1, \ldots, e_N$ form a basis (wolog of unit vectors) for X. That T is linear and bijective is trivial to show.

T is continuous since

$$\begin{aligned}
\|T\underline{x}\|_X &= \|x_1 e_1 + \ldots x_N e_N\| \\
&\leqslant |x_1| + \ldots + |x_N| \\
&= \sqrt{(\textstyle\sum_i |x_i|)^2} \\
&= \sqrt{\textstyle\sum_{ij} |x_i||x_j|} \\
&\leqslant \sqrt{\tfrac{1}{2}\textstyle\sum_{ij} |x_i|^2 + |x_j|^2} \\
&= \sqrt{N}\sqrt{\textstyle\sum_i |x_i|^2} \\
&= \sqrt{N}\|\underline{x}\|_2
\end{aligned}$$

Moreover T^{-1} is continuous, since let $f(\underline{x}) := \|T\underline{x}\|$, which is a composition of two continuous functions. Now the "sphere" $S = \{\, \underline{u} : \|\underline{u}\|_2 = 1 \,\}$ is a compact set (since it is closed and bounded in $\mathbb{C}^N$), so fS is also compact. Suppose $0 \in fS$; this would mean that $0 = \|T\underline{u}\|$ for some $\underline{u}$ with $\|\underline{u}\|_2 = 1$. But we get that $T\underline{u} = 0$, and T is invertible, so $\underline{u} = \underline{0}$, a contradiction. Hence $0 \notin fS$, so 0 is in the exterior of fS (since fS is closed), so it is contained in a ball $-c < 0 < c$ completely outside fS. This means that $c \leqslant \|T\underline{u}\|$, for any unit vector $\underline{u}$. Hence in general, $c\|\underline{x}\|_2 \leqslant \|T\underline{x}\|$ for any vector $\underline{x}$. By exercise, we get that T^{-1} is also continuous.

□

Theorem 2.4 **The Open Mapping Theorem**

Every onto continuous linear map $T : X \to Y$ between Banach spaces, maps open sets to open sets.

Proof: Let U be an open subset of X, and let $Tx \in TU$, so that $x \in B_\epsilon(x) \subseteq U$. We will show that $TB_1(0)$ contains a ball $B_r(0)$, from which follows that

$$Tx \in B_{r\epsilon}(Tx) = Tx + \epsilon B_r(0) \subseteq Tx + \epsilon TB_1(0) = TB_\epsilon(x) \subseteq TU$$

proving that TU is an open set.

Let $F = \overline{TB_1(0)}$, and suppose that it does not contain any open balls (i.e. it has no interior points). It follows that λF also contains no open balls for any $\lambda > 0$. Since F is bounded by $\|T\|$ it is possible to find a ball B_1 of radius less than 1 such that $\overline{B}_1 \cap F = \varnothing$. In general, suppose we have a ball B_n of radius at most $1/n$ such that $\overline{B}_n \cap nF = \varnothing$. Then $B_n \not\subseteq (n+1)F$, and so $B_n - (n+1)F$ is a non-empty open set, which must contain an open ball. By reducing its radius if necessary, we get a ball B_{n+1} of radius at most $1/(n+1)$ and such that $\overline{B_{n+1}} \cap (n+1)F = \varnothing$. In this way, we get a nested sequence of balls $\overline{B_{n+1}} \subseteq B_n$ with diminishing radii. Their centers y_n form a Cauchy sequence since $\|y_n - y_m\| \leqslant \max(1/n, 1/m)$, and so converge to some $y \in Y$. Now, for any $m \geqslant n, y_m \in B_n$, so $y \in \overline{B_n}$ for any n.

However $y = Tx$ for some $x \in X$ since T is onto. Choose $n > \|x\|$, to get $x \in nB_1(0)$, and so $y = Tx \in nTB_1(0) \subseteq nF$, which contradicts $y \in \overline{B_n} \subseteq (nF)'$. Thus F must contain an open ball $B_{8r}(a) = a + B_{8r}(0)$; by translating by $a \in F$, and scaling, we get $B_r(0) \subseteq \frac{1}{4}F = \overline{TB_{1/4}(0)}$.

Claim: $B_r(0) \subseteq TB_1(0)$. Let $y \in B_r(0) \subseteq \overline{TB_{1/4}(0)}$, so that there must be an $x_1 \in B_{1/4}(0)$ such that $\|y - Tx_1\| < r/2$; that is, $\|x_1\| < 1/4$ and $y - Tx_1 \in B_{r/2}(0) \subseteq \overline{TB_{1/8}(0)}$. But this implies that there is an $x_2 \in B_{1/8}(0)$ such that $\|y - Tx_1 - Tx_2\| < r/4$. We therefore get a sequence x_n such that

$$\|x_n\| < \frac{1}{2^{n+1}}, \quad \|y - T(x_1 + \ldots + x_n)\| < \frac{r}{2^n}$$

We conclude that $x := \sum_n x_n$ converges, with $\|x\| \leqslant \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$, and that $y = Tx$. Thus $x \in B_1(0)$ and $y = Tx \in TB_1(0)$, proving the claim.

□

Corollary **If $T : X \to Y$ is in addition also 1-1, then T^{-1} is continuous and X is isomorphic to Y.**

Theorem 2.5 **If Y is complete, then $B(X,Y)$ is a Banach space, with the norm defined by**

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

In particular X^* is a Banach space, with norm defined by

$$\|\phi\| = \sup_{x\neq 0} \frac{|\phi(x)|}{\|x\|}.$$

Proof: That $B(X,Y)$ is a vector space is a triviality. The norm is well-defined in the sense that if T is an operator, then $\|Tx\| \leqslant c\|x\|$ and $\|T\| \leqslant c$. Thus,

$$\|Tx\| \leqslant \|T\|\|x\|.$$

$$\begin{aligned}
\|S+T\| &= \sup_{x\neq 0} \tfrac{\|Sx+Tx\|}{\|x\|} \\
&\leqslant \sup_{x\neq 0} \tfrac{\|Sx\|}{\|x\|} + \sup_{x\neq 0} \tfrac{\|Tx\|}{\|x\|} \\
&= \|S\| + \|T\|
\end{aligned}$$

$$\begin{aligned}
\|\lambda T\| &= \sup_{x\neq 0} \tfrac{\|\lambda Tx\|}{\|x\|} \\
&= |\lambda|\|T\|
\end{aligned}$$

$$\begin{aligned}
\|T\| = 0 &\Leftrightarrow \|Tx\| = 0, \forall x = 0, \\
&\Leftrightarrow T = 0
\end{aligned}$$

The resulting normed vector space is complete. Let T_n be a Cauchy sequence in $B(X,Y)$, that is, $\|T_n - T_m\| \to 0$ as $n, m \to \infty$. So

$$\|T_n(x) - T_m(x)\| \leqslant \|T_n - T_m\|\|x\| \to 0$$

This means that $T_n(x)$ is a Cauchy sequence in Y, which is complete, so that $T_n(x) \to T(x)$ as $n \to \infty$ for each $x \in X$. We now show that T is linear and continuous:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty}(T_n(x)+T_n(y)) \\ &= T(x)+T(y) \end{aligned}$$

$$\begin{aligned} T(\lambda x) &= \lim_{n\to\infty} T_n(\lambda x) \\ &= \lim_{n\to\infty} \lambda T_n(x) \\ &= \lambda T(x) \end{aligned}$$

$$\begin{aligned} \|T(x)\| &\leqslant \|T_n(x)\| + \|T(x)-T_n(x)\| \\ &\leqslant \|T_n\|\|x\| + \epsilon\|x\| \\ &\leqslant c\|x\| \end{aligned}$$

since T_n is a Cauchy sequence and so is bounded.

Finally $T_n \to T$ since $\|(T_n - T)x\| \leqslant \|T_n - T_m\|\|x\| + \|T_m x - Tx\| \leqslant \epsilon\|x\| + \epsilon\|x\|$, so that $\|T_n - T\| \leqslant 2\epsilon$ and $\|T_n - T\| \to 0$ as $n \to \infty$.

□

Proposition 2.6 **If S and T are compatible operators, then so is ST, with $\|ST\| \leqslant \|S\|\|T\|$.**

Proof: That ST is linear is obvious, $ST(x+y) = S(Tx+Ty) = STx+STy$ and $ST(\lambda x) = S(\lambda Tx) = \lambda STx$. Also,

$$\|STx\| \leqslant \|S\|\|Tx\| \leqslant \|S\|\|T\|\|x\|$$

and taking the supremum on both sides gives $\|ST\| \leqslant \|S\|\|T\|$.

Corollary $B(X) := B(X,X)$ **is closed under multiplication (called a Banach algebra).**

2.0.4 Examples

The right-shift operator $R : \ell^1 \to \ell^1$ is defined by $R(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots)$. It is obviously linear, and also bounded since

$$\|R\mathbf{x}\| = \|(0, x_0, x_1, \ldots)\|_{\ell^1} = \sum_{n=0}^{\infty} |x_n| = \|\mathbf{x}\|_{\ell^1}$$

The multiplication operator $T : \ell^1 \to \ell^1$ defined by $T(x_0, x_1, \ldots) = (\alpha_0 x_0, \alpha_1 x_1, \ldots)$, where (α_n) is a bounded sequence, is also linear, and bounded since

$$\|T\mathbf{x}\| = \sum_{n=0}^{\infty} |\alpha_n x_n| \leqslant M \sum_{n=0}^{\infty} |x_n| = M\|\mathbf{x}\|$$

The **Fourier** transform on a function $f \in L^1[0,1]$ is defined to be the sequence of numbers $(\hat{f})$ where

$$\hat{f}(n) = \mathcal{F}f(n) = \int_0^1 e^{-2\pi inx} f(x)\,dx$$

Similarly the Fourier transform on a function $f \in L^1(\mathbb{R})$ is defined to be the function

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f(x)\,dx$$

It is easy to show that the Fourier transform is linear; moreover it is continuous with $\mathcal{F} : L^1[0,1] \to \ell^\infty$ and $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$, since

$$\|\hat{f}\| = \sup_n |\int_0^1 e^{-2\pi inx} f(x)\,dx| \leqslant \int_0^1 |f(x)|\,dx = \|f\|_{L^1[0,1]}$$

and similarly for the second case. In fact, one can show further that $\mathcal{F} : L^1[0,1] \to c_0$ and $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$.

