Ordinary Differential Equations

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(See also the Interactive Version: needs Acrobat Reader 9)

CHAPTER 1

Introduction

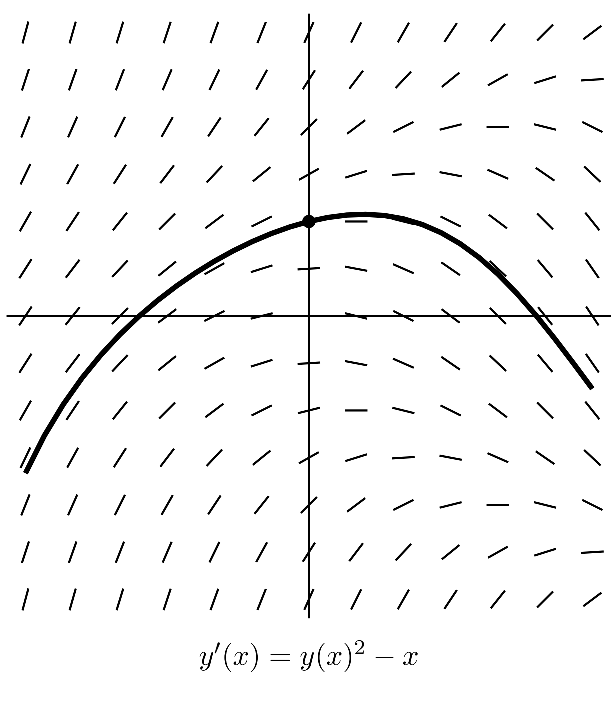
Definition An *ordinary* **differential equation** is an equation that specifies the derivative of a function $y : \mathbb{R} \to \mathbb{R}$ as

$$y'(x) = F(x, y(x)).$$

More generally, an *nth order ordinary differential equation* specifies the nth derivative of a function as

$$y^{(n)}(x) = F(x, y(x), \ldots, y^{(n-1)}(x)).$$

One can visualize a first-order o.d.e. by plotting the function $F(x, y)$ as slopes, for example,



$y'(x) = y(x)^2 - x$

A *solution* is then a function $y(x)$ that passes through the slopes. The main problem in o.d.e.'s (ordinary differential equations) is to find solutions given the differential equation, and to deduce something useful about them.

The simplest differential equation, $y' = f$, can be solved by integrating f to give $y(x) = \int f(x)\,\mathrm{d}x$.

Example: *Free fall.* Galileo's rule for free-fall is $\dot{y} = -gt$; integrating gives $y(t) = -\frac{1}{2}gt^2 + y_0$, where y_0 is arbitrary.

We learn several things from this simple example:

(a) Solving differential equations involves integration. Unlike differentiation, integration has no steadfast rules; one often has to *guess* the answer. Similarly, we expect that solving a differential equation will not be a straightforward affair. In fact many hard problems in mathematics and physics[1] involve solving differential equations.

(b) The solution is not unique: we can add any constant to y to get another solution. This makes sense — the equation gives us information about the derivative of y, and not direct information about y; we need to be given more information about the function, usually by being given the value of y at some point (called the initial point).

1.1 Separation of Variables

The simplest non-trivial differential equations which can be solved generically are of the type

$$y'(x) = f(x)g(y(x)).$$

These can be solved by *separating* the y-variable from the x (or t).

Examples: *Population Growth* and *Radioactive Decay*: $\dot{y} = ky$. Collecting the y-variables on the left, we get

$$\frac{\dot{y}}{y} = k, \quad \Rightarrow \int \frac{1}{y}\,\mathrm{d}y = kt + c$$

$$\Rightarrow \log y(t) = kt + c, \quad \Rightarrow y(t) = Ae^{kt}$$

where A is an arbitrary constant.

Newton's law of cooling: $\dot{T} = \alpha(T_0 - T)$.

$$\int \frac{1}{T_0 - T}\,\mathrm{d}T = \alpha t + c \Rightarrow -\ln(T_0 - T) = \alpha t + c$$
$$\Rightarrow T = T_0 + (T_1 - T_0)e^{-\alpha t}$$

[1] There is a \$1 million Clay prize for showing there is a unique solution to the Navier-Stokes equation.

Free fall with air resistance: $\dot{v} = -g - kv$.

$$\int \frac{1}{kv+g}\,\mathrm{d}v = -t + c$$

$$\Rightarrow \ \frac{1}{k}\log|kv+g| = -t + c$$

$$\Rightarrow \ v(t) = -g/k + Ae^{-kt}$$

In the general case of $y' = f(x)g(y)$,

$$\frac{y'(x)}{g(y(x))} = f(x)$$

$$\Rightarrow \ \int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

by the change of variable $x \mapsto y$. If the integrals can be worked out and inverted, we find

$$y(x) = G^{-1}(F(x) + c)$$

Note that it may not be immediately obvious that a function $F(x, y)$ is separable, e.g. $(x + y)^2 - x^2 - y^2$ and $\sin(x + y) + \sin(x - y)$ are separable. Other equations become separable only after a change of variable, e.g.

1. *Homogeneous equations.* $y' = F(y/x)$, use the change of variable $u := y/x$, so that $F(u) = y' = (ux)' = xu' + u$ and $u' = (F(u) - u)/x$.

2. Similarly, for $y' = F(ax+by+c)$ (a, b, c constant), use $u := ax+by+c$, giving $u' = a + by' = a + bF(u)$.

1.1.1 Exact Equations

More generally, if

$$y' = \frac{f(x,y)}{g(x,y)} \quad \text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = 0,$$

then there is a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = f$ and $\frac{\partial F}{\partial y} = -g$. It can be found by integrating $F = \int f(x, y)\,\mathrm{d}x = -\int g(x, y)\,\mathrm{d}y$. Hence $\frac{\mathrm{d}}{\mathrm{d}x}F(x, y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' = f - gy' = 0$ so the solution is given implicitly by $F(x, y) = c$.

1.2 Linear Equations

1.2.1 First Order

$$y'(x) + a(x)y(x) = f(x)$$

Multiply both sides by the *integrating factor* $I(x) := e^{\int a(x)\,\mathrm{d}x}$, which satisfies $I'(x) = a(x)I(x)$. Hence $(Iy)' = Iy' + aIy = If$, so integrating gives

$$y(x) = I(x)^{-1}\int^x I(s)f(s)\,\mathrm{d}s + AI(x)^{-1}.$$

Example. $y' = x + 2y$. The integrating factor is $I(x) = e^{\int -2} = e^{-2x}$,

$$\begin{aligned}(e^{-2x}y)' &= e^{-2x}(y' - 2y) = xe^{-2x}\\ \Rightarrow\ e^{-2x}y &= \int xe^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c\\ \Rightarrow\ y(x) &= ce^{2x} - \frac{1}{2}(x + \frac{1}{2})\end{aligned}$$

1.2.2 Second Order

$$y''(x) + ay'(x) + by(x) = f(x), \qquad a, b \text{ constant}$$

Factorizing the left-hand side gives $(D-\alpha)(D-\beta)y(x) = f(x)$ where α and β are the roots of the quadratic $D^2 + aD + b$ (sometimes called the *auxiliary equation*). If we write $v(x) := (D-\beta)y(x)$, the equation becomes first-order, so

$$y'(x) - \beta y(x) = v(x) = Ae^{\alpha x} + e^{\alpha x}\int e^{-\alpha s}f(s)\,\mathrm{d}s$$

$$\begin{aligned}y(x) &= Be^{\beta x} + e^{\beta x}\int e^{-\beta t}[Ae^{\alpha t} + e^{\alpha t}\int e^{-\alpha s}f(s)\,\mathrm{d}s]\,\mathrm{d}t\\ &= Be^{\beta x} + \frac{A}{\alpha-\beta}e^{\alpha x} + e^{\beta x}\iint e^{(\alpha-\beta)t-\alpha s}f(s)\,\mathrm{d}s\,\mathrm{d}t, \qquad (\alpha \neq \beta)\end{aligned}$$

If $\alpha = \beta$, the first integral becomes $Axe^{\beta x}$ instead, so the general solution is

$$y(x) = y_p(x) + \begin{cases} Ae^{\alpha x} + Be^{\beta x}, & \alpha \neq \beta \\ (A + Bx)e^{\alpha x} & \alpha = \beta \end{cases},$$

$$y_p(x) = e^{\beta x}\iint e^{(\alpha-\beta)t-\alpha s}f(s)\,\mathrm{d}s\,\mathrm{d}t$$

If the roots are complex, $\alpha = r + i\omega$, $\beta = r - i\omega$, then the expression $Ae^{\alpha x} + Be^{\beta x} = e^{rx}(Ae^{i\omega x} + Be^{-i\omega x}) = e^{rx}((A+B)\cos\omega x + i(A-B)\sin\omega x)$

can become real when $A+B$ is real (take $x=0$) and $A-B$ is imaginary (take $\omega x=\pi/2$), i.e., $B=\bar{A}$ and the solution involves $Ce^{rx}\cos\omega x+De^{rx}\sin\omega x$ for arbitrary real constants C,D.

Example. *Simple Harmonic Motion: Resonance* The equation

$$\ddot{y}+\omega^2 y=\cos\omega t$$

has solution

$$y(t)=A\cos\omega t+B\sin\omega t+(2\omega)^{-1}t\sin\omega t$$

which grows with t. This can occur, for example, in an oscillating *spring* which satisfies $\ddot{x}=-kx-\nu\dot{x}+f(t)$, or an *electrical* oscillator $L\ddot{I}+R\dot{I}+C^{-1}I=E(t)$.

Of course, this method *only* works when the coefficients are *constant*. Even the simple equation $y''=xy$ has solutions that cannot be written as combinations of elementary functions (polynomials, exponential, trigonometric, etc.)

1.2.3 Reduction to Linear Equations

Several equations can become linear with a correct change of variable:

Bernoulli's equation $y'=ay+by^n$ $(n\neq 1)$. Use the change of variable $u:=y^{1-n}$ to yield $u'+(n-1)au=b$.

