Categories

joseph.muscat@um.edu.mt
1 December 2012

1 Objects and Morphisms

A **category** is a class of *objects* A with *morphisms* $f : A \to B$ (a way of comparing/substituting/mapping/processing A to B) such that,

(i) given morphisms $f : A \to B$, $g : B \to C$, $gf : A \to C$ is also a morphism,

(ii) for compatible morphisms, $h(gf) = (hg)f$, and

(iii) each object A has a morphism $a : A \to A$ satisfying $af = f$, $ga = g$.

(Note: in a sense, an object A *is* the morphism a; so we can even do away with objects.)

Sets can be considered as 0-categories (only objects or elements), or as discrete categories with each object A having one morphism a.

The class of morphisms from A to B is denoted $\mathrm{Hom}(A, B)$; thus $\mathrm{Hom}(A, A)$ is a monoid.

Even at this abstract level there are at least three important categories:

1. logic (with statements as objects and $\Rightarrow$ as morphisms),
2. sets (with functions as morphisms),
3. computing (with data types and algorithms).

1.1 Morphisms

A **monomorphism** $f : A \to B$ satisfies

$$\forall C,\ \forall x, y \in \mathrm{Hom}(C, A), \quad fx = fy \ \Rightarrow\ x = y.$$

$$C \overset{x}{\underset{y}{\rightrightarrows}} A \xrightarrow{f} B$$

An **epimorphism** $f : A \to B$ satisfies

$$\forall C,\ \forall x, y \in \mathrm{Hom}(B, C), \quad xf = yf \ \Rightarrow\ x = y.$$

$$A \xrightarrow{f} B \overset{x}{\underset{y}{\rightrightarrows}} C$$

1. In particular, for a monomorphism f, $fg = f \ \Rightarrow\ g = \iota_A$; for an epimorphism $gf = f \ \Rightarrow\ g = \iota_B$.

2. The composition of monomorphisms is a monomorphism, and of epimorphisms an epimorphism.

3. Conversely, if fg is a monomorphism then so is g, and if it is an epimorphism then so is f.

A monomorphism $f : A \to B$ is also called a *sub-object* of B. Monomorphisms with the same codomain have a pre-order: let $f \leqslant g$ for $f : A \to C$, $g : B \to C$ when $f = gh$ for some (mono)morphism $h : A \to B$;

$$\begin{tikzcd} A \arrow[dr, "f"] \arrow[dd, dashed, "h"'] & \\ & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

It can be made into a poset by using the equivalence relation $f \cong g$ when $f \leqslant g \leqslant f$.

An **isomorphism** is an invertible morphism, i.e., f has an inverse g such that $fg = \iota_B, gf = \iota_A$. In this case, A and B are called *isomorphic* (an equivalence relation); iff $f \leqslant g \leqslant f$. An isomorphism $f : A \to A$ is called an *automorphism*; for example, any ι_A; the automorphisms of A form a group.

If $gf = \iota$ then f is called a *split monomorphism* or *section* (has a left-inverse), and g a *split epimorphism* or *retraction* (has a right-inverse). A morphism with left and right inverses is an isomorphism (since then, $g_1 = g_1 f g_2 = g_2$).

An *extremal monomorphism* is a monomorphism f such that the only way $f = ge$ with e an epimorphism is that e is an isomorphism (and g a monomorphism). An *extremal epimorphism* is an epimorphism f such that $f = eg$ with e a monomorphism $\Rightarrow$ e is an isomorphism (and g an epimorphism). Thus a monomorphism which is an extremal epimorphism, or an epimorphism which is an extremal monomorphism, is an isomorphism.

Let $f \perp g$ mean $gx = yf \;\Rightarrow\; \exists u \; x = uf, y = gu$. A *strong* monomorphism is one such that Epi $\perp f$.

$$\text{Isomorphisms} \subseteq \text{SplitMono} \subseteq \text{StrongMono} \subseteq \text{ExtremalMono} \subseteq \text{Monomorphisms}$$

$$\text{Isomorphisms} \subseteq \text{SplitEpi} \subseteq \text{StrongEpi} \subseteq \text{ExtremalEpi} \subseteq \text{Epimorphisms}$$

Proof. If f is a split monomorphism with $gf = \iota$, then f is a monomorphism and g an epimorphism. If $f = hk$ with k an epimorphism, then $ghk = \iota$ and $kghk = k$, so $kgh = \iota$; thus k has the inverse gh.

If $f = ge$ is a strong monomorphism and e epi, then $e \perp f$, so $f\iota = ge \;\Rightarrow\; \exists u \; \iota = ue, g = fu$. So e is split and an epi, hence an isomorphism.

A morphism $f : A \to A$ is called *idempotent* when $f^2 = f$; for example, the *split idempotents* $f = gh$ where $hg = \iota$.

An object is called *finite*, when every monomorphism $f : A \to A$ is an automorphism. In particular, if $B \subseteq A \cong B$ then $A = B$.