More generally, any transformation $Tf(\xi) = \int k(x,\xi)f(x)dx$ where $k(x,\xi)$ is a bounded function, is an operator.

2.1 The Dual Space

Functionals are simply row vectors when $X = \mathbb{R}^N$; thus X^* is isomorphic to $\mathbb{R}^N$ and is generated by a dual basis.

Examples: in ℓ^1, we have already seen that $\sum_n \alpha_n x_n$ is a functional when α_n are bounded. Also, $\int f(x)dx$ is a functional on L^1. Another example, for C, is given by $\delta_0(f) = f(0)$, which has norm $\|\delta_0\| = 1$.

There is a certain duality between X and its dual space X^* that we will explore below. The connection between the two is the following: **Definition**

The **annihilator** of a set of vectors A is the set of functionals $A^\circ = \{\, \phi \in X^* : \phi(x) = 0, \forall x \in A \,\}$.

2.1.1 The Hahn-Banach Theorem

Theorem 2.7 **Let Y be a subspace of a normed vector space, X. Then every functional $\phi \in Y^*$ can be extended to $\tilde{\phi} \in X^*$, that is $\tilde{\phi}(y) = \phi(y), \forall y \in Y$ and $\|\tilde{\phi}\|_{X^*} = \|\phi\|_{Y^*}$.**

Proof: Let us try to extend ϕ from a functional on Y to a functional on $Y + \langle v \rangle$ for any vector v. The only possibility is to let $\tilde{\phi}(y + \lambda v) = \phi(y) + \lambda c$

(where c would be equal to $\tilde{\phi}(v)$). Whatever c, this gives a linear map (easy to check). For certain values of c, this is also continuous:

$$
\begin{aligned}
|\tilde{\phi}(y+\lambda v)| &= |\phi(y)+\lambda c| \\
&= |\lambda||\phi(y/\lambda)+c|
\end{aligned}
$$

We thus want $|\phi(y/\lambda)+c| \leqslant \|\phi\|\|y/\lambda+c\|$, which is equivalent to finding a c such that

$$
-\phi(y)-\|\phi\|\|y+v\| \leqslant c \leqslant -\phi(y)+\|y+v\|, \qquad \forall y \in Y.
$$

Is this possible? Yes because $|\phi(y_1-y_2)| \leqslant \|\phi\|\|y_1-y_2\| \leqslant \|\phi\|(\|y_1+v\|+\|y_2+v\|)$, so we get (when ϕ is real-valued)

$$
\phi(y_1)-\phi(y_2) \leqslant \|\phi\|\|y_1+v\|+\|\phi\|\|y_2+v\|
$$

i.e.

$$
-\phi(y_2)-\|\phi\|\|y_2+v\| \leqslant -\phi(y_1)+\|\phi\|\|y_1+v\|, \qquad \forall y_1, y_2
$$

Hence there must be a constant c separating the two sides of the inequality. Thus $\|\tilde{\phi}\| \leqslant \|\phi\|$; in fact, equality holds because for $y \in Y$, the supremum of $|\tilde{\phi}(y)|/\|y\| = |\phi(y)|/\|y\|$ is $\|\phi\|$. A different proof (see reference book) shows that this is also the case when ϕ is complex-valued.

If X can be generated by a countable number of vectors v_n (X is called separable) then we can keep on extending until $\tilde{\phi}$ is a functional on X. But even if X needs an uncountable number of generating vectors, then can apply "Zorn's lemma" to conclude that the extension goes through to X (see book).

□

Corollary

$$
x=0 \Leftrightarrow \forall \phi \in X^*, \phi(x)=0
$$

Proof: Given $x \neq 0$, form the one-dimensional subspace $Y = [\![x]\!] = \{\, \lambda x : \lambda \in \mathbb{C} \,\}$. Let $\psi(\alpha x) = \alpha\|x\|$, then $\psi \in Y^*$ since

$$
|\psi(\alpha x)| = |\alpha|\|x\| = \|\alpha x\|
$$

in fact $\|\phi\| = 1$. So ϕ can be extended to a functional on all of X. Hence there is a $\phi = \tilde{\psi} \in X^*$ such that $\phi(x) = \|x\| \neq 0$.

The converse statement is trivial to prove.

□

Corollary

$$\|x\| = \sup_{\phi \neq 0} \frac{|\phi(x)|}{\|\phi\|}$$

Proof: Since $|\phi(x)| \leqslant \|\phi\|\|x\|$ we get that $\frac{|\phi(x)|}{\|\phi\|} \leqslant \|x\|$. We are left to show that the left hand expression approximates the right as closely as necessary. By the theorem, there is a $\phi \in X^*$ with $\phi(x) = \|x\|$ and $\|\phi\| = 1$, so for this functional, $|\phi(x)|/\|\phi\| = \|x\|$.

$\square$

2.1.2 X^{**}

Proposition 2.8 **Every normed vector space X is *embedded* in its double dual X^{**}. That is, there is a 1-1 linear isometry $J : X \to X^{**}$.**

Proof: Given $x \in X$, let $x^* : X^* \to \mathbb{C}$ be the map $x^*(\phi) = \phi(x)$. Then $x^* \in X^{**}$, ie.it is linear and bounded with $\|x^*\| = \|x\|$. Hence we can form the map $J : X \to X^{**}$ by $Jx = x^*$.

J is linear since for any $\phi \in X^*$,

$$(x+y)^*(\phi) = \phi(x+y) = \phi(x) + \phi(y) = x^*(\phi) + y^*(\phi).$$

$$(\lambda x)^*(\phi) = \phi(\lambda x) = \lambda\phi(x) = \lambda x^*(\phi).$$

J is isometric (preserves the norm) since

$$\|x^{**}(\phi)\| = |\phi(x)| \leqslant \|\phi\|\|x\|,$$

and there is a functional ψ such that $\psi(x) = \|x\|$ and $\|\psi\| = 1$, so that

$$\|x^{**}\| = \sup_{\phi \neq 0} \frac{x^{**}(\phi)}{\|\phi\|} = \|x\|.$$

That J is 1-1 follows from this isometry, since $x^{**} = y^{**} \Leftrightarrow (x-y)^{**} = 0 \Leftrightarrow \|(x-y)^{**}\| = 0 \Leftrightarrow \|x-y\| = 0 \Leftrightarrow x = y$.

$\square$

2.1.3 Completion of X

Given any normed vector space X, the double dual X^{**} is a Banach space. Hence the closure $\bar{J}X$ is a closed linear subspace of X^{**}, and so is itself complete i.e. a Banach space. It is called the **completion** of X, usually denoted $\hat{X}$.

2.1.4 Weak convergence

We already know what $T_n \to T$ means, namely, $\|T_n - T\| \to 0$. This of course implies pointwise convergence ie $\forall x,\ T_n x \to Tx$ since for each x, $\|T_n x - Tx\| = \|(T_n - T)x\| \leqslant \|T_n - T\|\|x\| \to 0$; but the converse is false ie it is possible to have pointwise convergence without $T_n \to T$. For this reason $T_n \to T$ is sometimes called **convergence in norm**.

Example: Let $T_n : \ell^1 \to \ell^1$ be defined by $T_n(x_i) = (0, \ldots, 0, x_n, x_{n+1}, \ldots)$ (well-defined, linear and continuous); we have pointwise convergence, $T_n \underline{x} \to \underline{0}$ since for each $\underline{x} = (x_i)$, $\|T_n \underline{x}\|_{\ell^1} = \|(0, \ldots, 0, x_n, x_{n+1}, \ldots)\|_{\ell^1} = |x_n| + |x_{n+1}| + \ldots \to 0$ since $\sum_{i=1}^{n} |x_i|$ converges. However $T_n \not\to 0$ since we can always find sequences $\underline{x} = (0, \ldots, 0, 1, 0, \ldots)$ for which $\|T_n \underline{x}\| = 1 = \|\underline{x}\|$ so that $\|T_n\| = 1 \not\to 0$.

There is yet another type of convergence, called **weak convergence**, defined as follows $T_n \rightharpoonup T \Leftrightarrow \forall x \in X, \forall \phi \in X^*,\ \phi T_n x \to 0$ as $n \to \infty$. Pointwise convergence implies weak convergence because $T_n x \to Tx$ and ϕ is continuous. However the converse is again false (see exercise in Chapter 3).

Proposition 2.9 **In finite dimensions, all these convergence types are equivalent.**

Proof: Let $A_n \rightharpoonup A$ where A_n, A are matrices. This means that $\underline{y}^\top (A_n - A)\underline{x} \to 0$ as $n \to \infty$. In particular if we let $\underline{y} = \underline{e}_i, \underline{x} = \underline{e}_j$ be basis vectors then $A_{n,ij} = \underline{e}_i A_n \underline{e}_j \to \underline{e}_i A \underline{e}_j = A_{ij}$, so that each component of the matrices A_n converges to the corresponding component in A. This then implies that $\|A_n - A\| \leqslant \sqrt{\sum_{ij} |A_{n,ij} - A_{ij}|^2} \to 0$.

□

2.2 The Adjoint $T^\top$

Definition The **adjoint** (or transpose) of a linear transformation $T : X \to Y$ is $T^\top : Y^* \to X^*$ defined by $T^\top \phi(x) = \phi(Tx)$ for any $\phi \in Y^*$.

It is well defined because

$$\|T^\top \phi(x)\| = \|\phi(Tx)\| \leqslant \|\phi\|\|Tx\| \leqslant \|\phi\|\|T\|\|x\|$$

so that $T^\top \phi \in X^*$. It is also linear since $T^\top(\phi + \psi)(x) = (\phi + \psi)(Tx) = \phi(Tx) + \psi(Tx) = T^\top \phi(x) + T^\top \psi(x)$ and $T^\top(\lambda\phi)(x) = \lambda\phi(Tx) = \lambda T^\top \phi(x)$.

If T is continuous and linear, then so is its adjoint, since by the above inequalities, $\|T^\top \phi\| \leqslant \|T\|\|\phi\|$; in fact $\|T^\top\| \leqslant \|T\|$.