Euler's equation $x^2y''+axy'+by=f(x)$, use $u(X):=y(x)$ where $X=\log x$, so that $xy'(x)=xu'(X)/x=u'(X)$, and similarly, $x^2y''(x)=u''(X)$, and the equation becomes $u''+au'+bu=f(e^X)$.

Riccati's equation $y'=a+by+cy^2$. First eliminate the by term by multiplying throughout by $I:=e^{-\int b}$ and letting $v:=Iy$, then substitute $v=-u'/cu$ to get $cu''-c'u'+ac^2u=0$.

1.3 Non-linear Equations

These are also common in applications. Here is a sample:

1. Newton's law of gravity: $\ddot{\boldsymbol{x}}=-GM/|\boldsymbol{x}|^2$.
2. Reaction rates of chemistry: $\dot{x}_i=\sum_{ij}\alpha_{ij}x_i^{n_i}x_j^{n_j}$
3. Lotka-Volterra predator-prey equations: $\dot{u}=uv-\lambda u$, $\dot{v}=-uv+\mu v$.
4. Solow's model in economics: $\dot{k}=sF(k)-nk$
5. Geodesics in geometry: $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}}=\frac{\partial L}{\partial x}$.

In general, their solution is not a straightforward affair at best. The most that we can aim for in this course is to describe their solutions qualitatively not solve them exactly.

CHAPTER 2

Existence of Solutions

Definition An **initial-value problem** is a first order o.d.e. whose solution satisfies an initial constraint:

$$y' = F(x, y) \quad \text{on } x \in [\alpha, \beta], \quad y(a) = Y$$

An initial value problem is said to be **well-posed** when

1. a solution exists
2. the solution is unique
3. the solution depends continuously on Y.

It has to be remarked straightaway that initial-value problems need not have a solution: for example, the equation $y' = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$, $y(0) = 0$, admits no differentiable solutions.

Even if a solution exists, it might not be unique. Consider the example $y' = \sqrt{y}$ with $y(0) = 0$; it has at least two solutions $y(x) = 0$ and $y(x) = x^2/4$. In fact it has many more, including

$$y(x) = \begin{cases} 0 & x < c \\ (x - c)^2/4 & x \geqslant c \end{cases}$$

Finally, even if there is only one solution, changing the initial point slightly might produce a drastic change in the solution. For example, if we take again $y' = \sqrt{y}$ with $y(0) = Y \neq 0$, then there is a unique solution $y(x) = (x - Y^{\frac{1}{2}})^2$ on the positive real line, until Y reaches 0 when the solution can drastically change to $y(x) = 0$.

The main theorem of this chapter, Picard's theorem, also called the Fundamental Theorem of O.D.E.'s, is that when F is a nice enough function of x and y, the initial value problem is well-posed.

2.1 Picard's Theorem

Definition A function $F(x,y)$ is called **Lipschitz** in y in a domain $A\times B$ when

$$\exists k>0 : \forall x\in A,\ \forall y_1,y_2\in B,\quad |F(x,y_1)-F(x,y_2)|\leqslant k|y_1-y_2|$$

Example: Functions that are continuously differentiable in y are Lipschitz on any bounded domain. By the mean-value theorem, for (x,y) in the bounded domain,

$$\left|\frac{F(x,y_1)-F(x,y_2)}{y_1-y_2}\right| = \left|\frac{\partial F}{\partial y}(x,\xi)\right| \leqslant k.$$

Thus $|F(x,y_1)-F(x,y_2)|\leqslant k|y_1-y_2|$.

Note that Lipschitz functions are continuous in y since, as $y_1\to y_2$, then $F(x,y_1)\to F(x,y_2)$. That is, the Lipschitz condition on functions is somewhere in between being continuous and continuously differentiable.

Theorem 2.1

(Picard-Lipschitz) Fundamental Theorem of O.D.E.s

Given the initial value problem

$$y'(x)=F(x,y(x)),\qquad y(a)=Y,$$

if F is continuous in x and Lipschitz in y in a neighborhood of the initial point $x\in(a-h,a+h), y\in(Y-l,Y+l)$, then the o.d.e. has a unique solution on some (smaller) interval, $x\in(a-r,a+r)$, that depends continuously on Y.

Proof. The o.d.e. with the initial condition is equivalent to the following integral equation:

$$y(x)=Y+\int_a^x F(s,y(s))\,\mathrm{d}s$$

Define the following Picard iteration scheme on the interval $x\in[\alpha,\beta]$:

$$\begin{aligned}
y_0(x) &:= Y\\
y_1(x) &:= Y+\int_a^x F(s,Y)\,\mathrm{d}s\\
y_2(x) &:= Y+\int_a^x F(s,y_1(s))\,\mathrm{d}s\\
&\dots\\
y_{n+1}(x) &:= Y+\int_a^x F(s,y_n(s))\,\mathrm{d}s
\end{aligned}$$

Notice that each of these functions is continuous in x.

Initially we assume that all the x and y encountered in the expressions are in a rectangle $(a-r, a+r) \times (Y-l, Y+l)$, where $r \leqslant h$, so that F is Lipschitz on them

$$|F(x,y) - F(x,\tilde{y})| \leqslant k|y-\tilde{y}|$$

We will later justify this assumption.

1. The iteration converges: for each $x \in (a-r, a+r)$, the sequence $y_n(x)$ converges. We say that y_n converges pointwise to some function y. First we prove the following by induction on n:

$$|y_{n+1}(x) - y_n(x)| \leqslant \frac{ck^n|x-a|^{n+1}}{(n+1)!}$$

When n is 0,

$$\begin{aligned} |y_1(x) - y_0(x)| &= |\int_a^x F(s,Y)\,\mathrm{d}s| \\ &\leqslant c|\int_a^x 1\,\mathrm{d}s| \\ &\leqslant c|x-a| \end{aligned}$$

where $c := \max_{s\in[a-r,a+r]} F(s,Y)$, which exists since F is continuous in s.

Assuming the claim for $n-1$,

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &= \left|\int_a^x F(s,y_n(s)) - F(s,y_{n-1}(s))\,\mathrm{d}s\right| \\ &\leqslant \int_a^x |F(s,y_n(s)) - F(s,y_{n-1}(s))|\,\mathrm{d}s \\ &\leqslant \int_a^x k|y_n(s) - y_{n-1}(s)|\,\mathrm{d}s \\ &\leqslant k\int_a^x \frac{ck^{n-1}|s-a|^n}{n!}\,\mathrm{d}s \\ &= \frac{ck^n}{n!}\frac{|x-a|}{n+1}^{n+1} \end{aligned}$$

Now $\displaystyle\sum_{n=0}^{\infty} \frac{ck^n}{(n+1)!}|x-a|^{n+1}$ converges. Therefore, by comparison, $\sum_{n=0}^{\infty} |y_{n+1}(x) - y_n(x)|$ also converges (absolutely). Hence,

$$\lim_{n\to\infty} y_n(x) = Y + \sum_{n=0}^{\infty}(y_{n+1}(x) - y_n(x))$$

converges to a function $y(x)$. Indeed the convergence is uniform in x because

$$\begin{aligned}
|y_{n+1}(x) - y_n(x)| &\leqslant \frac{c}{k}\frac{k^{n+1}|x-a|^{n+1}}{(n+1)!} \\
\Rightarrow \ |y(x) - y_N(x)| &\leqslant \sum_{n=0}^{N-1} |y_{n+1}(x) - y_n(x)| \\
&\leqslant \frac{c}{k}\sum_{n=N}^{\infty} \frac{k^{n+1}|x-a|^{n+1}}{(n+1)!} \\
&\leqslant \frac{c}{k}\frac{k^{N+1}h^{N+1}}{(N+1)!}e^{kh}
\end{aligned}$$

Recall that the uniform convergence of continuous functions is again continuous.

2. $y(x)$ is a solution: Pick y_n which is close to y. This is possible since y_n converges uniformly to y.

$$\forall \epsilon > 0, \ \exists N, \ \forall x \in [a-r, a+r], \quad n > N \ \Rightarrow \ |y_n(x) - y(x)| < \epsilon$$

Therefore,

$$\begin{aligned}
\left|\int_a^x F(s, y_n(s))\,\mathrm{d}s - \int_a^x F(s, y(s))\,\mathrm{d}s\right| &\leqslant \left|\int_a^x |F(s, y_n(s)) - F(s, y(s))|\,\mathrm{d}s\right| \\
&\leqslant \left|\int_a^x k|y_n(s) - y(s)|\,\mathrm{d}s\right| \\
&< k\epsilon|x-a| \\
&\leqslant \epsilon k h
\end{aligned}$$

which is as small as required. So taking the limit $n \to \infty$ of the expression

$$y_{n+1}(x) = Y + \int_a^x F(s, y_n(s))\,\mathrm{d}s$$

gives that y is a solution of the o.d.e.

$$y(x) = Y + \int_a^x F(s, y(s))\,\mathrm{d}s$$

3. y is unique: Suppose that $u(x)$ is another solution,

$$u(x) = Y + \int_a^x F(s, u(s))\,\mathrm{d}s.$$

y and u are bounded functions on $[a-r, a+r]$ since they are continuous (they are integrals of a continuous function).

$$|y(x) - u(x)| < C \quad \forall x \in [a-r, a+r]$$

Therefore,

$$\begin{aligned}|y(x)-u(x)| &= \left|\int_a^x F(s,y(s))-F(s,u(s))\,\mathrm{d}s\right| \\ &\leqslant \int_a^x k|y(s)-u(s)|\,\mathrm{d}s \\ &\leqslant kC|x-a|\end{aligned}$$

$$\begin{aligned}|y(x)-u(x)| &= \left|\int_a^x F(s,y(s))-F(s,u(s))\,\mathrm{d}s\right| \\ &\leqslant \int_a^x k|y(s)-u(s)|\,\mathrm{d}s \\ &\leqslant \int k^2C|s-a|\,\mathrm{d}s \\ &\leqslant Ck^2\frac{|x-a|^2}{2}\end{aligned}$$

Repeating this process, we get

$$|y(x)-u(x)| \leqslant \frac{Ck^n|x-a|^n}{n!} \to 0 \quad as \quad n\to\infty$$

We conclude that $y(x)=u(x)$ on $[a-r,a+r]$.