Example: For Sets, a monomorphism is a 1-1 function; an epimorphism is an onto function; such functions are automatically split; an isomorphism is thus a bijective function; isomorphic sets are those with the same number of elements; a set is finite in the category sense when it is finite in the set sense.

Functors (or *actions*) are maps between categories that preserve the morphisms (and so the objects),

$$Ff : FA \to FB, \quad F\iota_A = \iota_{FA}, \quad F(fg) = FfFg$$

They preserve isomorphisms.

1.2 Constructions

Subcategory: a subset of the objects and morphisms; a *full subcategory* is a subset of the objects, with all the corresponding morphisms.

Dual category: $\mathcal{C}'$ has the same objects but with reversed morphisms $f^\top : B \to A$, and $g^\top f^\top := (fg)^\top$; so $\mathcal{C}'' = \mathcal{C}$. Every concept in a theorem has a co-concept in its dual (eg monomorphisms correspond to epimorphisms); every theorem in a category has a dual theorem in the dual category. A functor between dual categories is called a *dual functor*; a functor from a dual category to a category is called *contra-variant*, $F(fg) = F(g)F(f)$.

A *dagger* category is one for which there is a functor $\dagger : \mathcal{C} \to \mathcal{C}'$, where

$$(fg)^\dagger = g^\dagger f^\dagger, f^{\dagger\dagger} = f.$$

(Set cannot be made into a dagger category because there is a morphism $\varnothing \to 1$ but not vice-versa).

Product of Categories: $\mathcal{C} \times \mathcal{D}$ the objects are pairs (X, Y) with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and the morphisms are (f, g), where

$$(f_1, g_1)(f_2, g_2) := (f_1 f_2, g_1 g_2), \quad \iota_{(X,Y)} = (\iota_X, \iota_Y).$$

The *projection* functors are $\mathcal{C} \times \mathcal{D} \to \mathcal{C}$, $(f, g) \mapsto f$, and $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$, $(f, g) \mapsto g$.

$$(\mathcal{C} \times \mathcal{D})' \sim \mathcal{C}' \times \mathcal{D}'$$

(The product is the categorical product in Category)

Quotient Category: given a category and an equivalence relation on morphisms (of same objects) $\sim$, then $\mathcal{C}/\sim$ is that category with the same objects and with equivalence classes of morphisms. The map $\mathcal{C} \to \mathcal{C}/\sim$ defined by $F : A \mapsto A, f \mapsto [f]$, is a functor.

Arrow Category: $\mathcal{C}^\to$ consists of the morphisms of $\mathcal{C}$ (as objects), with the morphisms $f \to g$ being pairs of morphisms (h, k), such that $kf = gh$,

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "h"'] & \cdot \arrow[d, "k"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

and composition $(h_1, k_1)(h_2, k_2) := (h_1 h_2, k_1 k_2)$, and identities (ι_A, ι_B). Monomorphisms are those pairs (h, k) where h and k are monomorphisms. For example, the arrow category of sets is the category of functions.

Slice Category (or comma category): $\mathcal{C} \downarrow B$ is the subcategory where the morphisms have the same codomain B and $k = \iota$; the morphisms simplify to h where $f = gh$; similarly for the morphisms with the same domain. An object A is called *projective* when every morphism $f : A \to B$ factors through any epimorphism $g : C \to B$, $f = gh$. Dually, A is called *injective* when $f : B \to A$ factors through any monomorphism $f = hg$.

1.3 Functors

Functors can be thought of as higher-morphisms acting on objects and morphisms; or as a model of $\mathcal{C}$ in $\mathcal{D}$.

(Examples: the *constant* functor, mapping objects to a single one, and morphisms to its identity; the mapping from a subcategory to the parent category; *forgetful functor* (when structure is lost) and *inclusion functor* (when structure is added, minimally); the mapping which sends A to the set $\mathrm{Hom}(B, A)$ and a morphism f to the function $g \mapsto f \circ g$ is a functor from any category to the category of sets; similarly for $A \mapsto \mathrm{Hom}(A, B)$ and $f \mapsto (g \mapsto g \circ f)$ (contra-variant).)

A functor is called *faithful* when it is 1-1 on morphisms (and hence objects) It is *full* when it is onto all morphisms in $Hom(FA, FB)$; it is called *dense*, when it is onto all objects up to isomorphism. It is an *isomorphism* on categories when it is bijective on the morphisms $Hom(FA, FB)$. A dense isomorphism is called an *equivalence*, and the two categories are said to be equivalent $\mathcal{A} \sim \mathcal{B}$.