Proposition 2.10

$$(S+T)^\top = S^\top + T^\top, (\lambda T)^\top = \lambda T^\top, \|T^\top\| = \|T\|.$$

Proof: $(S+T)^\top\phi(x) = \phi(Sx+Tx) = \phi(Sx)+\phi(Tx) = S^\top\phi(x)+T^\top\phi(x)$; since this is true for all x and all ϕ, the result follows.

Similarly, $(\lambda T)^\top\phi(x) = \phi(\lambda Tx) = \lambda\phi(Tx) = \lambda T^\top\phi(x)$.

$$\|T\| = \sup_{x\neq 0}\frac{\|Tx\|}{\|x\|} = \sup_{x\neq 0}\sup_{\phi\neq 0}\frac{|\phi(Tx)|}{\|x\|\|\phi\|}$$

$$= \sup_{\phi\neq 0}\sup_{x\neq 0}\frac{|\phi(Tx)|}{\|x\|\|\phi\|} = \sup_{\phi\neq 0}\sup_{x\neq 0}\frac{|T^\top\phi(x)|}{\|\phi\|\|x\|} = \sup_{\phi\neq 0}\frac{\|T^\top\phi\|}{\|\phi\|} = \|T^\top\|$$

□

Proposition 2.11

$$\ker T^\top = (\operatorname{im} T)^\circ, \quad \operatorname{im} T^\top \subseteq (\ker T)^\circ.$$

Proof: $T^\top\phi = 0 \Leftrightarrow \forall x, T^\top\phi(x) = 0 \Leftrightarrow \forall x, \phi(Tx) = 0 \Leftrightarrow \phi \operatorname{im} T = 0 \Leftrightarrow \phi \in (\operatorname{im} T)^\circ$.

Let $\psi \in \operatorname{im} T^\top$, ie $\psi = T^\top\phi$. Then for any $x \in \ker T$, we get that $\psi(x) = T^\top\phi(x) = \phi(Tx) = 0$.

□

Proposition 2.12 **T has a continuous inverse $\Leftrightarrow$ T and $T^\top$ are onto.**

Proof: $T^\top$ is onto $\Leftrightarrow$ T is 1-1, which implies that T is invertible and continuous by the open mapping theorem.

□

2.2.1 Exercises

1. Let $L : \ell^\infty \to \ell^\infty$ be defined by $L(x_n) = (x_{n+1})$. Show that L is linear and continuous.

2. For T invertible and linear, show that T^{-1} is continuous $\Leftrightarrow$ $c\|x\|_X \leqslant \|Tx\|_Y$, for some $c > 0$.

3. Show that if T and T^{-1} are operators, then $\|T^{-1}\| \geqslant \|T\|^{-1}$.

4. Let $T : \ell^1 \to \ell^1$ be defined by $T(x_n) = (x_n/n)$. Show that T is linear and continuous, and that T^{-1} is linear but not continuous. Show however that there are no sequences $\mathbf{x}$ such that $T\mathbf{x} = 0$.

5. Show that $(A+B)^\circ = A^\circ \cap B^\circ$, and $A^\circ + B^\circ \subseteq (A \cap B)^\circ$.

6. The Hahn-Banach theorem is trivial when Y is *dense* in X ie $\bar{Y} = X$, as follows: for $x = \lim_{n\to\infty} y_n$, define $\tilde{\phi}(x) := \lim_{n\to\infty} \phi(y_n)$. Show that $\tilde{\phi}$ is well-defined, linear and continuous with $\|\tilde{\phi}\| = \|\phi\|$.

7. Show that integration, $f \mapsto \int f$, is a functional in L^1 by showing that it is bounded.

8. Define the functional ϕ on the vector space of step functions in L^1, by $\phi(f) = \int f$. Use the Hahn-Banach theorem to show that this extends to a linear and continuous functional on L^1.

9. * Show that, for X and Y finite dimensional, then $\|T\| = \sqrt{\sum_{ij} |T_{ij}|^2}$.

10. A set is called *weakly bounded* when $\forall \phi \in X^*, \phi A$ is bounded. Show that a set is weakly bounded if, and only if, it is bounded. (Hint: use the fact that if $x^{**}(\phi) \leqslant c_\phi, \forall \phi$, then $\|x^{**}\| \leqslant c$ - called the uniform bounded principle)

11. In the embedding of X in X^{**}, show that $T^{\top\top} : X^{**} \to Y^{**}$ is an extension of $T : X \to Y$ in the sense that $T^{\top\top} x^{**} = (Tx)^{**}$.

12. * The corollary to the Hahn-Banach theorem, which states that there is a functional $\phi \in X^*$ such that $\phi(x) \neq 0$ can be strengthened as follows. Let M be any closed linear subspace, and let $x \notin M$. Let $Y = M \oplus [\![x]\!]$, and define $\phi(y) = 0$ for $y \in M$ and $\phi(\alpha x) = \alpha d(x, M)$. Show that ϕ is a functional on Y, and hence that it can be extended to X, while still having the properties that $\phi(x) \neq 0$ and $\phi(y) = 0$ for $y \in M$.

3 Sequence Spaces

3.1 The space ℓ^1

Proposition 3.1 ℓ^1 **is complete**

Proof: Let $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \ldots$ be an absolutely summable series in ℓ^1, ie $\sum_n \|\boldsymbol{x}_n\|_{\ell^1} = s$. Thus $|x_{ni}| \leqslant \|\boldsymbol{x}_n\|_{\ell^1}$, and so for each i, $\sum_n |x_{ni}|$ converges in $\mathbb{R}$ by comparison. Let $y_i = \sum_n x_{in}$.

$\boldsymbol{y} = \sum_n \boldsymbol{x}_n$ since

$$\begin{aligned} \|y - \textstyle\sum_n^N \boldsymbol{x}_n\|_{\ell^1} &= \textstyle\sum_{i=1}^\infty |y_i - \sum_n^N x_{ni}| \\ &= \textstyle\sum_{i=1}^\infty |\sum_{n=N+1}^\infty x_{ni}| \end{aligned}$$

But

$$\sum_{i=1}^K |\sum_{n=N+1}^M x_{ni}| \leqslant \sum_{i=1}^K \sum_{n=N+1}^M |x_{ni}| \leqslant \sum_{n=N+1}^\infty \|\boldsymbol{x}_n\|_{\ell^1} \to 0$$

as $N \to \infty$.

$\boldsymbol{y} \in \ell^1$ since

$$\sum_{i=1}^K |\sum_{n=1}^N x_{ni}| \leqslant \sum_{i=1}^K \sum_{n=1}^N |x_{ni}| \leqslant \sum_{n=1}^N \|\boldsymbol{x}_n\|_{\ell^1} \leqslant s$$

□

Proposition 3.2 **Every functional on** ℓ^1 **is of the type** $(x_n) \mapsto \sum_n y_n x_n$ **where** $(y_n) \in \ell^\infty$**,**

$$\ell^{1*} \approx \ell^\infty$$

Proof: Let $\boldsymbol{y} \in \ell^\infty$; then the map $\hat{\boldsymbol{y}} : \ell^1 \to \mathbb{C}$ defined by

$$\hat{\boldsymbol{y}}(\boldsymbol{x}) = \sum_n y_n x_n$$

is well-defined since

$$|\hat{\boldsymbol{y}}(\boldsymbol{x})| \leqslant \sum_n \|\boldsymbol{y}\| |x_n| \leqslant \|\boldsymbol{y}\|_{\ell^\infty} \|\boldsymbol{x}\|_{\ell^1}$$

; moreover it is a functional in ℓ^{1*} since

$$\hat{\boldsymbol{y}}(\boldsymbol{a} + \boldsymbol{b}) = \sum_n y_n(a_n + b_n) = \sum_n y_n a_n + \sum_n y_n b_n = \hat{\boldsymbol{y}}(\boldsymbol{a}) + \hat{\boldsymbol{y}}(\boldsymbol{b}),$$

$$\hat{\boldsymbol{y}}(\lambda \boldsymbol{x}) = \sum_n y_n(\lambda x_n) = \lambda \sum_n y_n x_n = \lambda \hat{\boldsymbol{y}}(\boldsymbol{x}),$$
$$|\hat{\boldsymbol{y}}(\boldsymbol{x})| \leqslant \|\boldsymbol{y}\|_{\ell^\infty} \|\boldsymbol{x}\|_{\ell^1}$$

Note it follows that $\|\hat{\boldsymbol{y}}\| \leqslant \|\boldsymbol{y}\|_{\ell^\infty}$.

Every functional is of this type: let $\phi \in \ell^{1^*}$ and let $y_n = \phi(\boldsymbol{e_n})$ where $\boldsymbol{e_n} = (\delta_{ni}) = (0, \ldots, 0, 1, 0, \ldots)$. Since $\boldsymbol{x} = \sum_n x_n \boldsymbol{e}_n$, then

$$\phi(\boldsymbol{x}) = \phi(\sum_n x_n \boldsymbol{e}_n) = \sum_n x_n y_n$$

by linearity and continuity of ϕ; also $|y_n| = |\phi(\boldsymbol{e}_n)| \leqslant \|\phi\|$ so that $\boldsymbol{y} \in \ell^\infty$, with $\|\boldsymbol{y}\|_{\ell^\infty} \leqslant \|\phi\|$.

Now let $J : \ell^\infty \to \ell^{1*}$ be the map $J\boldsymbol{y} = \hat{\boldsymbol{y}}$. The previous proofs have shown that this map is in fact onto ℓ^{1*}.

J is linear, since

$$\forall \boldsymbol{x} \in \ell^1, J(\boldsymbol{u}+\boldsymbol{v})(\boldsymbol{x}) = \sum_n (u_n+v_n)x_n = \sum_n u_n x_n + \sum_n v_n x_n = J\boldsymbol{u}(\boldsymbol{x}) + J\boldsymbol{v}(\boldsymbol{x}),$$

so $J(\boldsymbol{u}+\boldsymbol{v}) = J\boldsymbol{u} + J\boldsymbol{v}$. Similarly

$$J(\lambda \boldsymbol{y})(\boldsymbol{x}) = \sum_n (\lambda y_n) x_n = \lambda J\boldsymbol{y}(\boldsymbol{x}),$$

and $J(\lambda \boldsymbol{y}) = \lambda J \boldsymbol{y}$.