4. The solution depends continuously on Y: Let y be that unique solution of the o.d.e. with initial condition $y(a)=Y$; and let u be that unique solution with initial condition $u(a)=Y+\delta$. They satisfy the equations

$$y(x)=Y+\int_a^x F(s,y(s))\,\mathrm{d}s, \qquad u(x)=Y+\delta+\int_a^x F(s,u(s))\,\mathrm{d}s$$

So,

$$\begin{aligned}|y(x)-u(x)| &\leqslant |\delta|+\int_a^x |F(s,y(s))-F(s,u(s))|\,\mathrm{d}s \\ &\leqslant |\delta|+k\int_a^x |y(s)-u(s)|\,\mathrm{d}s \qquad (2.1)\end{aligned}$$

As y and u are continuous functions, their difference must be bounded, $|y(x)-u(x)|\leqslant C$ on $(a-r,a+r)$; so substituting into (2.1) we get

$$|y(x)-u(x)|\leqslant|\delta|+kC|x-a|$$

which is a slight improvement. This substitution can be repeated, and in general, we can show, by induction, that

$$\begin{aligned}|y(x)-u(x)| &\leqslant |\delta|\left(1+k|x-a|+\frac{k^2|x-a|^2}{2!}+\cdots+\frac{k^n|x-a|^n}{n!}\right)+\frac{Ck^{n+1}|x-a|^{n+1}}{(n+1)!} \\ &\to |\delta|e^{k|x-a|}\end{aligned}$$

Hence, $|y(x) - u(x)| \leqslant |\delta| e^{k|x-a|}$ which implies that $u(x) \to y(x)$ as $\delta \to 0$.

5. Now we have assumed that all the y's encountered remain in $(Y - l, Y + l)$ so that we could apply the Lipschitz inequality. We still have to show that this is the case.

Let $r = \min(h, l/M)$ where $M = \max_{x \in (a-h,a+h)} |F(x, y)|$.

For $x \in (a - r, a + r)$,

$$\begin{aligned} y_0(x) &= Y \in (Y - l, Y + l) \\ |y_{n+1}(x) - Y| &= |\textstyle\int_a^x F(s, y_n(s)) \frac{\mathrm{d}s}{\mathrm{d}|} \\ &\leqslant M|x - a| \leqslant Mr \leqslant l \end{aligned}$$

Therefore $y_{n+1}(x) \in (Y - l, Y + l)$ by induction on n.

□

Alternative Proof using Banach's Fixed Point Theorem

Consider the set of continuous functions on some bounded closed interval $I \subset \mathbb{R}$, and define $\|f\| := \max_{x \in I} |f(x)|$. It is easy to show that $\|f + g\| \leqslant \|f\| + \|g\|$. Note that $\|f_n - f\| \to 0$ precisely when f_n converges to f uniformly. If $\boldsymbol{f}$ is a vector function, then interpret $|\boldsymbol{f}(x)|$ as the Euclidean modulus of $\boldsymbol{f}$.

Banach's Fixed Point Theorem: If T is a contraction map on the interval I, that is, there is a constant $c < 1$ such that

$$\|T(y_1) - T(y_2)\| \leqslant c\|y_1 - y_2\|,$$

then the iteration $y_{n+1} := T(y_n)$ starting from any y_0, converges to some function y which is that unique fixed point of T, that is, $T(y) = y$.

Proof. (of Banach Fixed Point Theorem)

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|T(y_n) - T(y_{n-1})\| \\ &\leqslant c\|y_n - y_{n-1}\| \\ &\leqslant c^n\|y_1 - y_0\| \end{aligned}$$

using induction on n. Hence for $n > m$

$$\begin{aligned} \|y_n - y_m\| &\leqslant \|y_n - y_{n-1}\| + \cdots + \|y_{m+1} - y_m\| \\ &= \|T(y_{n-1}) - T(y_{n-2})\| + \cdots + \|T(y_m) - T(y_{m-1})\| \\ &\leqslant (c^{n-1} + \cdots + c^m)\|y_1 - y_0\| \\ &\leqslant \frac{c^m}{1 - c}\|y_1 - y_0\| \to 0 \quad \text{as } n, m \to \infty \end{aligned}$$

Thus, $|y_n(x) - y_m(x)| \leqslant \|y_n - y_m\| \to 0$ at each point x, hence there is convergence $y_n(x) \to y(x)$. Indeed this convergence is uniform in x, that is, $\|y_n - y\| \to 0$ as $n \to \infty$. It then follows that

$$\|T(y_n) - T(y)\| \leqslant c\|y_n - y\| \to 0$$

so that in the limit as $n \to \infty$, the equation $y_{n+1} = T(y_n)$ becomes $y = T(y)$.

This fixed point is unique, for if $u = T(u)$ as well, then

$$\begin{aligned} \|y - u\| &= \|T(y) - T(u)\| \leqslant c\|y - u\| \\ \therefore, \quad & 0 \leqslant (1-c)\|y - u\| \leqslant 0 \\ \therefore, \quad & \max_{x \in I} |y(x) - u(x)| = \|y - u\| = 0 \end{aligned}$$

and $y = u$ on the interval I.

□

Proof.(of the Fundamental Theorem of O.D.E.s) Let

$$T(\boldsymbol{y}) := \boldsymbol{Y} + \int_a^x \boldsymbol{F}(s, \boldsymbol{y}(s))\,\mathrm{d}s$$

on an interval $x \in [a-h, a+h]$ with h to be chosen later. Then

$$\begin{aligned} |T(\boldsymbol{y}_1) - T(\boldsymbol{y}_2)| &= |\int_a^x \boldsymbol{F}(s, \boldsymbol{y}_1(s)) - \boldsymbol{F}(s, \boldsymbol{y}_2(s))\,\mathrm{d}s| \\ &\leqslant \int_a^x |\boldsymbol{F}(s, \boldsymbol{y}_1(s)) - \boldsymbol{F}(s, \boldsymbol{y}_2)|\,\mathrm{d}s \\ &\leqslant \int_a^x k|\boldsymbol{y}_1(s) - \boldsymbol{y}_2(s)|\,\mathrm{d}s \\ &\leqslant k|x-a|\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \\ \therefore \quad \|T(\boldsymbol{y}_1) - T(\boldsymbol{y}_2)\| &\leqslant kh\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \end{aligned}$$

If h is chosen sufficiently small, i.e., $h < 1/k$, then T would be a contraction mapping. As before, consider the Picard iteration $\boldsymbol{y}_{n+1} := T(\boldsymbol{y}_n)$. Each new iterate is a continuous function because $\boldsymbol{F}$ and integration are continuous operations. The Banach fixed point theorem then guarantees that they converge uniformly to some function $\boldsymbol{y}$ ($\|\boldsymbol{y}_n - \boldsymbol{y}\| \to 0$ as $n \to \infty$). This unique function is the fixed point of T, that is, $\boldsymbol{y} = T(\boldsymbol{y}) = \boldsymbol{Y} + \int_a^x \boldsymbol{F}(s, \boldsymbol{y}(s))\,\mathrm{d}s$. Differentiating gives $\boldsymbol{y}'(x) = \boldsymbol{F}(x, \boldsymbol{y}(x))$ by the Fundamental Theorem of Calculus; also $\boldsymbol{y}(a) = \boldsymbol{Y}$.

If $\boldsymbol{F}$ is Lipschitz only in a neighborhood of the initial point $(a, \boldsymbol{Y})$, then the interval may need to be restricted further to ensure that each iterate $\boldsymbol{y}_n$ remains within this neighborhood. This follows by induction on n,

$$\begin{aligned} |\boldsymbol{y}_{n+1}(x) - \boldsymbol{Y}| &\leqslant \int_a^x |\boldsymbol{F}(s, \boldsymbol{y}_n(s))|\,\mathrm{d}s \\ &\leqslant hc \leqslant l, \end{aligned}$$

assuming $h \leqslant l/c$, where c is the maximum value of $|\boldsymbol{F}(x,\boldsymbol{y})|$ on the given rectangular neighborhood.

To show that $\boldsymbol{y}$ depends continuously on $\boldsymbol{Y}$, let $\boldsymbol{u}$ be the unique solution of $\boldsymbol{u}' = \boldsymbol{F}(x,\boldsymbol{u})$ with $\boldsymbol{u}(a) = \boldsymbol{Y} + \boldsymbol{\delta}$. Then $\boldsymbol{u} = \boldsymbol{Y} + \boldsymbol{\delta} + \int_a^x \boldsymbol{F}(s,\boldsymbol{u}(s))\,\mathrm{d}s$. Thus,

$$\begin{aligned} \|\boldsymbol{y}-\boldsymbol{u}\| &= \|T(\boldsymbol{y}) - T(\boldsymbol{u}) - \boldsymbol{\delta}\| \\ &\leqslant |\boldsymbol{\delta}| + \|T(\boldsymbol{y}) - T(\boldsymbol{u})\| \\ &\leqslant |\boldsymbol{\delta}| + c\|\boldsymbol{y}-\boldsymbol{u}\| \\ \therefore \quad \|\boldsymbol{y}-\boldsymbol{u}\| &\leqslant \frac{|\boldsymbol{\delta}|}{1-c} \end{aligned}$$

So $\boldsymbol{u} \to \boldsymbol{y}$ uniformly as $\boldsymbol{\delta} \to \boldsymbol{0}$.