A (left) **adjoint** of a functor is $F^* : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms e, i such that $e : FF^* \to 1, i : 1 \to F^*F$ and $\mathrm{Hom}(F^*A, B) \sim \mathrm{Hom}(A, FB)$; hence $(FG)^* = G^*F^*$. (For example, a forgetful functor and inclusion functor are adjoints, with i being the embedding)

2-Categories: Categories with functors as morphisms form a Category; the identity functor is the one which leaves objects and morphisms untouched; (there is an initial object namely $\varnothing$, and a terminal object, $\{\,.\,\}$ It has the additional structure of a 2-functor, called a "**natural transformation**" (or 'homotopy'), between functors on the same categories, $\tau : F \to G$; two such functors map an object $A \in \mathcal{C}$ to two objects FA and GA in $\mathcal{D}$, and a natural transformation determines a morphism $\tau_A : FA \to GA$ between the two, such that $\forall f : A \to B, (Gf)\tau_A = \tau_B(Ff)$ (so $Ff \sim Gf$). A *natural isomorphism* is a natural transformation for which τ_A are isomorphisms.

With these notions, two categories are equivalent when there are functors F and F^* such that $F^*F \sim 1, FF^* \sim 1$ (or equivalently when F and F^* are isomorphisms with $F^*F \sim 1$). The auto-equivalences of a category form a symmetric monoidal category.

More generally, a 2-category is a set of objects A, with morphisms $f : A \to B$, and 2-morphisms $\tau : f_1 \to f_2$ (for some $f_1, f_2 \in Hom(A, B)$); 2-morphisms can be combined either "vertically" by composition $\tau_2\tau_1$, (and must be associative, with an identity), or "horizontally" $\sigma \circ \tau$: $gf \mapsto \sigma(g)\tau(f)$, such that

$$\begin{tikzcd}
{} \arrow[rr, "f"] & {} \arrow[d, dotted, "\tau_1"'] & {} & {} \arrow[rr, "g"] & {} \arrow[d, dotted, "\sigma_1"] & {} \\
{} \arrow[rr] & {} \arrow[d, dotted, "\tau_2"'] & {} & {} \arrow[rr] & {} \arrow[d, dotted, "\sigma_2"] & {} \\
{} \arrow[rr] & {} & {} & {} \arrow[rr] & {} & {}
\end{tikzcd}$$

$$\tau_2\tau_1 \circ \sigma_2\sigma_1 = (\sigma_2 \circ \tau_2)(\sigma_1 \circ \tau_1).$$

A 2-category with 1 object gives rise to a monoidal category (of the morphisms and 2-morphisms of the object); a 2-category with 1 object and 1 morphism gives a commutative monoid (of 2-morphisms).

The functors themselves form a category $\mathcal{D}^{\mathcal{C}}$ where morphisms are the natural transformations. $\mathcal{C}^1 \sim \mathcal{C}$; $\mathcal{C}^2$ is the category of arrows on $\mathcal{C}$.

2 Limits

When a category maps under a functor $F : \mathcal{C} \to \mathcal{D}$ to another category, the image of an object may have morphisms that were not present in $\mathcal{C}$; an object $A \in \mathcal{D}$ may sometimes determine a unique (up to isomorphism) object (called a **universal**) U_A in $\mathcal{C}$, which makes $F(U_A)$ closest to A in the sense that there is a unique morphism $\phi_A : F(U_A) \to A$, such that

$$\forall f : F(B) \to A, \exists! g : B \to U_A, f = \phi_A F(g).$$

$$\begin{tikzcd}
U_A & & F(U_A) \arrow[dr, "\phi_A"] & \\
& & & A \\
B \arrow[uu, dashed, "g"] & & F(B) \arrow[uu, dashed, "F(g)"] \arrow[ur, dashed, "f"'] &
\end{tikzcd}$$

A *co-universal* is similarly an object $U_A \in \mathcal{C}$ with a morphism $\phi_A : A \to F(U_A)$ such that $\forall f : A \to F(B), \exists! g : U_A \to B, f = F(g)\phi_A$.

In particular, sub-categories $\mathcal{C}$ may have universal properties:

Terminal object 1: $\forall A, \exists! f : A \to 1$ (for the empty sub-category). **Initial object** 0: $\forall A, \exists! f : 0 \to A$.

$$0 \longrightarrow A \longrightarrow 1$$

$(0, 0)$ is an initial object in $\mathcal{C} \times \mathcal{D}$. For example, $\{0\}$ and $\varnothing$ are the terminal and initial objects of sets; True and False are the ones for logic.

Isomorphism The closest objects for an object A with its identity morphism (the category 1), are its isomorphic copies. For example, sets with the same cardinality are isomorphic, while statements $A \Leftrightarrow B$ are so in logic.

Products: For the subcategory 2 (with only the identity morphisms), the closest object of A and B is $A \times B$, with morphisms $\pi_A : A \times B \to A$, π_B :

$A \times B \to B$ such that any other morphisms $p_A : C \to A$, $p_B : C \to B$ factor out through a unique morphism $g : C \to A \times B$, $p_A = \pi_A g$, $p_B = \pi_B g$.

$$\begin{tikzcd}
 & & A \\
C \arrow[urr, dashed, "p_A"] \arrow[r, dashed, "g"] \arrow[drr, dashed, "p_B"'] & A \times B \arrow[ur, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
 & & B
\end{tikzcd}$$

$1 \times A \cong A$; $A \times B \cong B \times A$; $(A \times B) \times C \cong A \times (B \times C)$.