J is isometric since by the previous proofs we have $\|\hat{\boldsymbol{y}}\| \leqslant \|\boldsymbol{y}\| \leqslant \|\hat{\boldsymbol{y}}\|$. It follows that J is 1-1 as well.

3.2 The space ℓ^∞

Proposition 3.3 **ℓ^∞ is complete.**

Proof: Let $(\boldsymbol{x}_n)$ be a Cauchy sequence in ℓ^∞, ie $\|\boldsymbol{x}_n - \boldsymbol{x}_m\|_{\ell^\infty} \to 0$ as $n, m \to \infty$.

For each i, we get that $|x_{ni} - x_{mi}| \leqslant \|\boldsymbol{x}_n - \boldsymbol{x}_m\| \to 0$, so $(x_{ni} - x_{mi})$ is a Cauchy sequence in $\mathbb{C}$, and so converges. Let $y_i = \lim_{i\to\infty} x_{ni}$.

$\boldsymbol{y} \in \ell^\infty$ since

$$|x_{ni}| \leqslant \|\boldsymbol{x}_n\| < C$$

since Cauchy sequences are always bounded.

$\boldsymbol{x}_n \to \boldsymbol{y}$ in ℓ^∞ since, by choosing m large enough depending on i,

$$|y_i - x_{ni}| \leqslant |y_i - x_{mi}| + |x_{mi} - x_{ni}| \leqslant \frac{1}{n} + \|\boldsymbol{x}_m - \boldsymbol{x}_n\| \to 0,$$

as $n \to \infty$, independently of i.

$\square$

Proposition 3.4 **The space of convergent sequences,**

$$c = \{\, (x_n) : \lim_{n\to\infty} x_n = x, \exists x \,\}$$

is a complete subspace.

Proof: $c \subset \ell^\infty$ since $x_n \to x$ implies that (x_n) is bounded. Moreover it is easily shown to be linear.

c is closed in ℓ^∞ since let $\boldsymbol{y}$: let $\boldsymbol{y}$ be a limit point of c ie we can find $\boldsymbol{x}_n \in c$ such that $\boldsymbol{x}_n \to \boldsymbol{y}$. Now consider the limits of each of these sequences $x_n = \lim_{i\to\infty} x_{n,i}$. These form a Cauchy sequence since $|x_n - x_m| = \lim_{i\to\infty} |x_{n,i} - x_{m,i}| \leqslant \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_{\ell^\infty} \to 0$. So $x_n \to y$ for some $y \in \mathbb{C}$.

Finally, we can show that $\lim_{i\to\infty} y_i = y$ since

$$\begin{aligned} |y_i - y| &\leqslant |y_i - x_{n,i}| + |x_{n,i} - x_n| + |x_n - y| \\ &\leqslant 3\epsilon \end{aligned}$$

for large enough n and i. Thus $\boldsymbol{y} \in c$, and c is a closed set in ℓ^∞.

□

Corollary The space of sequences with limit 0,

$$c_0 = \{\, (x_n) : \lim_{n\to\infty} x_n = 0 \,\}$$

is a complete subspace.

Proof: It is obvious that $c_0 \subset c$.

c_0 is closed in c since if $\boldsymbol{x}_n \to \boldsymbol{y}$ with $\boldsymbol{x}_n \in c_0$, then by the above proof, $\lim_{i\to\infty} y_i = 0$.

Hence c_0 is closed and so complete.

□

Note that $\ell^{\infty *}$ is a complicated space not isomorphic to ℓ^1.

Proposition 3.5

$$c_0^* \approx \ell^1$$

Proof: (i) Every functional on c_0 is of the type $(x_n) \mapsto \sum_n y_n x_n$ with $(y_n) \in \ell^1$. First, given $(y_n) \in \ell^1$, this is well-defined because $|sum_n y_n x_n| \leqslant \|\boldsymbol{x}\|_{\ell^\infty} \|\boldsymbol{y}\|_{\ell^1}$; secondly, this map is a functional because it is linear (as shown in the previous proposition) and continuous (by the above inequality). Moreover let ϕ be any functional on c_0, and let $y_n = \phi(\boldsymbol{e}_n)$; then $\phi(\boldsymbol{x}) = \phi(\sum_n x_n \boldsymbol{e}_n) = \sum_n y_n x_n$ by linearity and continuity of ϕ, and

$$\sum_{n=1}^{N} |y_n| = \sum_{n=1}^{N} \pm\phi(\boldsymbol{e}_n) = \phi(\sum_{n=1}^{N} \pm\boldsymbol{e}_n) \leqslant \|\phi\| \|(\pm 1)\|_{\ell^\infty} = \|\phi\|$$

hence $\boldsymbol{y} \in \ell^1$.

(ii)Let $J : \ell^1 \to c_0^*$ be defined by $J\boldsymbol{y}(\boldsymbol{x}) = \sum_n y_n x_n$. Then J is linear as in the previous proof, and it is isometric since $\|J\boldsymbol{y}\| \leqslant \|\boldsymbol{y}\|_{\ell^1}$ has already been shown, and $|J\boldsymbol{y}(\pm\boldsymbol{e}_i)| = \sum_i |y_i|$ (where $\pm y_i = |y_i|$, so

$$\|J\boldsymbol{y}\| \geqslant \sum_i |y_i| = \|\boldsymbol{y}\|$$

□

3.3 The space ℓ^2

Proposition 3.6 **(i)**

$$|\sum_n x_n y_n| \leqslant \sqrt{\sum_n |x_n|^2} \sqrt{\sum_n |y_n|^2}$$

(ii) The function $\sqrt{\sum_n |x_n|^2}$ is a norm, since

$$\sqrt{\sum_n |x_n + y_n|^2} \leqslant \sqrt{\sum_n |x_n|^2} + \sqrt{\sum_n |y_n|^2}$$

.

Proof: (i) It is easy to show that $|ab| \leqslant (a^2 + b^2)/2$ for any real numbers a, b. Hence,

$$(\sum_n a_n b_n)^2 = \sum_{i,j} a_i b_i a_j b_j \leqslant \sum_{i,j} (a_i^2 b_j^2 + a_j^2 b_i^2)/2 = \sum_{i,j} a_i^2 b_j^2 = \sum_i a_i^2 \sum_j b_j^2$$

It follows, that for complex numbers x_n, y_n,

$$|\sum_n x_n y_n| \leqslant \sum_n |x_n||y_n| \leqslant \sqrt{\sum_n |x_n|^2} \sqrt{\sum_n |y_n|^2}$$

(ii)

$$\sum_n |x_n + y_n|^2 = \sum_n |x_n|^2 + |y_n|^2 + (\bar{x}_n y_n + x_n \bar{y}_n) \leqslant \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| = (\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2$$

□

Proposition 3.7 ℓ^2 **is complete.**

Proof: Let (x_n) be an absolutely summable series, ie $\sum_n \|\boldsymbol{x}\|_{\ell^2}$ converges. Fix i, then

$$|\sum_n x_{ni}| \leqslant \sum_n |x_{ni}| \leqslant \sum_n \|\boldsymbol{x}_n\|_{\ell^2}$$

so can define $y_i = \sum_n x_{ni}$.

Now consider

$$\begin{aligned}\textstyle\sum_{i=1}^M |\sum_{n=A}^N x_{ni}|^2 &\leqslant \textstyle\sum_{i=1}^M (\sum_{n=A}^N |x_{ni}|)^2\\ &= \textstyle\sum_{i=1}^M \sum_{n=A}^N \sum_{m=A}^N |x_{ni}||x_{mi}|\\ &\leqslant \textstyle\sum_{n=A}^N \sum_{m=A}^N \sqrt{\sum_{i=1}^M |x_{ni}|^2}\sqrt{\sum_{i=1}^M |x_{mi}|^2}\\ &= \textstyle(\sum_{n=A}^N \sqrt{\sum_{i=1}^M |x_{ni}|^2})^2\end{aligned}$$

So as $M, N \to \infty$ we get

$$\sum_{i=1}^{\infty} |\sum_{n=A}^{\infty} x_{ni}|^2 \leqslant (\sum_{n=A}^{\infty} \|\boldsymbol{x}_n\|)^2$$

Taking $A = 1$ shows that $\boldsymbol{y} \in \ell^2$; taking $A = N + 1$ shows that $\sum_n \boldsymbol{x}_n \to \boldsymbol{y}$ in ℓ^2.

□

Proposition 3.8

$$\ell^{2*} \sim \ell^2$$

Proof: Exercise.

3.4 Function Spaces

In a similar fashion, but using theorems from integration, it can be shown that L^1, L^2 and L^∞ are Banach spaces, and that $L^{1*} \approx L^\infty$ and $L^{2*} \approx L^2$. The proofs are very similar, and use the analogous inequalities

$$\|fg\|_{L^1} \leqslant \|f\|_{L^1}\|g\|_{L^\infty},$$

$$\|fg\|_{L^1} \leqslant \|f\|_{L^2}\|g\|_{L^2},$$

called the Hölder's inequality, as well as the Monotone Convergence Theorem.

3.5 Exercises

1. Show that $\boldsymbol{x} = \sum_n x_n \boldsymbol{e}_n$, where $\boldsymbol{e}_n = (\delta_{ni})$, holds in the spaces ℓ^1, ℓ^2, and ℓ^∞.

2. Show that $\ell^{2*} \sim \ell^2$ by repeating the proofs for ℓ^1 and c_0.

3. Prove that $\|fg\|_{L^1} \leqslant \|f\|_{L^1}\|g\|_{L^\infty}$.

4. Show that the adjoint of the Fourier transform is a map from ℓ^1 to $L^\infty[0,1]$ such that $\mathcal{F}^\top(a_n) = \sum_n a_n e^{-2\pi i n x}$.