□

Example

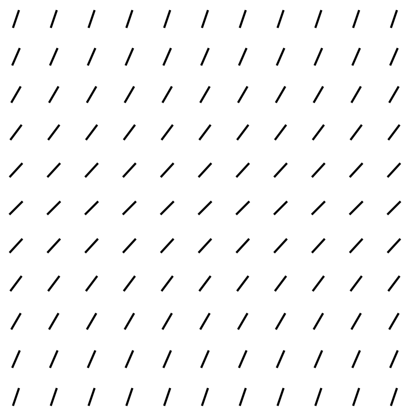
The o.d.e. $y'(x) = \frac{\sqrt{x+y}}{x-y}$, $y(0) = 1$ has a unique solution in some interval about 0.

Solution. The function $F(x,y) := \frac{\sqrt{x+y}}{x-y}$ is continuous in x when $x-y \neq 0$ and $x+y \geqslant 0$, and Lipschitz in y when $|\frac{\partial F}{\partial y}| = |\frac{3x+y}{2(x-y)^2\sqrt{x+y}}| \leqslant k$; both these conditions are met in the square $(-\frac{1}{3},\frac{1}{3}) \times (1-\frac{1}{3}, 1+\frac{1}{3})$ because the lines $y = \pm x$ are avoided. Picard's theorem then assures us that there is a solution in some interval about 0.

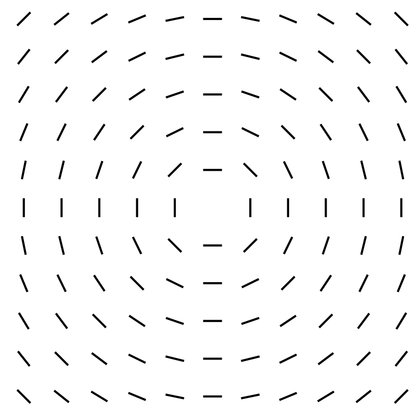
To find a specific interval, we need to ensure $h \leqslant l/\max|F|$; in this case $\max|F| \leqslant \frac{\sqrt{5/3}}{1/3} < 4$, so taking $r := \frac{1/3}{4} = 1/12$, we can assert that there is a unique solution on $-1/12 < x < 1/12$. (Of course, the real interval on which there is a solution could be larger; solving this equation numerically gives a solution at least on $-1.6 < x < .31$.)

Note that if $F(x,y)$ is Lipschitz for *all* values of y (i.e., $l = \infty$), then the equation is well-posed on at least $[a-h, a+h]$, without the need to restrict to a smaller interval.

It shouldn't come as a surprise that even if $F(x,y)$ is “well-behaved” at all points $(x,y) \in \mathbb{R}^2$, the solutions may still exist on only a finite interval. The following are two such examples:



$y' = 1 + y^2$



$y' = -x/y$

2.2 Extensions of Picard's theorem

2.2.1 Vector valued functions

A system of first order differential equations are of the type

$$\begin{aligned} u'(x) &= f(x, u(x), v(x), \dots) \qquad u(a) = U \\ v'(x) &= g(x, u(x), v(x), \dots) \qquad v(a) = V \\ &\dots \end{aligned}$$

Writing $\boldsymbol{y}(x) := \begin{pmatrix} u(x) \\ v(x) \\ \vdots \end{pmatrix}$, we find that

$$\boldsymbol{y}'(x) = \begin{pmatrix} u'(x) \\ v'(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} f(x, u(x), v(x)) \\ g(x, u(x), v(x) \\ \vdots \end{pmatrix} = \boldsymbol{F}(x, \boldsymbol{y}(x)).$$

This is a *vector* first-order equation, and indeed, Picard's theorem can be generalized to this case. All we need is an analogue of the Lipschitz condition.

We say that $\boldsymbol{F}$ satisfies a Lipschitz condition in $\boldsymbol{y}$ when,

$$|\boldsymbol{F}(x, \boldsymbol{y}_1) - \boldsymbol{F}(x, \boldsymbol{y}_2)| \leqslant k|\boldsymbol{y}_1 - \boldsymbol{y}_2|$$

where $|\cdot|$ denotes the norm or modulus of a vector.

With this definition, we can repeat all the steps of the main proof, taking care to change all y's into vector $\boldsymbol{y}$'s.

If $\boldsymbol{F}$ is continuous in x and Lipschitz in $\boldsymbol{y}$ in a region $x \in A$, $\boldsymbol{y} \in B$ containing $x = a$, $\boldsymbol{y} = \boldsymbol{Y}$, then the o.d.e. $\boldsymbol{y}'(x) = \boldsymbol{F}(x, \boldsymbol{y}(x))$ with initial *condition $\boldsymbol{y}(a) = \boldsymbol{Y}$ is well-posed on a (smaller) interval about a; that is, there is a unique solution $\boldsymbol{y}(x)$ which depends continuously on $\boldsymbol{Y}$.*

Proof. Exercise: Go through the proof of Picard's theorem again, using vector variables throughout.

There is a simpler criterion which is equivalent to the Lipschitz condition:

If every component $F_i(x, \boldsymbol{y})$ is Lipschitz in $\boldsymbol{y} = \begin{pmatrix} u \\ v \\ \vdots \end{pmatrix}$

$$|F_i(x, \boldsymbol{y}_1) - F_i(x, \boldsymbol{y}_2)| \leqslant k_i(|u_1 - u_2| + |v_1 - v_2| + \cdots)$$

then the vector function $\boldsymbol{F}$ is Lipschitz in $\boldsymbol{y}$.

Proof. The following inequalities hold for any positive real numbers $a_1, \ldots, a_n$,

$$\left|\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right| \leqslant a_1 + \cdots + a_n \leqslant \sqrt{n}\left|\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right|$$

So

$$\begin{aligned} |\boldsymbol{F}(x, \boldsymbol{y}_1) - \boldsymbol{F}(x, \boldsymbol{y}_2)| &\leqslant \sum_i |F_i(x, \boldsymbol{y}_1) - F_i(x, \boldsymbol{y}_2)| \\ &\leqslant \sum_i k_i(|u_1 - u_2| + |v_1 - v_2| + \cdots) \\ &\leqslant K|\boldsymbol{y}_1 - \boldsymbol{y}_2| \end{aligned}$$

where $K := \sqrt{n}\sum_i k_i$.

Exercise: Prove the converse.

Example

Show that the system of equations

$$\begin{aligned} u'(x) &= xu(x) - v(x), \quad & u(0) = 1 \\ v'(x) &= u(x)^2 + v(x), \quad & v(0) = 0 \end{aligned}$$

has a unique solution in some interval about $x = 0$.

This is an equation $\boldsymbol{y}'(x) = \boldsymbol{F}(x, \boldsymbol{y}(x))$ where $\boldsymbol{F}(x, u, v) = \begin{pmatrix} xu - v \\ u^2 + v \end{pmatrix}$. It is obviously continuous in x everywhere. Consider the components of the function $\boldsymbol{F}$:

$$\begin{aligned} |(xu_1 - v_1) - (xu_2 - v_2)| &\leqslant |x||u_1 - u_2| + |v_1 - v_2| \\ &\leqslant (k_1 + 1)(|u_1 - u_2| + |v_1 - v_2|) \end{aligned}$$

as long as $|x| \leqslant k_1$;

$$\begin{aligned}|(u_1^2+v_1)-(u_2^2+v_2)| &\leqslant |u_1+u_2||u_1-u_2|+|v_1-v_2| \\ &\leqslant (2k_2+1)(|u_1-u_2|+|v_1-v_2|)\end{aligned}$$

as long as $|u_1|, |u_2| \leqslant k_2$. This is enough to show that $\boldsymbol{F}$ is Lipschitz in the region $(-1,1)\times(-1,1)$, say. Hence, Picard's theorem assures us there is a unique solution in some (smaller) neighborhood.

2.2.2 Higher order o.d.e.'s

Consider the nth order ordinary differential equation with initial conditions

$$\begin{gathered} y^{(n)}(x) = F(x, y(x), \ldots, y^{(n-1)}(x)) \\ \text{given } y(a) = Y_0, y'(a) = Y_1, \ldots, y^{(n-1)}(a) = Y_{n-1} \end{gathered}$$

It can be reduced to a first order vector o.d.e. as follows:

Let $\boldsymbol{y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix}$. Differentiating gives:

$$\boldsymbol{y}'(x) = \begin{pmatrix} y'(x) \\ y''(x) \\ \vdots \\ F(x, y(x), \ldots, y^{(n-1)}(x)) \end{pmatrix} = \boldsymbol{F}(x, y(x), \ldots, y^{(n-1)}(x))$$

with $\boldsymbol{y}(a) = \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{n-1} \end{pmatrix}$.

Therefore, as long as $\boldsymbol{F}$ is Lipschitz in the vector sense, and continuous in x, then the differential equation is well-posed.

If $F(x, y, y', \ldots)$ is continuous in x and Lipschitz in y in the sense that

$$|F(x, y_1, y_1', \ldots) - F(x, y_2, y_2', \ldots) \leqslant k(|y_1 - y_2| + |y_1' - y_2'| + \ldots),$$

then the differential equation

$$y^{(n)}(x) = F(x, y(x), \ldots, y^{(n-1)}(x)) \quad \textit{given } y(a) = Y_0, y'(a) = Y_1, \ldots, y^{(n-1)}(a) = Y_{n-1}$$

has a unique solution in some neighborhood of a.

Example

Show that the equation

$$y''(x) + \cos x\, y'(x) - \sin x\, y(x) = \tan x, \qquad y(0) = 1, y'(0) = 0$$

has a unique solution in some interval about 0.

Solution: Convert it to a vector first-order equation,

$$\begin{aligned} u'(x) &= v(x) & u(0) = 1 \\ v'(x) &= \tan x + \sin x\, u(x) - \cos x\, v(x) \qquad & v(0) = 0 \end{aligned}$$

The right-hand side is continuous in x in the interval $-\pi/2 < x < \pi/2$, and Lipschitz in u, v,

$$|(\tan x + \sin x\, u_1 - \cos x\, v_1) - (\tan x + \sin x\, u_2 - \cos x\, v_2)| \leqslant |u_1 - u_2| + |v_1 - v_2|.$$

Picard's theorem then implies there is a unique solution in some interval about 0.