For example, the usual product $A \times B$, and the statement A AND B are the products for sets and logic respectively.

More generally, starting with a discrete category, the closest object of A_i is $\prod_i A_i$, with $\pi_i : \prod_i A_i \to A_i$ i.e., if $p_i : X \to A_i$ are morphisms then there is a morphism $h : X \to \prod_i A_i$ with $p_i = \pi_i h$. A repeated product gives $A^{\mathcal{C}}$ (starting with a constant functor from a discrete category).

A **relation** on objects A, B is a monomorphism $R : \rho \to A \times B$.

Sums (or Co-products): $\coprod_i A_i$ is the dual of the product in the dual category i.e., it is the closest object with morphisms $\pi_i : A_i \to \coprod_i A_i$. For example, $A+B$ (disjoint union) and A OR B.

Equalizer: starting from the category with two objects A, B, and morphisms $f_i : A \to B$, their equalizer is the closest object E with (extremal mono-)morphism

$$\mathrm{eq} : E \to A, \qquad \forall i, j,\ f_i \mathrm{eq} = f_j \mathrm{eq}.$$

$$\begin{tikzcd}
E \arrow[r, "\mathrm{eq}"] & A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & B
\end{tikzcd}$$

For example, for Sets, $\{\, x : f_1(x) = f_2(x) \,\}$.

Equalizers are monomorphisms: let $e = \mathrm{eq}$, if $xe = ye$ then $xef = xeg$, so $\exists! u,\ xe = ue,\ x = u$; similarly $y = u = x$.

Co-equalizer: similarly an (extremal epi-)morphism

$$\mathrm{coeq} : Y \to E, \quad \forall i, j,\ \mathrm{coeq} f_i = \mathrm{coeq} f_j.$$

For example, the co-equalizer of a relation on a set X is the partition on it (for an equivalence relation, this partition is compatible with the relation).

Pullback (fibre product): starting from the category with objects X_i and morphisms $f_i : X_i \to Z$, then the pullback is the (unique…) closest object $\prod_Z X_i$ with morphisms

$$\pi_i : \prod_Z X_i \to X_i, \quad f_i \pi_i = \pi_Z.$$

$$\begin{tikzcd}[row sep=small] & A \arrow[dr, "f"] & \\ A \times_Z B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & & Z \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

The equalizer is a special case when the morphisms start from the same object. If Z is the terminal object, then $\prod_Z X_i = \prod_i X_i$. For example, the pullback on sets is $X \times_Z Y = \{ (x, y) : f(x) = g(y) \}$; in particular when g is the identity, $X \times_Z Y = f^{-1}Y$.

Pullback lemma: pullbacks form squares $(X \times_Z Y, X, Z, Y)$; if two adjacent squares form pullbacks, then so does the outer rectangle; if the outer rectangle and the right (or bottom) square are pullbacks, then so is the left (or upper) square.

Pullbacks preserve monomorphisms: If $fu = vg$ with f mono, and $gx = gy$, then $fux = vgx = vgy = fuy$, so $ux = uy$ and $x = y$ by uniqueness of pullbacks.

Push-out is that closest object $\coprod_Z X_i$ with

$$\pi_i : X_i \to \coprod_Z X_i, \quad \pi_i f_i = \pi_Z.$$

For example, for sets, the push-out $X \cup_Z Y$ is the set $X \cup Y$ with the elements $f(z) \in X$ and $g(z) \in Y$ identified.

Inverse Limit: starting from the subcategory of a chain of objects A_i with morphisms $f_{j,i}$ (such that $f_{k,i} = f_{k,j} f_{j,i}$), the inverse limit is the closest object $\lim_{\leftarrow} A_i$ with morphisms

$$\pi_i : \lim_{\leftarrow} A_i \to A_i, \qquad \pi_j = f_{j,i}\pi_i.$$

$$\lim_{\leftarrow} A_i \xrightarrow{\pi_i} \cdots A_3 \xrightarrow{f_{23}} A_2 \xrightarrow{f_{12}} A_1$$

(More generally, can start with a topology of objects rather than a chain.) The pullback is a special case. For example, the inverse limit of sets X_i is the set of sequences $x_i \in X_i$ such that $x_j = f_{j,i}(x_i)$.

Co-limit (Direct Limit) is similar with $\lim_{\rightarrow} A_i$ and morphisms

$$\pi_i : A_i \to \lim_{\rightarrow} A_i, \qquad \pi_i = \pi_j f_{j,i}.$$

More generally, for any subcategory, or any functor, $F : \mathcal{C} \to \mathcal{D}$ there may be a **limit** object $\lim F$ in $\mathcal{D}$ with (unique) morphisms $\pi_A : \lim F \to A$ $(A \in \mathcal{C})$ such that for any $f : A \to B$, $A, B \in \mathcal{C}$,

$$f\pi_A = \pi_B$$

$$\begin{tikzcd} \lim F \arrow[r, "\pi_A"] \arrow[rr, bend right, "\pi_B"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

and it is the closest such object in the sense that for any other $C \in \mathcal{D}$ with $fp_A = p_B$ then $\exists! u : C \to \lim F, \pi_A u = p_A$. A limit, if it exists, is unique up to isomorphism.