5. Let R be the right-shift operator on ℓ^1. What is $R^n \underline{x}$? Show that $R^n \underline{x} \not\rightharpoonup \underline{0}$, but that $R^n \rightharpoonup 0$. (Recall that $\phi(\underline{x}) = \sum_i y_i x_i$ for some sequence $(y_i) \in \ell^\infty$)

4 Banach Algebras

Definition A Banach algebra is a Banach space X, over $\mathbb{C}$, with an associative multiplication of vectors with unity such that

$$(S+T)U = SU + TU, \quad S(T+U) = ST + SU,$$

$$(\lambda S)T = \lambda(ST) = S(\lambda T),$$
$$\|ST\| \leqslant \|S\|\|T\|, \quad \|1\| = 1.$$

Proposition 4.1 **Multiplication is continuous.**

Proof: follows immediately from

$$T_n S_n - TS = (T_n - T)S_n + T(S_n - S),$$

so that as $T_n \to T$ and $S_n \to S$, then $T_n S_n \to TS$.

$\square$

4.0.1 Examples

$\mathbb{C}^N$ with pointwise multiplication.

$C(K)$ the space of continuous functions on a compact set; this contains the closed sub-algebra of holomorphic functions when $K \subseteq \mathbb{C}$.

$B(X)$ for any Banach space X.

Definition The morphisms of Banach algebras are those continuous maps which preserve the vector structure (linear) and the multiplication,

$$\Phi(ST) = \Phi(S)\Phi(T)$$

In particular, the complex morphisms $\phi : X \to \mathbb{C}$.

Theorem 4.2 **Every Banach algebra can be embedded as a closed subalgebra of $B(X)$.**

Proof: Let $L_a(x) = ax$ be left-multiplication by a. Then $L_a \in B(X)$ since multiplication is distributive and continuous.

$$L_a(x+y) = a(x+y) = ax + ay = L_a(x) + L_a(y),$$

$$L_a(\lambda x) = a(\lambda x) = \lambda L_a(x),$$
$$\|L_a(x)\| = \|ax\| \leqslant \|a\|\|x\|,$$

so that $\|L_a\| \leqslant \|a\|$. Now $L_{a+b} = L_a + L_b$, and $L_{ab} = L_a L_b$, $L_{\lambda a} = \lambda L_a$, $L_1 = I$ are obvious.

$$L_{a+b}(x) = (a+b)x = ax + bx = L_a(x) + L_b(x),$$

$$L_{\lambda a}(x) = (\lambda a)x = \lambda L_a(x),$$
$$L_{ab}(x) = (ab)x = a(bx) = L_a L_b(x),$$
$$L_1(x) = 1x = I(x),$$
$$\|L_a\| = \|a\|$$

since $a = a1 = L_a 1$, so $\|a\| = \|L_a 1\| \leqslant \|L_a\|$. So, the space of such operators is a sub-algebra of $B(X)$, and the mapping $L : X \to B(X)$ defined by $L : a \mapsto L_a$ is an isometric morphism of Banach algebras.

Moreover it is closed, since let $L_{a_n} \to T$ in $B(X)$; then $L_{a_n} x = a_n x = (L_{a_n} 1)x$ and so as $n \to \infty$, $Tx = T1x$ by continuity, i.e. $T = L_{T1}$. Thus the sub-algebra is complete.

□

4.1 Differentiation and Integration

Definition A function f between Banach algebras is said to be **differentiable** at T when there is a continuous linear map $f'(T)$ such that for H in a neighborhood of T,

$$f(T+H) = f(T) + f'(T)(H) + o(H)$$

i.e. $\|o(H)\|/\|H\| \to 0$ as $H \to 0$.

A function is **holomorphic** when it is differentiable in the sense above with $H = z \in \mathbb{C}$, i.e.

$$f(T+z) = f(T) + f'(T)z + o(z)$$

Note - if $f(T)$ is not defined, but $f(T+H) = A + B(H) + o(H)$, then can redefine f at T to make it differentiable (T is called a 'removable' singularity).

Example: any polynomial is differentiable;

Integration is also well-defined on paths in a Banach algebra via

$$\int_\gamma F(z)dz := \int_\gamma F(z(t))\dot{z}ds.$$

We shall assume the following facts about integration 1. it is linear, with $\int TF(z)dz = T\int F(z)dz$, 2. $\|\int F(z)dz\| \leqslant \int \|F(z)\|ds$.

Theorem 4.3 **Cauchy's Theorem**

Let F be a function from $\mathbb{C}$ into a Banach algebra, which is holomorphic in a bounded region U having a finite number of differentiable curves as boundary. Then,

$$\oint F(z)dz = 0$$

Proof: At any holomorphic point, $F(z+h) = F(z) + F'(z)h + o(h)$, and for $z+h$ in a sufficiently small disk $B_\delta(z)$, we have $\|o(h)\| < \epsilon\delta$. Thus, for any triangle inside this disk we get

$$\begin{aligned}\oint_\triangle F(w)dw &= \oint_{(\triangle - z)} F(z+h)dh \\ &= \oint F(z) + F'(z)h + o(h)dh\end{aligned}$$

so that $\|\oint_\triangle F(w)dw\| \leqslant \epsilon\delta Perimeter(\triangle) \leqslant 6\epsilon\delta^2$.

Now any triangle can be partitioned into a number of triangles each smaller than δ. Since the triangle is totally bounded, a single δ can be found that suffices for the whole triangle (using a Lebesgue net). Thus $\oint F(w)dw$ over the whole triangle is the sum of integrals over the δ-triangles. The number of these does not exceed $(L/2\delta)^2$ where L is the largest side of the triangle, so that

$$\|\oint_\triangle F(w)dw\| \leqslant 6\epsilon\delta^2(L/2\delta)^2 = c\epsilon,$$

proving that the integral vanishes over triangles.

More generally, the curves can be approximated by many-sided polygons, and the interior can be triangulated, proving the theorem.

$\square$

Corollary Cauchy Residue Theorem

The integral over a closed path depends only on those areas inside, where f is not holomorphic,

$$\oint F(z)dz = \sum_i Residue_i(F)$$

Proof: enclose the non-holomorphic areas by curves, to form one holomorphic region.

Corollary If F is holomorphic in a path-connected region, then the integral $\int_a^z F(w)dw$ is well-defined, and is holomorphic.

4.2 Power Series

Any power series $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence given by $R = \liminf_n |a_n|^{-1/n}$.

Lemma For any T, the sequence $\|T^n\|^{1/n}$ converges to a number denoted by $r(T)$.

Proof: It is clear that $0 \leqslant \|T^n\|^{1/n} \leqslant \|T\|$. Let $r(T)$ be the infimum value of $\|T^n\|^{1/n}$. We can find an N such that $r(T) \leqslant \|T^N\|^{1/N} < r(T) + \epsilon$. Although it is not true that the sequence of numbers is decreasing, notice that $\|T^{qN}\|^{1/qN} \leqslant \|T^N\|^{1/N}$. More generally, for any n, let $n = qN + r$ with $0 \leqslant r < N$ (by the remainder theorem), then we get $0 \leqslant r/n < N/n \to 0$ as $n \to \infty$ so that

$$\begin{aligned}
r(T) \leqslant \|T^n\|^{1/n} &\leqslant \|T^N\|^{q/n} \|T\|^{r/n} \\
&\leqslant \|T^N\|^{\frac{1}{N}(1-\frac{r}{n})} \|T\|^{r/n} \\
&\leqslant (\|T^N\|^{\frac{1}{N}} + \epsilon)(1 + \epsilon) \\
&\leqslant r(T) + (\|T\| + 3)\epsilon
\end{aligned}$$

for n large enough.

$\square$

Theorem 4.4 **Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Then if $r(T) < R$, the series $f(T) := \sum_{n=0}^{\infty} a_n T^n$ converges absolutely, and diverges for $r(T) > R$.**

Proof: We are given that $r(T) < R$, so for $n > N$ large enough,

$$\|a_n T^n\| = \left(\frac{\|T^n\|^{1/n}}{|a_n|^{-1/n}} \right)^n \leqslant \left(\frac{r(T) + \epsilon}{R - \epsilon} \right)^n = \alpha^n$$

where $\alpha < 1$. Note that when $R = \infty$, replace the denominator with any positive real number. The series $\sum_n \|a_n T^n\|$ converges by comparison with geometric series on the right.

Similarly, when $r(T) > R$ the series diverges since

$$\|a_n T^n\| = \left(\frac{\|T^n\|^{1/n}}{|a_n|^{-1/n}}\right)^n \geqslant \left(\frac{r(T)-\epsilon}{R+\epsilon}\right)^n \geqslant 1$$

for an infinite number of n.

$\square$

Note that power series are continuous in T.

Proposition 4.5 **A power series is holomorphic within its radius of convergence.**

Proof: $\sum_n a_n(T+z)^n = \sum_n a_n T^n + \sum_n n a_n T^{n-1} z + o(z)$, and the radius of convergence of the coefficients na_n is $\liminf_n |na_n|^{-1/n} = R$ since $n^{1/n} \to 1$ as $n \to \infty$.

$\square$

Proposition 4.6 **(i)** $\sum_n a_n T^n = 0$ **within a positive radius of convergence** $\Leftrightarrow$ $a_n = 0, \forall n$**.**

(ii) There is only one power series such that $f(T_n) = A_n$ **with** $T_n \to 0$**.**

Proof: Let $\sum_n a_n T^n = 0$ for $T = T_n \to 0$ not divisors of zero and not nilpotents; then $a_0 = 0$ follows from $T_n \to 0$; suppose $a_0, \ldots, a_{m-1} = 0$, then $0 = \sum_n a_n T_n^n = T_n^m(a_m + a_{m+1}T_n + \ldots)$, so that $a_m = 0$ by letting $T_n \to 0$.

Suppose $f(T_n) = A_n = g(T_n)$ for two power series f, g. Let $h = f - g$, so that $h(T) = \sum_n a_n T^n$ and $0 = h(T_n)$, implying that $h = 0$.

$\square$

For example, $e^T = 1 + T + T^2/2! + \ldots$ converges for any operator T, and satisfies $\|e^T\| \leqslant e^{\|T\|}$. It can be shown (exercise) that when (and only when) S and T commute, we get $e^{S+T} = e^S e^T$, from which follows that e^T is invertible with inverse e^{-T}.

Similarly, for $\|T\| < 1$, we can define $\log(1+T) = T - T^2/2 + T^3/3 + \ldots$.