Exercises 2.2

1. Find regions for which the following functions $F(x, y)$ are Lipschitz in y:

$$1 - xy, \qquad (x+y)^2, \qquad y + e^x/y, \qquad \sin(x+y), \qquad \tan(xy).$$

2. Show that $\sqrt{y}$ is not Lipschitz in y in any region that includes $y = 0$.

3. By first proving that $F(y) = y + \frac{1}{y}$ is Lipschitz in a certain region, show that the equation

$$y' = y + \frac{1}{y} \quad y(1) = 1$$

is well-posed in a neighborhood of $x = 1, y = 1$ and find the first three approximations in the Picard iteration scheme.

4. Consider the simultaneous differential equations,

$$\begin{aligned} u'(x) &= 2 - u(x)v(x), \qquad & u(0) = -1, \\ v'(x) &= u^2(x) - xv(x), & v(0) = 2. \end{aligned}$$

By considering $(u(x), v(x))$ as a vector function, apply the vector version of the Picard iteration scheme to find the first three iterates. Does the o.d.e. satisfy a Lipschitz condition, and what conclusions do we draw from Picard's theorem?

5. Repeat for
$$\begin{aligned} u' &= uv, \quad u(0) = 0, \\ v' &= u + v, \; v(0) = 1 \end{aligned}$$

6. Apply Picard's method for the o.d.e.
$$y'(x) - xy(x)^2 = 1, \qquad y(0) = 0$$
to get the first four approximations to the solution. Find the best Lipschitz constant for $x \in [-h, h]$ and $y \in [-l, l]$. On what interval about 0 does Picard's theorem guarantee that a solution exists?

7. Show that the following second-order equation is well-posed in a neighborhood of $x = 1$,
$$xy'' - y' + x^2/y = 0, \quad y(1) = 1, y'(1) = 0.$$

8. Show that the differential equation $yy'' + (y')^2 = 0$ with $y(0) = 1$ and $y'(0) = 1$ has a unique solution on some interval about 0. Find the solution (hint: divide the equation by yy').

9. If the Lipschitz function $F(x, y)$ is periodic (with period T), and it happens that y repeats the initial condition, i.e.,
$$F(x + T, y) = F(x, y), \quad y(a + T) = y(a)$$
show that the unique solution is also periodic.

10. A Lienard equation is one of the form $y'' = a'(y)y' + b(y)$. Use the substitution $\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} y \\ y' - a(y) \end{pmatrix}$ to reduce it to a first-order equation $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v + a(u) \\ b(u) \end{pmatrix}$.

11. When a second order equation $y'' = F(y, y')$ is reduced to a first order equation using the substitution $v = y'$, show that we get $v\frac{\mathrm{d}v}{\mathrm{d}y} = v' = F(y, v)$. One can solve this first, then solve $y' = v(y)$. Use this method to solve $y'' = 2y'y$.

12. Suppose $F(x, y)$ is Lipschitz in y and continuous in x, and let y be a solution of $y' = F(x, y)$, $y(0) = Y$, existing on an interval I. Show that it must also be unique on I, i.e., no other solution can be identical to it on any subinterval $J \subseteq I$.

CHAPTER 3

Linear Ordinary Differential Equations

Definition A **first order linear o.d.e.** is one of type:

$$\boldsymbol{y}' = A(x)\boldsymbol{y} + \boldsymbol{f}(x).$$

It can be rewritten in short as $L\boldsymbol{y} = \boldsymbol{f}$ where $L = D - A$.

The operator L is a linear transformation on the vector space of real or complex functions, since taking derivatives and multiplying by matrices are both linear operations. That is,

$$\begin{aligned} L(\boldsymbol{y}_1 + \boldsymbol{y}_2) &= L\boldsymbol{y}_1 + L\boldsymbol{y}_2 \\ L(\alpha\boldsymbol{y}) &= \alpha L\boldsymbol{y}_1 \end{aligned}$$

Examples.

1. Two thermal layers lose heat according to Newton's cooling law,

$$\begin{aligned} \dot{u} &= -k_1(u - v) \\ \dot{v} &= -k_2 v + k_1(u - v). \end{aligned}$$

2. Two connected springs (or pendulums) attached at their ends satisfy the equations

$$\begin{aligned} m_1\dot{v} &= -k_1 x + k_2(y - x), \qquad & m_2\dot{w} &= -k_2(y - x) \\ \dot{x} &= v, & \dot{y} &= w \end{aligned}$$

So

$$\begin{pmatrix} \dot{x} \\ \dot{v} \\ \dot{y} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{pmatrix} \begin{pmatrix} x \\ v \\ y \\ w \end{pmatrix}$$

Proposition 3.1

(Special case of Picard's theorem)

> **If $A(x)$ and $\boldsymbol{f}(x)$ are continuous in x on a bounded closed interval, then the differential equation**
>
> $$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}$$
>
> **is well-posed in that interval.**

PROOF. We need to check that $\boldsymbol{F}(x, \boldsymbol{y}) := A(x)\boldsymbol{y} + \boldsymbol{f}$ is Lipschitz in $\boldsymbol{y}$.

$$\begin{aligned}\|(A\boldsymbol{y}_1 + \boldsymbol{f}) - (A\boldsymbol{y}_2 + \boldsymbol{f})\| &= \|A(\boldsymbol{y}_1 - \boldsymbol{y}_2)\| \\ &= \|A\boldsymbol{u}\|\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|\end{aligned}$$

where $\boldsymbol{u} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|$ is a unit vector.

$$\begin{aligned}\|A(x)\boldsymbol{u}\| = \left\| \begin{pmatrix} a_{11}(x)u_1 + \cdots + a_{1n}(x)u_n \\ \vdots \\ a_{n1}(x)u_1 + \cdots + a_{nn}(x)u_n \end{pmatrix} \right\| &\leqslant |a_{11}(x)u_1| + \cdots + |a_{nn}(x)u_n| \\ &\leqslant |a_{11}(x)| + \cdots + |a_{nn}(x)| \\ &\leqslant K\end{aligned}$$

since $|u_i| \leqslant 1$ and $a_{ij}(x)$ are continuous, hence bounded, functions on an interval $[\alpha, \beta]$.

□

In particular, when A is a constant matrix (independent of x), the equation $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{Y}$, has a unique solution for all x. In this case the solution can be written explicitly as $\boldsymbol{y}(x) = e^{xA}\boldsymbol{Y}$, where $e^{xA} = I + xA + (xA)^2/2! + \cdots$.

PROOF. Suppose $\boldsymbol{y}$ is any solution of $\boldsymbol{y}' = A\boldsymbol{y}$, and let $\boldsymbol{v}(x) := e^{-xA}\boldsymbol{y}(x)$.

$$\boldsymbol{v}'(x) = -Ae^{-xA}\boldsymbol{y}(x) + e^{-xA}\boldsymbol{y}'(x) = -A\boldsymbol{v}(x) + A\boldsymbol{v}(x) = \boldsymbol{0}$$

so $\boldsymbol{v}(x) = \boldsymbol{v}(0) = \boldsymbol{Y}$, and $\boldsymbol{y}(x) = e^{xA}\boldsymbol{Y}$. We now show that $\boldsymbol{y}(x) := e^{xA}\boldsymbol{Y}$ is indeed a solution: $\boldsymbol{y}'(x) = Ae^{xA}\boldsymbol{Y} = A\boldsymbol{y}$ and $e^{0A} = I$, so $\boldsymbol{y}$ satisfies both the differential equation and the condition.

Note that we did not need to resort to Picard's theorem to show uniqueness of this solution.

□

3.1 Homogeneous and Particular Solutions

We can use the methods and terminology of linear transformations to tackle linear differential equations. The first thing that comes to mind is to try to find the inverse L^{-1} of the operator L. However in differential equations, these operators are not invertible, e.g. the D operator sends all the constant functions to 0. We are therefore led naturally to the study of the kernel of L:

Definition The **kernel** of L is the set $\ker L = \{\, \boldsymbol{u} : L\boldsymbol{u} = \boldsymbol{0}\,\}$.

The equation $L\boldsymbol{u} = \boldsymbol{0}$, that is $\boldsymbol{y}' = A\boldsymbol{y}$, is termed a **homogeneous o.d.e.**.

By choosing any initial condition $\boldsymbol{y}(a) = \boldsymbol{Y}$, Picard's theorem shows that there exist solutions to the equation $L\boldsymbol{u} = \boldsymbol{0}$. The kernel $\ker L$ is therefore non-trivial, so that L is not 1-1 (because both $\boldsymbol{y} = \boldsymbol{u}$ and $\boldsymbol{y} = \boldsymbol{0}$ are mapped by L to $L\boldsymbol{y} = \boldsymbol{0}$), hence not invertible.

A **particular solution** is a single function $\boldsymbol{y}_P$ which is a solution of $L\boldsymbol{y}_P = \boldsymbol{f}$, among the many that exist.

The following proposition states that the *general solution* of a linear o.d.e. splits up into two parts:

Proposition 3.2

Every solution of $L\boldsymbol{y} = \boldsymbol{f}$ is the sum of the particular solution and a solution of the homogeneous equation.