A **co-limit** is similar with $f_A : F(A) \to \mathrm{colim} F$ such that

$$\forall f : A \to B, \quad f_B F(f) = f_A.$$

In general, any functor from a category with an initial object to $\mathcal{C}$ has a limit; and any functor from a category with a terminal object has a co-limit.

A **complete** category is one in which every subcategory (or functor) has a limit. For example, the category of sets is complete and co-complete.

So, every functor has an adjoint $F^* : \mathcal{D} \to \mathcal{C}$ mapping $A \mapsto U_A$ and $f \mapsto g$; so that $FF^* \sim 1$, and similarly $F^*F \sim 1$.

A functor is said to be **continuous** when it preserves limits (e.g. right-adjoints) i.e., $\forall G, \lim(FG) = F(\lim G)$. It is *co-continuous* (e.g. left-adjoints) when it preserves co-limits.

The existence of products $A \times B$ and equalizers implies the existence of all finite limits. The $\mathrm{Hom}(A, .)$ functor is continuous, so it represents these limits by sets (and $\mathrm{Hom}(., A)$ takes colimits to limits).

A family of **zero morphisms** 0 are such that

$$\forall f, g, \quad 0f = g0$$

for example, when $0 \cong 1$ (called a *zero object*), $0 : A \to 0 \to B$ are zero morphisms.

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "0"'] \arrow[dr, "0"] & {} \arrow[d, "0"] \\ {} \arrow[r, "g"'] & {} \end{tikzcd}$$

In this case, the *kernel* of a morphism is the equalizer of f and 0 i.e., the closest (mono)morphism $k : K \to A$ such that $fk = 0$.

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

The co-kernel is the co-equalizer ie the closest (epi)morphism $k' : B \to K'$ such that $k'f = 0$.

A **pre-sheaf** is a contra-variant functor from a pre-order (or topology) to a category $F : O \to \mathcal{C}$ (the $F(x)$ are called *sections* of F over x) such that $x \leqslant y \;\Rightarrow\;$ there is a restriction morphism $F(x) \to F(y)$ with $\mathrm{res}_{x,x} = \iota_{F(x)}$ and $x \leqslant y \leqslant z \;\Rightarrow\; \mathrm{res}_{y,x}\mathrm{res}_{z,y} = \mathrm{res}_{z,x}$.

A **sheaf** is a continuous pre-sheaf (preserves limits). On a topological space X, the *stalk* at $x \in X$ is the direct limit of the open neighborhoods of x. So there is a morphism $F(U) \to F_x$ for $x \in U$ open (if the morphism is a function $f \mapsto f_x$, where f_x is called the germ at x). The *etale* space E is the space of stalks, with the continuous map $E \to X$, $F_x \mapsto x$. (the set of sheaves form a topos, with $\Omega =$ the disjoint union of all open sets) The space E is locally homeomorphic to X (i.e., there are isomorphic open sets in E and X that cover F_x and x).

For example, a sheaf of sets is a *bundle*, i.e., a collection of disjoint sets A_i with a map $\pi : \bigcup_i A_i \to I$, $\pi^{-1}(i) = A_i$; the category of bundles over I is the same as the comma category.

2.1 Monoidal Categories

Objects have an associative functor *tensor* product $A \otimes B$ and an object I (called *unit*) such that

$$I \otimes A \cong A \cong A \otimes I$$

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

$$(A \otimes I) \otimes B \cong A \otimes B \cong A \otimes (I \otimes B)$$

(the isomorphisms in the first two lines are called the two *unitor* and one *associator* natural isomorphisms; more generally, any product of n objects are isomorphic to each other). Product of morphisms $f \otimes g : A \otimes B \to C \otimes D$.

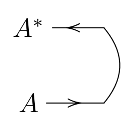
The tensor product is like treating two objects in parallel; so a morphism $f : A \otimes \ldots \otimes B \to C \otimes \ldots \otimes D$ takes n objects and "maps" them to m objects, and looks like a Feynman diagram. The unit object is null, so $f : I \to A$ "creates" one object. The tensor product is different from the categorical product in that there need not be projections.

The morphisms $Hom(I, I)$ now have two operations: $(f \otimes g)(h \otimes k) = (fh) \otimes (gk)$; but from universal algebras, this implies that $f \otimes g = fg$ and is commutative.

Set with $\times$ is monoidal (in fact cartesian-closed); Set with disjoint union is also monoidal.

The (right) **dual** of an object A is another object A^* (unique up to isomorphism), such that there are "annihilation/creation" morphisms

$$A \otimes A^* \to I, \qquad I \to A^* \otimes A,$$

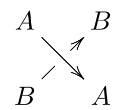


called the *co-unit* of A and the *unit* of A, respectively, satisfying the zig-zag equations, i.e., creating then annihilating A and A^* leaves nothing I; (A^* can be represented as a line in the opposite direction of A; A is called the left dual of A^*).