Proposition 4.7 **If** $\|T\| < 1$ **then**

$$(1-T)^{-1} = 1 + T + T^2 + \ldots$$

is continuous, with $\|(1-T)^{-1}\| \leqslant (1 - \|T\|)^{-1}$**.**

Proof: $\|T^n\| \leqslant \|T\|^n$ and the right-hand terms are summable when $\|T\| < 1$, so $\sum_n \|T^n\| \leqslant 1/(1-\|T\|)$. This implies that $\sum_n T^n$ also converges (since the Banach algebra is complete) to, say, S. Now $(1-T)S = \lim_{n\to\infty}(1-T)(1+T+\ldots+T^n) = \lim_{n\to\infty}(1-T^{n+1}) = 1$ since $\|T^n\| \to 0$ as $n \to \infty$. Similarly $S(1-T) = 1$, proving that S is the inverse of $(1-T)$.

□

Corollary An affine map $f(x) = a+Tx$ with $\|T\| < 1$ has a unique fixed point $z = f(z)$.

Proof: x is a fixed point $\Leftrightarrow x = f(x) = a+Tx \Leftrightarrow a = (1-T)x \Leftrightarrow x = (1-T)^{-1}a$.

□

Of course, the result also follows from the Banach contraction mapping theorem, since $d(f(x), f(y)) = \|a + Tx - a - Ty\| \leqslant \|T\|d(x, y)$.

4.3 Group of Invertible Elements

The invertible elements form a group, which contains 1 and the surrounding unit ball of elements, extended to the interior of the "double cone" $\{\, \lambda T : \lambda \neq 0, \|T - 1\| < 1 \,\}$, as well as all the operators e^T.

Proposition 4.8 **The group of invertible elements is an open set, and the map $T \mapsto T^{-1}$ is continuous.**

Proof: Let T be an invertible element, and let $\delta < \|T^{-1}\|^{-1}$. Then for $\|H\| < \delta$, we get $(T+H)^{-1} = (1+T^{-1}H)^{-1}T^{-1}$, which exists by the previous theorem since $\|-T^{-1}H\| \leqslant 1$. This shows that the group is an open set.

Moreover we have

$$(T + H)^{-1} = T^{-1} - T^{-1}HT^{-1} + T^{-1}HT^{-1}HT^{-1} + \ldots$$

This shows that $\|(T + H)^{-1} - T^{-1}\| \leqslant \|H\|\|T^{-1}\|^2 + \|H\|^2\|T^{-1}\|^3 + \ldots \leqslant \frac{\|H\|\|T^{-1}\|^2}{1-\|T^{-1}\|\|H\|}$ which converges to 0 as $H \to 0$.

□

The proof in fact shows that the map $T \mapsto T^{-1}$ is differentiable; indeed this shows that the group of invertibles is essentially what is called a 'Lie group', a topic that has a vast literature devoted to it.

In particular note that for $H = h1$, we have $(T+h)^{-1} = T^{-1} - hT^{-2} + \ldots$, from which follows that the map $z \mapsto (T - z)^{-1}$ is holomorphic whenever $z \notin \sigma(T)$. Indeed, for $z \notin \sigma(T)$ we have

$$(T - z + h)^{-1} = (T - z)^{-1} - h(T - z)^{-2} + O(|h|^2)$$

so that

$$\lim_{h\to 0}\frac{(T-z+h)^{-1}-(T-z)^{-1}}{h}=-(T-z)^{-2}$$

Proposition 4.9 **Let R_n be invertible elements that converge to an element T on the boundary of the group of invertible elements. Then**

(i) $\|R_n^{-1}\| \geqslant 1/\|R_n - T\|$;

(ii) T is a generalized divisor of zero T, i.e. there are unit elements S_n such that

$$TS_n \to 0, \text{ OR } S_nT \to 0, \text{ as } n \to \infty$$

Proof: Let $R_n \to T$ be invertible elements converging to a boundary element T. Since the group of invertible elements is open, T cannot be invertible, whereas R_n and its surrounding ball of radius $\|R_n^{-1}\|^{-1}$ are invertible. Thus $\|R_n - T\| \geqslant \|R_n^{-1}\|^{-1}$.

Let $S_n = R_n^{-1}/\|R_n^{-1}\|$; then

$$\begin{aligned}\|TS_n\| &= \|TR_n^{-1}\|/\|R_n^{-1}\| \\ &= \|(T-R_n)R_n^{-1}+1\|/\|R_n^{-1}\| \\ &\leqslant 2\|T-R_n\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

□

The generalized divisors of zero are also called topological divisors of zero, and are not invertible (exercise).

The set of non-invertible elements is connected (through the origin), and may disconnect the group of invertible elements.

Proposition 4.10 **The maximally connected subset of invertible elements containing 1, is an open normal subgroup generated by e^T for all T.**

Proof: Let T be in the maximally connected subset containing 1, denoted by G_1. Then the connected ball $B_\epsilon(T)$ around T is also in G_1 (because maximally connected), so that G_1 is open. Moreover for any T, TG_1 is also connected because multiplication is continuous, so when $T \in G_1$ we must get $TG_1 \subseteq G_1$; hence for any $S, T \in G_1$, we have $TS \in G_1$. Similarly, taking the inverse is a continuous mapping which fixes 1 and maps the connected set G_1 to a connected set containing 1; thus $T^{-1} \in G_1$. Also, by the same reasoning $T^{-1}G_1T \subseteq G_1$, which makes G_1 a normal subgroup.

The inverse of e^T is e^{-T} from their definition as power series. Hence the group generated by exponentials $e^T, \ldots, e^S$ consists of finite products of the type $e^{\pm T} \ldots e^{\pm S}$. Moreover the elements of the type e^T form a connected set since the Banach algebra is connected and the exponential is continuous. Multiplication is also continuous so that the group generated by exponentials is connected. As $e^0 = 1$, it must be a subgroup of G_1.

Claim: This group is open and closed, and so must equal G_1. The elements near to $1 = e^0$ are all exponentials i.e. they all have a logarithm (check! theorem 10.30 of Rudin?), and so any element T generated by the exponentials also has a neighborhood inside this group. Its complement must also be open, because it consists of open cosets of the group.

$\square$

4.3.1 Exercises

1. Show that if $ST = 1 = TR$ then $S = R = T^{-1}$.

2. Find Banach algebras of dimensions 1, 2, 3 (over $\mathbb{C}$)

3. Show that for any morphism Φ between Banach algebras, $\Phi(1) = 1$ and $\Phi(T^{-1}) = \Phi(T)^{-1}$. Deduce that for $\|T\| = 1$ and $|\lambda| < 1$ then $\|\Phi(\lambda T)\| < 1$ and hence that Φ must be continuous.

4. Let $f(z) = \sum_n a_n z^n$, and let $|f|(z) = \sum_n |a_n| z^n$. Show that $\|f(T)\| \leqslant |f|(\|T\|)$.

5. Show that (i) $r(\lambda T) = |\lambda| r(T)$, (ii) $r(ST) = r(TS)$, (iii) $r(T^n)^{1/n} = r(T)$, (iv) if $ST = TS$ then $r(T+S) \leqslant r(T) + r(S)$ and $r(TS) \leqslant r(T)r(S)$ (hint: for the addition take $T = \alpha T_0$ with $r(T_0) = 1$ and similarly for S).

6. Show that when $ST = TS$, $e^S e^T = e^{S+T}$. Deduce that the inverse of e^T is e^{-T}. Show further that the map $t \mapsto e^{tT}$ is a group homomorphism from $\mathbb{R}$ to the group of invertible elements, which is differentiable.

7. Show that $\|(1-T) - (1-T)^{-1}\| \leqslant \frac{\|T\|^2}{1-\|T\|}$ (use $(1-T)^{-1} = 1+T+\ldots$).

8. Show that the iteration $x_{n+1} = a + Tx_n$, starting from any x_0, converges to the fixed point $z = a + Tz$. (Hint: prove first that $\|x_{n+1} - x_n\| \leqslant \|T\| \|x_n - x_{n-1}\|$, and then that (x_n) is a Cauchy sequence)

9. Suppose that $Ax = b$ is a matrix equation, with A being practically a diagonal matrix, in the sense that $A = D+B$, with D being the diagonal of A, and $\|B\| < \|D\|$. Hence $Dx = b-Bx$ and $x = D^{-1}b-D^{-1}Bx$. Let $Tx = D^{-1}b - D^{-1}Bx$ and show that $\|T\| < 1$ so that there is a unique fixed point. Hence describe a recursive algorithm (due to Jacobi) for finding the solution of the equation.

10. Similarly suppose that $Ax = b$ with A being practically a lower triangular matrix, in the sense that $A = L + B$ where L is lower triangular and $\|B\| < \|L\|$. The inverse of a triangular matrix, such as L, is fairly easy to compute. Describe a recursive algorithm (Gauss-Siedel) for finding the solution x.

11. Let $Tf(x) = \int_a^b k(x,y)f(y)dy$ be a mapping between function spaces. Show that if $k(x,y)$ is bounded (ie in $L^\infty[a,b]^2$), then T is an operator mapping $L^1[a,b]$ to itself and $L^\infty[a,b]$ to itself. Show further that $\|T\| \leqslant \|k\|_{L^\infty}|b-a|$ in both cases. Deduce that if $|k(x,y)| < 1/|b-a|$ then the equation $Tf + g = f$ has a unique solution $\sum_n T^n g$ (find the kernel of T^n).

4.4 Spectrum of T

Definition The **spectrum** of an element T in a Banach algebra is defined as the set

$$\{\, \lambda \in \mathbb{C} : T - \lambda \text{ is not invertible} \,\}$$

Proposition 4.11 **The spectrum of T is a non-empty compact subset of $\mathbb{C}$ bounded by $r(T)$.**

Proof: If $|\lambda| > r(T)$, then $T - \lambda = -\lambda(1 - T/\lambda)$ is invertible. So any spectral values must satisfy $|\lambda| \leqslant r(T)$.

Let $\lambda \notin \sigma(T)$, ie $T-\lambda$ is invertible. For any $z \in \mathbb{C}$, $\|(T-z)-(T-\lambda)\| = |z-\lambda|$, so that for z close enough to λ, we have $T - z$ also invertible. This shows that λ is an exterior point of the spectrum, and hence that the spectrum is closed.