PROOF. Suppose that $L\boldsymbol{y} = \boldsymbol{f}$.
$L(\boldsymbol{y} - \boldsymbol{y}_P) = L\boldsymbol{y} - L\boldsymbol{y}_P = \boldsymbol{f} - \boldsymbol{f} = \boldsymbol{0}$
$\therefore\ \boldsymbol{y} - \boldsymbol{y}_P \in \ker L$
$\therefore\ \boldsymbol{y} - \boldsymbol{y}_P = \boldsymbol{y}_H$ for some $\boldsymbol{y}_H \in \ker L$
$\therefore\ \boldsymbol{y}(x) = \boldsymbol{y}_P(x) + \boldsymbol{y}_H(x)$

□

This suggests that we divide the original problem

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}, \quad \boldsymbol{y}(a) = \boldsymbol{Y}$$

into two parts:

the homogeneous equation:	the particular equation:
$L\boldsymbol{y}_H = \boldsymbol{0}$ with $\boldsymbol{y}_H(a) = \boldsymbol{Y}$	$L\boldsymbol{y}_P = \boldsymbol{f}$ with $\boldsymbol{y}_P(a) = \boldsymbol{0}$

Note. The initial condition for the particular solution was chosen to be $\mathbf{0}$, but this is quite arbitrary. If it is easier to solve the particular equation for other initial conditions, say $\boldsymbol{y}_P(0) = \boldsymbol{V}$, then this can be done as long as one compensates in the homogeneous solution with $\boldsymbol{y}_H(0) = \boldsymbol{Y} - \boldsymbol{V}$.

3.1.1 The Homogeneous Equation

In this section we will investigate how many linearly independent homogeneous solutions there can be; in other words we will find $\dim \ker L$.

Let us denote by $\boldsymbol{u}_1$ the solution of the linear equation with some specific initial condition, say $\boldsymbol{u}_1(0) = \boldsymbol{e}_1$, the first canonical basis vector. Then we can solve the equation $\boldsymbol{y}' = A\boldsymbol{y}$ for any multiple of this condition $\boldsymbol{y}(0) = \alpha\boldsymbol{e}_1$: the unique solution is $\boldsymbol{y} = \alpha\boldsymbol{u}_1$. Thus varying the initial condition along a one-dimensional subspace of $\mathbb{R}^N$ gives a one-dimensional space of solutions of the homogeneous equation. It is clear that if we consider an independent initial condition, $\boldsymbol{y}(0) = \boldsymbol{e}_2$, then the solution $\boldsymbol{u}_2$ will not be in this subspace, i.e., it will be linearly independent also. The following proposition generalizes this argument and makes it rigorous:

Proposition 3.3

> **Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ be a basis for $\mathbb{R}^N$. The set of functions $\boldsymbol{u}_1(x), \ldots, \boldsymbol{u}_N(x)$, which satisfy**
>
> $$L\boldsymbol{u}_i = \mathbf{0}, \quad \boldsymbol{u}_i(0) = \boldsymbol{e}_i,$$
>
> **is a basis for $\ker L$; so its dimension is N.**

Proof. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be a basis for $\mathbb{R}^n$, and let $\boldsymbol{u}_i(x)$ be the unique solutions of the following o.d.e.'s.

$$\begin{array}{ll} L\boldsymbol{u}_1 = \mathbf{0} & \boldsymbol{u}_1(a) = \boldsymbol{e}_1 \\ L\boldsymbol{u}_2 = \mathbf{0} & \boldsymbol{u}_2(a) = \boldsymbol{e}_2 \\ \ldots & \\ L\boldsymbol{u}_N = \mathbf{0} & \boldsymbol{u}_N(a) = \boldsymbol{e}_N \end{array}$$

More compactly, they can be written as $L\boldsymbol{u}_i = \mathbf{0} \quad \boldsymbol{u}_i(a) = \boldsymbol{e}_i$. Note that Picard's theorem assures us that $\boldsymbol{u}_i(x)$ exists and is unique.

These solutions can be combined together to form a matrix $U_x = [\boldsymbol{u}_1 \ldots \boldsymbol{u}_n]$ that therefore satisfies the equation $U' = AU$.

$\boldsymbol{u}_1(x), \ldots, \boldsymbol{u}_N(x)$ *span* $\ker L$: Let $\boldsymbol{y}(x) \in \ker L$; that is $L\boldsymbol{y} = 0$. Let $\boldsymbol{Y} = \boldsymbol{y}(a)$; we can write $\boldsymbol{Y} = \sum_i \alpha_i\boldsymbol{e}_i$ since the latter form a basis for $\mathbb{R}^N$. Define the function

$$\boldsymbol{v}(x) := \alpha_1\boldsymbol{u}_1(x) + \alpha_2\boldsymbol{u}_2(x) + \cdots + \alpha_N\boldsymbol{u}_N(x).$$

Therefore, since L is a linear operator, and $L\boldsymbol{u}_i = \boldsymbol{0}$,

$$L\boldsymbol{v}(x) = \alpha_1 L\boldsymbol{u}_1(x) + \cdots + \alpha_N L\boldsymbol{u}_N(x) = \boldsymbol{0}.$$

Also

$$\begin{aligned}\boldsymbol{v}(a) &= \alpha_1\boldsymbol{u}_1(a) + \cdots + \alpha_N\boldsymbol{u}_N(a)\\ &= \alpha_1\boldsymbol{e}_1 + \cdots + \alpha_N\boldsymbol{e}_N\\ &= \boldsymbol{Y}\end{aligned}$$

Hence both $\boldsymbol{y}$ and $\boldsymbol{v}$ are solutions of $L\boldsymbol{y} = 0$ with the *same* initial conditions. By uniqueness of solution, this is possible only if

$$\boldsymbol{y}(x) = \boldsymbol{v}(x) = \alpha_1\boldsymbol{u}_1(x) + \cdots + \alpha_N\boldsymbol{u}_N(x)$$

Therefore every function in $\ker L$ can be written as a linear combination of the $\boldsymbol{u}_i$.

The functions $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ are linearly independent: Suppose that

$$\alpha_1\boldsymbol{u}_1(x) + \cdots + \alpha_N\boldsymbol{u}_N(x) = \boldsymbol{0} \quad \forall x,$$

then it follows that at $x = a$,

$$\alpha_1\boldsymbol{e}_1 + \cdots + \alpha_N\boldsymbol{e}_N = \boldsymbol{0}.$$

But the vectors $\boldsymbol{e}_i$ are linearly independent so that $\alpha_i = 0$. Hence the functions $\boldsymbol{u}_i(x)$ are linearly independent.

We have just shown that $\boldsymbol{u}_i(x)$ form a basis for $\ker L$, the space of homogeneous solutions. Its dimension is therefore N, the number of variables in $\boldsymbol{y}$.

□

Originally we wanted to solve the homogeneous equation, $L\boldsymbol{y}_H = \boldsymbol{0}$, $\boldsymbol{y}_H(a) = \boldsymbol{Y}$. We now know that $\boldsymbol{y}_H$ is a linear combination of the N functions $\boldsymbol{u}_i(x)$,

$$\boldsymbol{y}_H(x) = \alpha_1\boldsymbol{u}_1(x) + \cdots + \alpha_n\boldsymbol{u}_n(x) = U_x\boldsymbol{\alpha}.$$

To determine the unknown vector $\boldsymbol{\alpha}$, we substitute the initial condition,

$$\boldsymbol{Y} = \boldsymbol{y}_H(a) = U_a\boldsymbol{\alpha},$$

which implies that $\boldsymbol{\alpha} = U_a^{-1}\boldsymbol{Y}$. Hence,

$$\boldsymbol{y}_H = U_x U_a^{-1}\boldsymbol{Y}.$$

Note that if the initial basis for $\mathbb{R}^N$ were chosen to be the standard basis, then U_a, and hence U_a^{-1}, is just the identity matrix, making the above expression slightly simpler.

3.1.2 Solutions of Homogeneous Equation in Two Variables

Method 1.

First work out $y_1', y_1'', \ldots, y_1^{(n)}$ in terms of the variables y_i' using the given N equations. Invert this matrix to get an Nth order differential equation for y_1.

Example 1. Solve $u' = v$, $v' = -u$. Differentiate the first equation to get $u'' = v'$, and then use the second equation to write it in terms of u i.e., $u'' = -u$ which can be solved. v can then be obtained from $v = u'$.

Example 2. Solve $u' = u + 2v$, $v' = u - v$. We get $u'' = u' + 2v' = u' + 2(u - v) = u' + 2u - (u - u') = u + 2u'$, which can be solved.

Method 2.

This second method works only for the case when the coefficients of the matrix A are constant. Any matrix can be made triangular by a suitable change of variables, $A = P^{-1}TP$ where T is a triangular matrix and P is a matrix of eigenvectors. Making the substitution $\boldsymbol{Y}(x) := P\boldsymbol{y}(x)$ gives the equation $\boldsymbol{Y}' = P\boldsymbol{y}' = PAP^{-1}\boldsymbol{Y} = T\boldsymbol{Y}$. This set of equations $\boldsymbol{Y}' = T\boldsymbol{Y}$ can be solved line by line starting from the bottom.

This method is preferable to the first one when there are a large number of equations, but for two equations or so the first method is just as good and more practical.

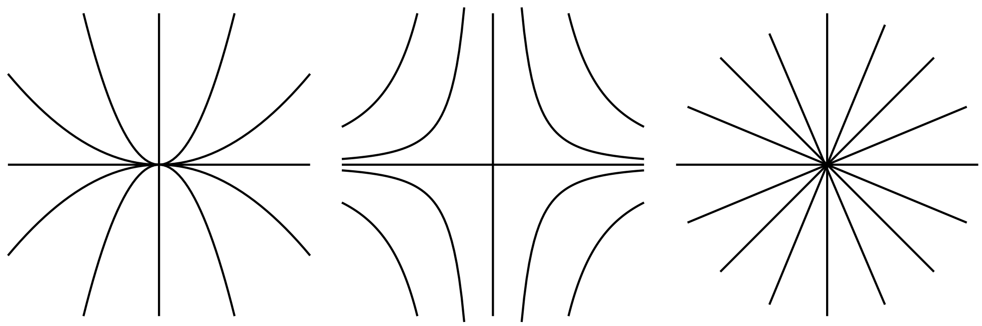
We will in fact use this method to classify the different possible differential equations for $N = 2$. We assume, from Linear Algebra, that any real 2×2 matrix A, has zero, one, or two eigenvalues λ and μ, with the following properties:

- Two distinct eigenvalues λ and μ; then A is diagonalizable.
- One eigenvalue $\lambda = \mu$; then A may not be diagonalizable, but can be made triangular.
- Zero real eigenvalues; the eigenvalues are complex, $\lambda = r + i\omega$, μ is its conjugate $r - i\omega$; A is diagonalizable but with complex eigenvectors.