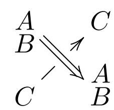
2.1.1 Braided Monoidal categories

A monoidal category in which there is a natural isomorphism that switches objects around,

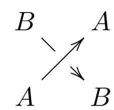
$$A \otimes B \cong B \otimes A,$$



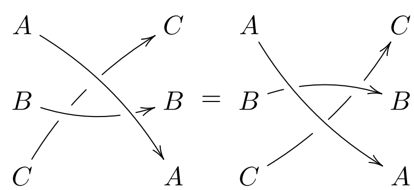
such that all permutations of products become isomorphic, e.g. $(A \otimes B) \otimes C \cong C \otimes (B \otimes A)$, i.e.,



It need not be its own inverse! Its inverse is:



The Yang-Baxter equation states



Left duals are duals.

A braided monoidal category is called *symmetric* when the switching isomorphism is its own inverse.

2.2 Closed Monoidal Categories

A monoidal category is **closed** when every set of morphisms $\mathrm{Hom}(A, B)$ has an associated object B^A, with

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, C^A)$$

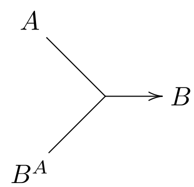
(or alternatively $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, C^B)$) (via "currying" natural isomorphisms). That is, every morphism can be treated as an object (without inputs). In particular $f : A \to B$ is associated to $I \to B^A$.

For example, in sets, the powerset axiom asserts that $Hom(A, B)$ is a set B^A; in logic the distinction is between the morphism $A \vdash B$ and the object $A \;\Rightarrow\; B$.

A monoidal category is **compact** (or *autonomous*) when every object has a dual and a left dual. In this case it is closed, with $A^B := B^* \otimes A$, i.e., $A^* \cong \mathrm{Hom}(A, I)$; in particular the unit I corresponds to a unit inside $A^* \otimes A$.

The reverse of currying, changing an object into a morphism, is an *evaluation* morphism

$$\mathrm{eval} : A \otimes B^A \to B, \qquad \mathrm{eval}(f \otimes \iota_A) = f.$$



(So morphisms of two variables become morphisms of one variable.)

For example, in sets (and functional programming languages), $\text{eval}(f, x) = f(x)$; in logic, it is modus ponens, A and $A \Rightarrow B$ gives B.

2.3 Cartesian-closed categories

Finite products exist and are closed, i.e., every functor $\times A$ has a right-adjoint $\ ^A$, called exponentiation,

$$\text{Hom}(A \times B, C) \cong \text{Hom}(B, C^A)$$

This means that every morphism $f : \prod_i A_i \to C$ can be represented by an ordered set of morphisms $f_i : A_i \to C$.

It is thus symmetric braided monoidal, with $\otimes$ being $\times$ and the unit being the terminal object 1; but has more properties in that it can *duplicate* objects via $\Delta : A \to A \times A$; and *delete* objects by mapping to 1, i.e., $! : A \mapsto 1$; every morphism $f : 1 \to A \times B$ is of the type $(1, 1) : 1 \to A, 1 \to B$.

(e.g. the adjoint of $X \mapsto (X, X)$ is $(X, Y) \mapsto X \times Y$.)

$f \times g : A \times B \to C \times D$ can be defined as that unique morphism induced by $f\pi_A, g\pi_B$. In particular, $(1_a, 1_b) = 1_{a \times b}$. Similarly, can define the sum $f + g$.

2.3.1 Evaluation

$$\text{eval} : A \times B^A \to B, \qquad \text{eval}(f \times \iota_A) = f.$$

An *element* or *point* of A is a morphism $x : 1 \to A$; so $\text{eval}(f, x) = fx$.

In particular a morphism $f : A \to B$ corresponds to an element $1 \to B^A$ (called the *name* of f).

In such categories, dual concepts lose their symmetry:

There are no morphisms $A \to 0$ unless $A \cong 0$, in particular if $0 \cong 1$, then all objects are isomorphic; $0 \to A$ is monic.

$$0 \times A \cong 0, \ A^1 \cong A, \ A^0 \cong 1, \ 1^A \cong 1$$

(proofs: there is only one morphism $0 \to B^A$, so only one morphism $0 \times A \to B$ so $0 \cong 0 \times A$, and $A \to 0 \times A \cong 0 \to A$ forces them to be isomorphisms; *eval* $: A^1 \to A$ is an isomorphism; $1 \to A^0$ corresponds to $0 \cong 1 \times 0 \to A$ which is unique, so $1 \to A^0$ and $A^0 \to 1$ are inverses; $1^A \to 1$ must be ι and $1 \to 1^A$ corresponds to $A \to 1$ also unique; any map $B \to 0$ is a unique isomorphism so $fg = fh \Rightarrow g = h$)

$$X^{A+B} \cong X^A \times X^B, \qquad (A \times B)^C \cong A^C \times B^C,$$

$$(C^A)^B \cong C^{A\times B}, \qquad X \times (A+B) \cong X\times A + X \times B$$