Closed and bounded sets in $\mathbb{C}$ are compact.

Suppose $\sigma(T) = \varnothing$, so that $(T-z)^{-1}$ is holomorphic everywhere. Then by Cauchy's theorem, we get that the integral over any circle vanishes $\oint (T-z)^{-1}dz = 0$. But $(T-z)^{-1} = -\frac{1}{z} - \frac{1}{z^2}T + \ldots$ for $|z| > \|T\|$, so that for a circle of radius larger than $r(T)$ we get

$$\oint (T-z)^{-1} dz = \oint -\frac{1}{z} - \frac{1}{z^2}T + \ldots dz = \oint \frac{1}{z} dz = 2\pi i \neq 0,$$

so it must be the case that the spectrum does not vanish. (Alternative proof: if the map $\lambda \mapsto (T-\lambda)^{-1}$ is differentiable everywhere, then since $(T-\lambda)^{-1} \to 0$ as $\lambda \to \infty$, by Liouville's theorem we get $(T-\lambda)^{-1} = S$ for all λ, a contradiction)

$\square$

Proposition 4.12 **Spectral Radius Formula**

The largest extent of the spectrum, called its spectral radius, is $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

Proof: We have already shown that $\sigma(T)$ is bounded by $r(T)$. We need to show that there are spectral values with magnitude $r(T)$.

Invert the spectrum to form the set $S = \{\, z^{-1} : z \in \sigma(T) \,\}$. Thus $z \notin S \Leftrightarrow (T-z^{-1})$ is invertible, and so the largest radius of that disk in which $(T-z^{-1})^{-1}$ exists is $1/r$, where r is the spectral radius of T. Inside this disk, $(T-1/z)^{-1}$ is holomorphic, and so can be written as a power series, which turns out to be $(T-1/z)^{-1} = -z\sum_n (zT)^n$. But the radius of convergence of the series $\sum_n (Tz)^n$ is the same as that of the series $\sum_n \|T^n\| z^n$, that is $\liminf_n \|T^n\|^{-1/n} = 1/r(T)$. Thus we must have $1/r_\sigma(T) \leqslant 1/r(T)$, ie $r(T) \leqslant r_\sigma(T)$.

$\square$

Note that if λ is a boundary point of the spectrum, then $T-\lambda$ is at the boundary of the group of invertible elements, and so is a topological divisor of zero.

Theorem 4.13 **Spectral Mapping Theorem**

The spectrum of $f(T)$ is equal to the set $\{\, f(\lambda) : \lambda \in \sigma(T) \,\}$; ie

$$\sigma(f(T)) = f(\sigma(T))$$

Proof: We start by considering $f(x) = p(x)$ a polynomial. Let $p(x) - \lambda = (x-a)(x-b)\ldots$ where $a, b, \ldots$ are the roots, dependent on λ. Whether λ is a spectral value of $p(T)$ or not depends on whether $p(T) - \lambda = (T-a)(T-b)\ldots$ is invertible or not.

Suppose that $\lambda \notin p(\sigma(T))$ i.e. $p(z) - \lambda \neq 0$ for any $z \in \sigma(T)$. Hence any $z \in \sigma(T)$ is not one of the roots $a, b, \ldots$ and so the factors $(T - a)$ are all invertible. This shows that $\lambda \notin \sigma(p(T))$.

Conversely suppose that $\lambda \notin \sigma(p(T))$, so that $p(T) - \lambda$ has an inverse S. Then $S(T - a)(T - b) \ldots = 1 = (T - a)(T - b) \ldots S$; thus for any root a, there are elements R and R' such that $(T - a)R = 1 = R'(T - a)$. But $R' = R'(T - a)R = R$, so that any factor $T - a$ is invertible. Thus all the roots of $p(z) - \lambda$ are not in the spectrum. But if $\lambda = p(z)$ then z is a root, and so $\lambda = p(a) \notin p(\sigma(T))$.

Now let $f(T)$ be a power series $\sum_n a_n T^n$. Then $\sigma(\sum_{n=0}^N a_n T^n) = \sum_{n=0}^N \sigma(T)^n$ converges separately to $\sigma(f(T))$ and $f(\sigma(T))$.

$\square$

4.4.1 Exercises

1. Show that the resolvent $\mathbb{C} - \sigma(T)$ is open by proving that the map $\lambda \mapsto T - \lambda$ is continuous, and using the fact that the invertible elements form an open set.

2. Suppose that $p(T) = 0$ for some polynomial p. Show that $\sigma(T)$ consists of roots of p.

3. Use the spectral radius formula to show that the spectral radii of TS and ST are the same.

4. Use the spectral mapping theorem to show that if $e^T = 1$ then $T = 2\pi i S$ where $\sigma(S) \subset \mathbb{Z}$.

5. Let T be a nilpotent operator ie $\exists n : T^n = 0$; show that its spectrum consists of just the origin.

6. Suppose that the only non-invertible element of a Banach algebra is 0 (called a division algebra). Using the fact that the spectrum is non-empty, show that $T = \lambda$ for some $\lambda \neq 0$, and hence that the Banach algebra is isomorphic to $\mathbb{C}$.

7. Suppose that the only topological divisor of zero is 0. Show that the spectrum of any element is a single point (consider its boundary) and so each non-zero element is invertible. Deduce that the Banach algebra is $\mathbb{C}$.

4.5 Holomorphic Functions $f(T)$

Theorem 4.14 **Taylor series If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on a disk $B_r(z_0)$ that includes $\sigma(T)$ then f is a power series about z_0 with**

$$f(T) = \sum_n a_n (T - z_0)^n = \frac{1}{2\pi i} \oint f(w)(w - T)^{-1} dw.$$

Laurent series If f is holomorphic on a disk $B_r(z_0)$ except at its center $z_0 \notin \sigma(T)$ and the disk includes $\sigma(T)$, then the (Laurent) series converges

$$\sum_{n=-\infty}^{\infty} a_n (T - z_0)^n = \frac{1}{2\pi i} \oint f(w)(w - T)^{-1} dw.$$

In both cases, $a_n = \frac{1}{2\pi i} \oint f(w)(w - T)^{-n-1} dw$.

Proof: (i) $(w - T)^{-1} = (w - z_0 + z_0 - T)^{-1} = (w - z_0)^{-1}(1 - \frac{T - z_0}{w - z_0})^{-1} = \sum_n (w - z_0)^{-1-n}(T - z_0)^n$ since $r(T - z_0) < R = |w - z_0|$ by the spectral mapping theorem.

Thus, $\frac{1}{2\pi i} \oint f(w)(w - T)^{-1} dw = \sum_n (T - z_0)^n \frac{1}{2\pi i} \oint f(w)(w - T)^{-1-n} dw = \sum_n a_n (T - z_0)^n$, with $a_n = \frac{1}{2\pi i} \oint f(w)(w - T)^{-1-n} dw$.

By Cauchy's theorem, we can take any curve surrounding the spectrum of T.

(ii) Continuing the analysis of (i), we also get that on a small circle around z_0, $(w - T)^{-1} = (w - z_0 + z_0 - T)^{-1} = -(T - z_0)^{-1}(1 - (w - z_0)(T - z_0)^{-1})^{-1} = -\sum_n (w - z_0)^n (T - z_0)^{-n-1}$. Thus taking the integral in a clockwise direction, we get $\frac{1}{2\pi i} \oint f(w)(w - T)^{-1} dw = \sum_n a_{-n}(T - z_0)^{-n-1}$ where $a_{-n} = \frac{1}{2\pi i} \oint f(w)(w - z_0)^n dw$.

□

Corollary If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in a neighborhood of a point, then it is a power series, and thus infinitely many times differentiable, with

$$|f^{(n)}(z_0)| \leqslant \frac{n! M}{r^n}$$

Proof: $|f^{(n)}(z_0)/n!| = |a_n| \leqslant \frac{1}{2\pi} \|f\|_{L^\infty(B_r(z_0))} r^{-n-1} 2\pi r = M/r^n$.

□

Corollary (Fundamental theorem of Algebra)
Every polynomial $p(z)$ has n roots.

Proof: Suppose that $p(z)$ has no roots, so that $1/p(z)$ is holomorphic. Now $p(z) \to \infty$, in fact for $|z|$ large enough, $|p(z)| \geqslant |z|^n - |a_{n-1}z^{n-1} + \ldots + a_0| \geqslant |z|^n/c$. So

$$\frac{1}{|p(0)|} \leqslant \frac{1}{2\pi} \oint \frac{1}{|p(w)|}|w|^{-1}ds \leqslant c/R^n \to 0$$

as $R \to \infty$, which is a contradiction.

$\square$

Corollary (Liouville) if $F(z)$ is holomorphic throughout $\mathbb{C}$ then it must be unbounded or constant.

Proof: Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic but bounded by M on $\mathbb{C}$. Then for any z, $|f'(z)| \leqslant M/R \to 0$ as $R \to \infty$, so $f' = 0$.

More generally, if $F(z)$ is holomorphic, then so is $\phi \circ F : \mathbb{C} \to \mathbb{C}$ for any functional ϕ. If it is bounded, then $\phi(F(z)) = \phi(F(0))$ for any ϕ, which implies that $F(z) = F(0)$ for all z.

$\square$

Corollary Two holomorphic functions on $\mathbb{C}$ must agree on a connected set if they agree on an interior disk.

Proof: let A be the set on which $f = g$ locally i.e. $z \in A \Leftrightarrow f = g$ on $B_\epsilon(z)$. Then $A \neq \varnothing$ by hypothesis, A is open by definition, and A is closed, since suppose z is a limit point, with $z_n \to z$ and $z_n \in A$. Then f and g are power series about z with $f(z_n) = g(z_n)$; hence $f = g$ throughout the radius of convergence, and so $z \in A$. It follows that A is all of the connected set.

$\square$

Taylor's theorem implies that for any polynomial $p(T) = \frac{1}{2\pi i}\oint p(w)(w - T)^{-1}dw$; also $(T-\lambda)^{-1} = \frac{1}{2\pi i}\oint (w-\lambda)^{-1}(w-T)^{-1}dw$ for $\lambda \notin \sigma(T)$.