Let us consider each possibility:

Eigenvalues real and distinct

The matrix A is diagonalizable, $\boldsymbol{Y}' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \boldsymbol{Y}$ has solution $\boldsymbol{Y}(x) = \boldsymbol{a}e^{\lambda x} + \boldsymbol{b}e^{\mu x}$. The exponential $e^{\lambda x}$ grows when $\lambda > 0$, it decreases when $\lambda < 0$ and remains constant when $\lambda = 0$. One therefore gets various pictures of the solution for various combinations of the values of λ and μ.



Node $\lambda, \mu > 0$ OR $\lambda, \mu < 0$ — **Saddle** $\mu < 0 < \lambda$ — **Focus** $\lambda = \mu$

If both eigenvalues are positive, the solutions form what is called a **source node**; if both are negative they form a **sink node**; if one is positive and the other negative the origin is called a **saddle point**; for the special case in which the eigenvalues are equal but A is still diagonalizable, the solution is termed a **proper node** (source or sink).

In general, an equation whose solutions converge towards the origin is called a **sink**, while one for which the solutions diverge away from the origin is called a **source**.

Eigenvalues complex conjugates

In the case that $\lambda = r + i\omega$, $\mu = r - i\omega$, the solutions in general are $\boldsymbol{A}e^{rx}e^{i\omega x} + \boldsymbol{B}e^{rx}e^{-i\omega x}$. But these include complex-valued solutions; such a linear combination is real when

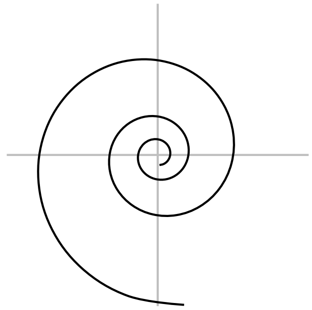
$$0 = \text{Im}(\boldsymbol{A}e^{i\omega x} + \boldsymbol{B}e^{-i\omega x}) = (\boldsymbol{A}_2 + \boldsymbol{B}_2)\cos\omega x + (\boldsymbol{A}_1 - \boldsymbol{B}_1)\sin\omega x$$

where $\boldsymbol{A} = \boldsymbol{A}_1 + i\boldsymbol{A}_2$, $\boldsymbol{B} = \boldsymbol{B}_1 + i\boldsymbol{B}_2$, which holds if, and only if, $\boldsymbol{B} = \overline{\boldsymbol{A}}$. This implies $\boldsymbol{A}e^{i\omega x} + \boldsymbol{B}e^{-i\omega x} = 2\boldsymbol{A}_1\cos\omega x + 2\boldsymbol{A}_2\sin\omega x$. In this case the solution is called a **spiral** and can be written as

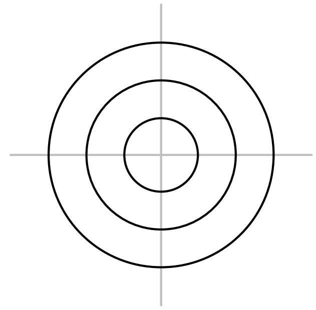
$$\boldsymbol{Y}(x) = \boldsymbol{a}e^{rx}\cos\omega x + \boldsymbol{b}e^{rx}\sin\omega x$$

The solutions spiral in or out towards $\mathbf{0}$ (sink or source) depending on whether the real part r is negative or positive respectively.

The special case in which $\text{Re}(\lambda) = r = 0$ is called a **center**.



spiral
$\lambda, \mu = r \pm i\omega$



center
$\lambda, \mu = \pm i\omega$

Eigenvalues real, equal and not diagonalizable

In this case, the equations take the form

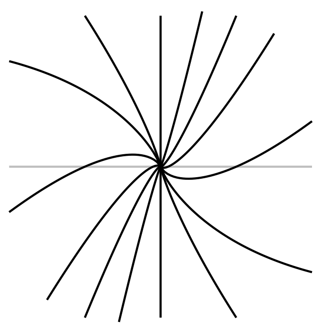
$$\boldsymbol{Y}' = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} \boldsymbol{Y} \qquad (a \neq 0)$$

which has solution

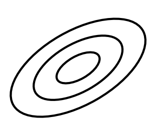
$$\boldsymbol{Y}(x) = \boldsymbol{a}e^{\lambda x} + \boldsymbol{b}xe^{\lambda x}$$

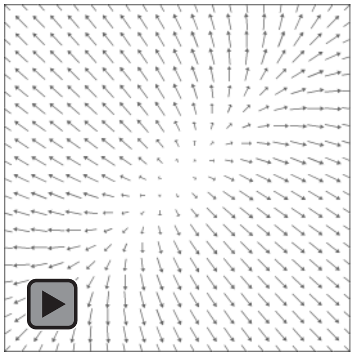
The solution therefore depends on the sign of λ as in the first case.

The solution in this case is called an **deficient node**.

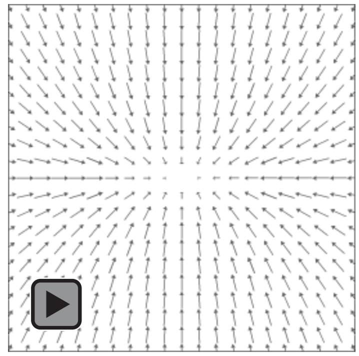


Note: These are the plots of U against V; if one were to convert back to u and v the plot would look like squashed/rotated versions of them. For example, a center might look like

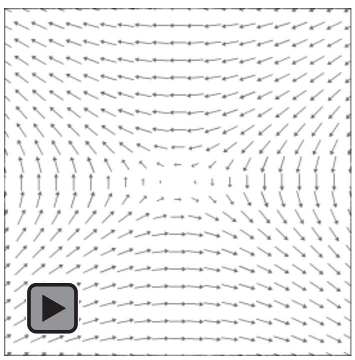




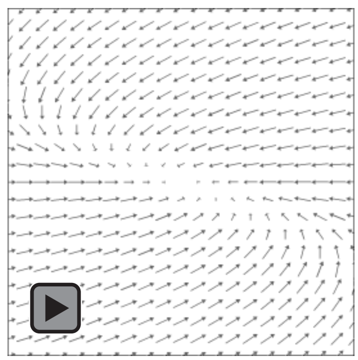
node: source



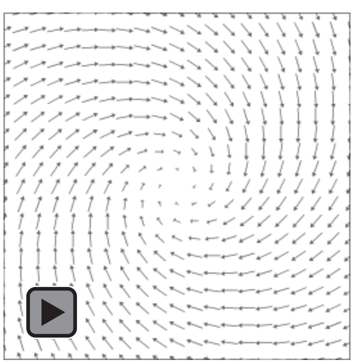
sink



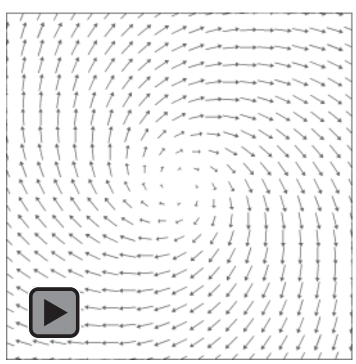
saddle



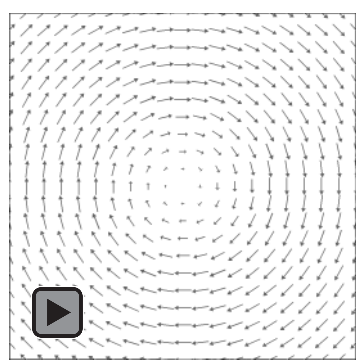
focus



inspiral



outspiral



center

Figure 3.1: Click inside a diagram to start a solution.

3.2 The Particular Solution

3.2.1 Variation of Parameters

Consider a linear equation in two variables, $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(x)$, and suppose the solutions of the homogeneous equation are $a\boldsymbol{u}(x) + b\boldsymbol{v}(x)$, where a and b are arbitrary scalars. It was Lagrange who had the idea of trying to find a solution of the full equation of the type $\boldsymbol{y}(x) := a(x)\boldsymbol{u}(x) + b(x)\boldsymbol{v}(x)$. We find

$$\begin{aligned}\boldsymbol{y}' &= a'\boldsymbol{u} + a\boldsymbol{u}' + b'\boldsymbol{v} + b\boldsymbol{v}' \\ &= a'\boldsymbol{u} + b'\boldsymbol{v} + aA\boldsymbol{u} + bA\boldsymbol{v} \\ &= A\boldsymbol{y} + a'\boldsymbol{u} + b'\boldsymbol{v}\end{aligned}$$

If we can find functions $a(x)$ and $b(x)$ such that $a'\boldsymbol{u} + b'\boldsymbol{v} = \boldsymbol{f}$ then the differential equation is completely solved.

More generally, suppose we have at our disposal the N homogeneous solutions $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ each satisfying $\boldsymbol{u}_i' = A\boldsymbol{u}_i$, and which can be combined together to form a matrix $U_x = [\boldsymbol{u}_1 \ldots \boldsymbol{u}_n]$ that therefore satisfies the equation $U' = AU$.