(Proofs: the inclusions $A, B \to A+B$ give $X^{A+B} \to X^A \times X^B$; conversely, $X^A \times X^B \to X^{A+B}$ correspond to $A+B \to X^{X^A\times X^B}$ i.e., to two inclusion maps, and hence the projections $X^A \times X^B \to X^A, X^B$;

The projections $A\times B \to A, B$ give rise to a map $(A\times B)^C \to A^C \times B^C$; its inverse is $A^C \times B^C \to (A\times B)^C$ which corresponds to $C \times A^C \times B^C \to A \times B$ i.e., to $C\times A^C \times B^C \to A, B$, i.e., the projections $A^C \times B^C \to A^C, B^C$;

$C^{A\times B} \to (C^A)^B$ corresponds to $B \times C^{A\times B} \to C^A$, i.e., the evaluation map $A \times B \times C^{A\times B} \to X$, similarly $(C^A)^B \to C^{A\times B}$ corresponds to the double evaluation $B \times A \times (C^A)^B \to C$.;

The maps $A + B \to (X\times A + X\times B)^X$ correspond to the inclusions $X \times A, X\times B \to X\times A + X \times B$) There is a functor mapping morphisms $f : X_1 \to X_2$ to $Ff : X_1^Y \to X_2^Y$ defined by $(Ff)g = fg$ for $g : Y \to X_1$. There is another contra-variant functor (restriction?) mapping morphisms $f : Y_1 \to Y_2$ to $Ff : X^{Y_2} \to X^{Y_1}$, defined by $(Ff)g = gf$.

2.4 Topos

A category with finite limits, exponentials (i.e., cartesian-closed), and a sub-object classifier.

A **sub-object classifier** is an object Ω (unique up to isomorphism) and a morphism True $: 1 \to \Omega$ such that monomorphisms $f : A \to B$ ("sub-objects") correspond to unique morphisms

$$\chi_f : B \to \Omega, \qquad \chi_f f = A \to 1 \to \Omega$$

$$\begin{tikzcd}
 & & 1 \arrow[d, shift left, "\text{True}"] \\
A \arrow[rru] \arrow[r, "f"'] & B \arrow[ru] \arrow[rd, "\chi_f"'] & \\
 & & \Omega \arrow[u, shift left]
\end{tikzcd}$$

In particular True corresponds to $\chi_{\text{True}} = \iota_\Omega$, and the unique monomorphism $0 \to \Omega$ corresponds to a morphism $\neg : \Omega \to \Omega$; hence False $:= \neg True : 1 \to \Omega$.

For example, for sets $\Omega = 2$; sub-objects $B : I \to X$ correspond to subsets $B \subseteq X$; subsets are maps $A \to 2$ and correspond to the characteristic maps $\chi_A : 1 \to 2^A$; a *singleton* is a map $A \to 2^A$.

Other logical connectives are defined in terms of their characteristic maps:

AND $: \Omega\times\Omega \to \Omega$	$(\text{True}, \text{True}) : 1 \to \Omega \times \Omega$
OR $: \Omega\times\Omega \to \Omega$	$(\text{True}_\Omega, \iota_\Omega), (\iota_\Omega, \text{True}_\Omega) : \Omega + \Omega \to \Omega \times \Omega$
$\Rightarrow : \Omega\times\Omega \to \Omega$	$2 \to \Omega\times\Omega$(where 2 is the category $0 \leqslant 1$)
complement of f	$\neg\chi_f$
intersections $f \cap g$	$\chi_{f\cap g} := \chi_f$ AND χ_g
unions $f \cup g$	$\chi_{f\cup g} := \chi_f$ OR χ_g.

But there may be several truth values, i.e., Ω may have several elements $1 \to \Omega$, not just True and False.

Ω is injective, i.e., for any monomorphism $f : A \to B$ and any morphism $g : A \to \Omega$, there is a morphism $\bar{g} : B \to \Omega$ such that $g = \bar{g}f$. Ω^A can be thought of as a "dual" of A; the Fourier map $\hat{} : A \to \Omega^{\Omega^A}$ defined by $\hat{x}(f) = fx$;

$f \cong g \Leftrightarrow \chi_f = \chi_g$; the sub-objects of A form a bounded lattice, $Sub(A) \cong Hom(A, \Omega)$. A morphism is an isomorphism $\Leftrightarrow$ it is both mono and epi (called a bi-morphism) (since an epi monomorphism $f : A \to B$ is the equalizer of χ_f and Trueι_B). Every morphism factors as $f = gh$ where h is epi and g is mono (via the object fA obtained by the pushout of f with itself). The pull-back of an epimorphism is also epi. Coproducts preserve pullbacks. (implies finite co-limits also exist)

Every category can be extended to a topos. The product of topoi is a topos. A comma category $\mathcal{C}/A$ of a topos is also a topos; its elements are bundles of elements (i.e., sections) of A.