The integral $\frac{1}{2\pi i}\oint f(w)(w-T)^{-1}dw$ exists for any T for which f is holomorphic in a neighborhood of $\sigma(T)$ (since $\|(w-T)^{-1}\|$ is continuous in w and bounded since the path is compact; hence $\|f(T)\| \leqslant \frac{1}{2\pi}\int |f(w)|\|(w-T)^{-1}\|ds \leqslant M$.

Definition For any function $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic in a neighborhood of $\sigma(T)$,

$$f(T) := \frac{1}{2\pi i}\oint f(w)(w-T)^{-1}dw$$

Note that f is differentiable (and so continuous) at T (for H sufficiently small, $f(T+H)$ is defined since $\sigma(T+H) \subseteq \sigma(T) + B_\epsilon$; also, $f(T+H) = f(T) + \frac{1}{2\pi i} \oint f(w)(w-T)^{-1}H(w-T)^{-1}dw + o(H)$)

One can also discuss the integral about a part of $\sigma(T)$; in particular $\frac{1}{2\pi i} \oint (w-T)^{-1}dw$ is a projection map on T.

Theorem 4.15
$$\sigma(f(T)) = f(\sigma(T))$$

Proof: Let $\lambda \notin f(\sigma(T))$, so $(f(z)-\lambda)^{-1}$ is holomorphic on the open set $U - \{\, z : f(z) = \lambda \,\}$, which contains $\sigma(T)$. Thus $(f(T)-\lambda)^{-1}$ exists, and $\lambda \notin \sigma(f(T))$.

Conversely, let $\lambda \in \sigma(T)$, and let $F(z) := \frac{f(z)-f(\lambda)}{z-\lambda}$ which is holomorphic on $\sigma(T)$ and at λ. So since $(z-\lambda)F(z) = f(z) - f(\lambda)$ we get $(T-\lambda)F(T) = f(T) - f(\lambda)$. But $T - \lambda$ is not invertible, and so neither is $f(T) - f(\lambda)$ i.e. $f(\lambda) \in \sigma(f(T))$.

□

Examples: We can define $\log T$ whenever $0 \notin \sigma(T)$ and one can draw a contour from 0 to outside $\|T\|$ (so that one can enclose $\sigma(T)$ by a simple closed path in which $\log(z)$ is holomorphic), but this definition is not unique. Note that $e^{\log T} = T$.

Furthermore, can define $T^{1/n}$ (again not unique) with $(T^{1/n})^n = T$.

4.5.1 Exercises

1. Show that $(f+g)(T) = f(T) + g(T)$, and that $f \circ g(T) = f(g(T))$. It is also true, but more difficult to show that $(fg)(T) = f(T)g(T)$.

2. Show that $f(z1) = f(z)1$ where $z \in \mathbb{C}$ and 1 is the identity element. Hence $f(T)$ is truly an extension of the complex holomorphic function $f(z)$.

4.6 $B(X)$

T invertible means there is an operator $T^{-1} \in B(X)$, that is the inverse is continuous and linear. Now if an operator $T \in B(X)$ is bijective, then T^{-1} will always be linear, but not necessarily continuous. So one must be careful in interpreting the term "invertible" in $B(X)$.

If λ is an eigenvalue, ie $Tx = \lambda x$ for some $x \neq 0$ (called an eigenvector) then $(T-\lambda)x = 0$, hence $T - \lambda$ is not 1-1, and so not invertible.

What if $T-\lambda$ is 1-1? If it is also onto, then by the open mapping theorem, a bijective operator has a continuous inverse, and λ is not a spectral value. So for spectral values for which $T-\lambda$ is 1-1, it cannot also be onto.

If $T-\lambda$ is 1-1, and its image is dense in X, ie $\overline{\text{im}(T-\lambda)} = X$, then λ is said to be part of the continuous spectrum. In particular we can find x_n such that $(T-\lambda)x_n \to 0$, ie $Tx_n \approx \lambda x_n$.

If $T-\lambda$ is 1-1, and its image is not even dense in X, then λ is said to be part of the residual spectrum.

Proposition 4.16 **The eigenvectors of distinct eigenvalues are linearly independent.**

Proof: Let v_i be eigenvectors associated with the distinct eigenvalues λ_i. Then if $\sum_{i=1}^N \alpha_i v_i = 0$ implies

$$0 = \sum_{i=1}^{N} \alpha_i (T-\lambda_N) v_i = \sum_i \alpha_i(\lambda_i - \lambda_N)v_i = \sum_{i=1}^{N-1} \alpha_i(\lambda_i-\lambda_N)v_i = \sum_{i=1}^{N} \beta_i v_i$$

Thus by induction we get $\beta_i = 0$ ie $\alpha_i = 0$ for $i < N$. Hence $\alpha_N v_N = 0$ which implies $\alpha_N = 0$ as well.

$\square$

4.6.1 Example

1. Let $L(x_n) = (x_{n+1})$ be the left-shift operator on ℓ^1. The eigenvalues are given by $T\mathbf{x} = \lambda\mathbf{x}$ ie $(x_{n+1}) = \lambda(x_n)$, ie $\forall n, x_{n+1} = \lambda x_n$. This is a recurrence relation, which can easily be solved to give $x_n = \lambda^n x_0$. For this to be an eigenvector in ℓ^1 we need that $\sum_n |x_n| = |x_0| \sum_n |\lambda|^n$ to be finite, and this implies that $|\lambda| < 1$. Conversely, for any such $\lambda \neq 0$, the sequence (λ^n) is an eigenvector. Hence the spectrum consists of the unit closed ball, since the norm is 1.

2. Let $T : L^1[0,1] \to L^\infty[0,1]$ be defined by $Tf(x) = \int_{1-x}^1 f(s)ds$. Then T is trivially linear; it is continuous since $\|Tf\|_{L^\infty} = \sup_x |\int_{1-x}^1 f(s)ds| \leqslant \|f\|_{L^1}$. Thus $\|T\| \leqslant 1$.

For the eigenvalues, we get $\int_{1-x}^1 f(s)ds = \lambda f(s)$. Differentiating twice gives $f''(x) + \frac{1}{\lambda^2} f(x) = 0$ with boundary conditions $f(0) = 0 = f'(1)$. Thus $f(x) = B\sin(x/\lambda)$ where $\lambda = 2/k\pi$, k odd.

Proposition 4.17

$$\sigma(T^\top) = \sigma(T)$$

Proof: let $\lambda \notin \sigma(T)$, then one can easily check that $T^\top - \lambda$ has the inverse $(T-\lambda)^{-1\top}$, showing that $\lambda \notin \sigma(T^\top)$.

Conversely, let $\lambda \notin \sigma(T^\top)$, then by the previous proof $\lambda \notin \sigma(T^{\top\top})$. But X is embedded in X^{**}, so that $T^{\top\top}$ equals T when restricted to X. However, we need to show that its inverse can also be restricted to X. We can deduce that $T-\lambda$ is 1-1, and we now show it is also onto. Suppose otherwise, so we can find a non-zero functional in $\mathrm{im}(T-\lambda)^\circ = \ker(T^\top - lambda)$; hence $T^\top - \lambda$ is not 1-1 contrary to our assumption that $\lambda \notin \sigma(T^\top)$. Thus $T-\lambda$ is 1-1 and onto, and hence its inverse equals the restriction of the inverse of $(T-\lambda)^{\top\top}$. Thus $\lambda \notin \sigma(T)$.

□

4.6.2 Exercises

1. Show that if λ is an eigenvalue of T then $f(\lambda)$ is an eigenvalue of $f(T)$, with the same eigenvector.

2. Let $T : \ell^2 \to \ell^2$ be defined by $T(x_n) = (\alpha_n x_n)$ where α_n are dense in a compact set K. Show that its spectrum is K and that there is no residual spectrum.

3. Let $T : C[0,1] \to C[0,1]$ be defined by $Tf(x) = xf(x)$. Show that T is an operator, find its norm and show that its spectrum consists of only the residual part.

4. Let $T : C[0,1] \to C[0,1]$ be defined by $Tf(x) = \int_0^x f$. Show that $T^n f = \int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)dy$, and that $\|T^n\| \leqslant \frac{|x|^n}{n!}$. Deduce, using the spectral radius formula, that its spectrum is just the origin.

5 C^*-Algebras

Definition A C^*-algebra is a Banach algebra with an **involution** map $* : X \to X$ with the properties:

$$(T+S)^* = T^* + S^*, (\lambda T)^* = \bar{\lambda} T^*, (ST)^* = T^* S^*, T^{**} = T, \|T^*T\| = \|T\|^2$$

Proposition 5.1 **The involution map is continuous, and**

$$\|T^*\| = \|T\|$$

Proof: We have $\|T\|^2 = \|T^*T\| \leqslant \|T\|\|T^*\|$, so that $\|T\| \leqslant \|T^*\|$; furthermore, $\|T^*\| \leqslant \|T^{**}\| = \|T\|$.

□

Proposition 5.2

$$\sigma(T^*) = \overline{\sigma(T)}$$

Proof: If T is invertible, then so is T^* since $T^*(T^{-1})^* = (T^{-1}T)^* = 1^* = 1$, and similarly $(T^{-1})^*T^* = 1$. Thus, if $T - \lambda$ is invertible, then so is $T^* - \bar{\lambda}$, and vice-versa.

□

It is a theorem that every C^*-algebra can be embedded into $B(H)$ for some Hilbert space H.

5.1 Normal and Self-adjoint elements

Definition An element T is called **normal** when $T^*T = TT^*$, **self-adjoint** when $T^* = T$, and **unitary** when $T^* = T^{-1}$.

Every element can be written as $T = A + iB$ with A and B self-adjoint.

5.1.1 Exercises

1. Show that $T + T^*$, T^*T and TT^* are self-adjoint for any T.

Note: For $B(H)$, the group of invertible elements is connected, and generated by two exponentials. Proof: every $T = RU$ where R is self-adjoint and U unitary. Thus since logarithms exist, $R = e^S$ and $U = e^{iH}$ for self-adjoint operators S, H; in particular any element of the group of invertible elements is of this form. Moreover the map $t \mapsto e^{tS}e^{itH}$ gives a connected path starting from 1 and ending at T, so that the group is path-connected, and so connected.

□