Every topos has power objects $P(A) := \Omega^A$, meaning objects $P(A)$ and ϵ_A and a monomorphism $\in : \epsilon_A \to P(A) \times A$ such that every *relation* (i.e., monomorphism) $r : R \to B \times A$ has an associated unique morphism $f_r : B \to P(A)$ such that $R \to B \times A \to P(A) \times A = R \to \epsilon_A \to P(A) \times A$.

$$\begin{tikzcd}
\epsilon_A \arrow[r, "\in"] & A \times P(A) \\
R \arrow[u] \arrow[r, "r"] & A \times B \arrow[u]
\end{tikzcd}$$

$\Omega \cong P(1)$. Conversely every category with finite limits and power objects is a topos.

2.4.1 Well-pointed topos

A topos that satisfies the extensionality axiom, elements are epi:

$$\forall x : 1 \to A, fx = gx \;\Rightarrow\; f = g.$$

A morphism is mono $\Leftrightarrow$ it is 1-1, i.e., $fx = fy \;\Rightarrow\; x = y$ for all $x, y : 1 \to A$.

A morphism is epi $\Leftrightarrow$ it is onto, i.e., $\forall y : 1 \to B, \exists x : 1 \to A, fx = y$.

The only non-empty object (i.e., without any elements $1 \to A$) is the initial object (since $\chi_{1_A} \neq \chi_{0_A}$). The only elements of Ω are $True$ and $False$ (bivalent), and $\Omega \cong 1+1$ (Boolean). In fact a topos is well-pointed $\Leftrightarrow$ the only non-empty object is the initial one, and $\Omega \cong 1 + 1$.

The arrow category $Set^{\to}$ is neither Boolean nor bivalent; Set^2 is Boolean but not bivalent; the category of actions of a monoid (that is not a group) is bivalent but not Boolean.

2.4.2 With Axiom of Choice

A category is called **balanced** when f is an isomorphism $\Leftrightarrow$ it is a monomorphism and an epimorphism.

A category satisfies an **Axiom of Choice** when every epimorphism is right-invertible (splits). So balanced.

For example, in sets, every monomorphism has a left-inverse, except for $0 \to A$; the axiom of choice says that every epimorphism has a right-inverse.

Strong Axiom of Choice: $\forall f, \exists g, f = fgf$.

A topos with the axiom of choice has the *localic* property: $\exists i : C \to 1$ monomorphism and $g_1 \neq g_2 \Rightarrow \exists f : C \to A, g_1 f \neq g_2 f$.

Also every object has a complement $X = A + A'$.

2.5 Pre-additive Categories

When $\mathrm{Hom}(A, B)$ is an abelian group, distributive over composition of morphisms ie $f(g + h) = fg + fh, (f + g)h = fh + gh$. (then $\mathrm{Hom}(A, A)$ is a ring)

Can be extended to an Abelian category.

2.5.1 Additive Categories

A pre-additive category with finite products and sums;

2.5.2 Abelian Categories

an additive category in which every morphism has a kernel and a co-kernel (so there is a zero object), and every monomorphism is a kernel and every epimorphism is a co-kernel.

2.6 Concrete category

one in which the objects are sets and the morphisms are functions; ie a category which has a faithful functor $\mathcal{C} \to \mathrm{Sets}$ (called the forgetful functor).

2.6.1 Category of Sets

One can even consider set theory from the categorical point of view with the following axioms:

1. Sets and functions form a category;
2. Sets have finite limits and co-limits;
3. Sets allow exponentiation;
4. Sets have a sub-object classifier (so form a topos); this is a form of comprehension axiom;
5. With a morphism $T : 1 \to 2$;
6. Sets are Boolean in the sense that the truth-value object 2 is given by $1 + 1$;

7. 2 has two elements (up to isomorphism);

8. Axiom of Choice (every epimorphism has a right-inverse);

9. There is an infinite (inductive) set.

It then follows that for every $A \neq 0, \exists A \to 1$ epimorphism and $\exists x : 1 \to A$ morphisms (since $A \to 1$ is unique, which gives $A \to B \to 1$ where $A \to B$ is an epimorphism; but $A \neq 0 \Rightarrow B \neq 0$, so $B = 1$; the axiom of choice gives a morphism $x : 1 \to A$); every monomorphism $A \to B$ induces a "complement" monomorphism $A' \to B$ (the pullback of $B \to \Omega$ along $F : 1 \to \Omega$).

3 Research Questions

Most grand questions in pure mathematics are of the following type:

1. Syntax: given a set of mathematical structures/examples, to find a minimal set of axioms common to all.

2. Semantics: given a set of axioms, to discover all mathematical examples satisfying them; classify all possible spaces X in a category i.e., give a concrete description of the spaces, up to isomorphism.

This problem may be too hard or even impossible to answer, so the first attempt is to restrict X to the smaller ones, or else ask an easier question

2a. Find a way of distinguishing spaces: given any two spaces X, Y is there a way of showing whether they are isomorphic or not?

2b. Can one show whether X is isomorphic to a known space?

2c. In particular is X isomorphic to the trivial space?