We can use these homogeneous solutions to find a particular solution satisfying $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}$. Let $\boldsymbol{a}$ be defined by

$$\boldsymbol{y}(x) = U_x\boldsymbol{a}(x).$$

Therefore,

$$\begin{aligned}\boldsymbol{y}' &= U'\boldsymbol{a} + U\boldsymbol{a}' \\ &= AU\boldsymbol{a} + U\boldsymbol{a}' \\ &= A\boldsymbol{y} + U\boldsymbol{a}' \\ \text{therefore} \quad U\boldsymbol{a}' &= \boldsymbol{f} \\ \Rightarrow \quad \boldsymbol{a}'(x) &= U_x^{-1}\boldsymbol{f}(x) \\ \Rightarrow \quad \boldsymbol{a}(x) &= \int_a^x U_s^{-1}\boldsymbol{f}(s)\,\mathrm{d}s + \boldsymbol{c} \\ \Rightarrow \quad \boldsymbol{y}(x) &= \underbrace{\int_a^x U_x {U_s}^{-1}\boldsymbol{f}(s)\,\mathrm{d}s}_{\boldsymbol{y}_P(x)} + \underbrace{U_x\boldsymbol{c}}_{\boldsymbol{y}_H(x)}\end{aligned}$$

So one particular solution is $\boxed{\boldsymbol{y}_P(x) = \int_a^x U_x U_s^{-1}\boldsymbol{f}(s)\,\mathrm{d}s.}$ The matrix $U_x {U_s}^{-1}$ is called the Green's function for the system of equations. It can be calculated once and used for different functions $\boldsymbol{f}$.

To fully justify the above proof, we need to show that U_x is invertible.

PROOF. Let $W(x) := \det[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N]$, called the *Wronskian* of the functions $\boldsymbol{u}_i$.

Jacobi's formula states that $W' = (\operatorname{tr} A)W$. To show this recall Laplace's formula for the determinant $\det U = \sum_k u_{ik}(\text{adj}U)_{ki}$, where adj$U$ is the classical adjoint of U; then taking the derivative on both sides gives

$$\begin{aligned}
W' &= \sum_{ij} \frac{\partial}{\partial u_{ij}} \det U . u'_{ij} \\
&= \sum_{ij} (\sum_k \delta_{jk} (\text{adj}U)_{ki} + u_{ik} 0) u'_{ij} \\
&= \sum_{ij} (\text{adj}U)_{ji} u'_{ij} \\
&= \operatorname{tr}(\text{adj}U)U' = \operatorname{tr}(\text{adj}U)AU \\
&= \operatorname{tr} U(\text{adj}U)A = \operatorname{tr}(\det U)A = (\operatorname{tr} A)W.
\end{aligned}$$

(Note also that, for any matrix A, $\det e^A = e^{\operatorname{tr} A}$; this can be proved by considering a basis in which A is triangular.)

Solving gives $W(x) = W(0)e^{\int^x \operatorname{tr} A}$; as $W(0) = \det U_0 \neq 0$, it follows that $W(x)$ is never 0 and U_x is invertible.

□

Example 1

Consider the equations

$$\begin{aligned}
u' &= u - v + f(x) \\
v' &= 2u - v + g(x).
\end{aligned}$$

When written in vector form we get

$$\boldsymbol{y}' = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \boldsymbol{y} + \boldsymbol{f},$$

where $\boldsymbol{y} := \binom{u}{v}$ and $\boldsymbol{f} := \binom{f}{g}$.

We first find a basis for the homogeneous solutions,

$$\boldsymbol{y}' = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \boldsymbol{y}, \text{ which is just } \begin{aligned} u' &= u - v \\ v' &= 2u - v \end{aligned}.$$

These can be solved as follows,

$$u'' = u' - v' = u' - (2u - v) = u' - 2u + (u - u') = -u$$

$$\begin{aligned} \Rightarrow u(x) &= A\cos x + B\sin x \\ v(x) &= u(x) - u'(x) = A(\cos x + \sin x) + B(\sin x - \cos x) \\ \boldsymbol{y}(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} &= A\begin{pmatrix} \cos x \\ \cos x + \sin x \end{pmatrix} + B\begin{pmatrix} \sin x \\ \sin x - \cos x \end{pmatrix}. \end{aligned}$$

We can therefore take $\boldsymbol{u}_1(x) = \left(\begin{smallmatrix} \cos x \\ \cos x + \sin x \end{smallmatrix}\right)$ and $\boldsymbol{u}_2(x) = \left(\begin{smallmatrix} \sin x \\ \sin x - \cos x \end{smallmatrix}\right)$ as our basis for $\ker L$. The particular solution would therefore be

$$\begin{aligned} \boldsymbol{y}_P(x) = \begin{pmatrix} u_P(x) \\ v_P(x) \end{pmatrix} &= \begin{pmatrix} \cos x & \sin x \\ (\cos x + \sin x) & (\sin x - \cos x) \end{pmatrix} \\ &\qquad \times \int^x \begin{pmatrix} (\cos s - \sin s) & \sin s \\ (\cos s + \sin s) & -\cos s \end{pmatrix} \begin{pmatrix} f(s) \\ g(s) \end{pmatrix} \, \mathrm{d}s \end{aligned}$$

For example, if $f(x) = 1$ and $g(x) = -1$, the integral works out to

$$\int^x \begin{pmatrix} \cos s - 2\sin s \\ 2\cos s + \sin s \end{pmatrix} \mathrm{d}s = \begin{pmatrix} \sin x + 2\cos x \\ 2\sin x - \cos x \end{pmatrix},$$

$$\begin{aligned} \boldsymbol{y}_P(x) = \begin{pmatrix} u_P(x) \\ v_P(x) \end{pmatrix} &= \begin{pmatrix} \cos x & \sin x \\ (\cos x + \sin x) & (\sin x - \cos x) \end{pmatrix} \begin{pmatrix} \sin x + 2\cos x \\ 2\sin x - \cos x \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

Of course, we could have got this result much simpler by putting a trial solution, but the point here is that there is a general formula which applies for any function $\boldsymbol{f}$.

Example 2

Solve the equations

$$\begin{aligned} u' &= 2u + v + 3e^{2x} \qquad & u(0) = 3 \\ v' &= -4u + 2v + 4xe^{2x} \qquad & v(0) = 2 \end{aligned}$$

First solve the homogeneous equation:

$$u'' = 2u' + v' = 4u' - 8u$$

to get the roots $2 \pm 2i$, hence

$$\boldsymbol{y}_H(x) = \begin{pmatrix} u_H(x) \\ v_H(x) \end{pmatrix} = Ae^{2x}\begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix} + Be^{2x}\begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}$$

Putting the two independent solutions into the matrix U and working out its inverse we get

$$U^{-1}\boldsymbol{f} = \frac{1}{2}e^{-2x}\begin{pmatrix}2\cos 2x & -\sin 2x\\ 2\sin 2x & \cos 2x\end{pmatrix}\begin{pmatrix}3e^{2x}\\ 4xe^{2x}\end{pmatrix} = \begin{pmatrix}3\cos 2x - 2x\sin 2x\\ 3\sin 2x + 2x\cos 2x\end{pmatrix}$$

Its integral is $\left(\begin{smallmatrix}\sin 2x + x\cos 2x\\ -\cos 2x + x\sin 2x\end{smallmatrix}\right)$, so the general solution is given by

$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{y}_H + U_x\int^x U_s^{-1}\boldsymbol{f}(s)\,\mathrm{d}s\\ &= Ae^{2x}\begin{pmatrix}\cos 2x\\ -2\sin 2x\end{pmatrix} + Be^{2x}\begin{pmatrix}\sin 2x\\ 2\cos 2x\end{pmatrix} + e^{2x}\begin{pmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{pmatrix}\begin{pmatrix}\sin 2x + x\cos 2x\\ -\cos 2x + x\sin 2x\end{pmatrix}\\ &= e^{2x}\begin{pmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} + e^{2x}\begin{pmatrix}x\\ -2\end{pmatrix}\end{aligned}$$

To satisfy the initial conditions, we need

$$\begin{pmatrix}3\\ 2\end{pmatrix} = \boldsymbol{y}(0) = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} + \begin{pmatrix}0\\ -2\end{pmatrix},$$

that is, $A = 3$, $B = 2$.

In practice, the integrals need not give well-known functions. Also, the method for solving the homogeneous equation can only be used for equations with the coefficients of u, v being constants. The following cannot be solved this way:

Legendre	$u'(x) = v(x)/(1-x^2)$ $v'(x) = -2u(x)$	*Hermite*	$u'(x) = e^{x^2}v(x)$ $v'(x) = -\lambda e^{-x^2}u(x)$
Bessel	$u'(x) = v(x)/x$ $v'(x) = (\lambda/x - x)u(x)$	*Laguerre*	$u'(x) = v(x)/x$ $v'(x) = (x-1)u(x) + v(x)$

Exercises 3.4

1. Solve the simultaneous equations

$$\begin{aligned}u'(x) &= v(x) + x\\ v'(x) &= u(x) - 1\end{aligned}$$

given the initial conditions $u(0) = 1$ and $v(0) = 2$.
(Answer: $u(x) = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$, $v(x) = -x + \frac{3}{2}e^x + \frac{1}{2}e^{-x}$.)

2. Solve the simultaneous equations

$$\begin{aligned}u'(x) &= u(x) + v(x) + 1\\ v'(x) &= -u(x) + v(x) + x\end{aligned}$$

with initial conditions $u(0) = 1, v(0) = 0$. (Assume $\int xe^{-x}e^{ix}\,\mathrm{d}x = \frac{1}{2}e^{-x}(((x+1)\sin x - x\cos x) + i((1+x)\cos x + x\sin x))$.)

(Answer: $u(x) = \frac{1}{2}e^{x}(2\cos x + \sin x) + x/2$, $v(x) = \frac{1}{2}e^{x}(\cos x - 2\sin x) - (x+1)/2$.)

3. Show that the solution of the first-order equation $y' + a(x)y = f(x)$, for scalar functions $y(x)$, as solved by the method in this chapter, is the same solution as given by the integrating factor method.

4. Use the Picard iteration scheme to show that the solution of $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{Y}$, with A a constant matrix (independent of x), is $\boldsymbol{y}(x) = e^{xA}\boldsymbol{Y}$.

   Show that $U_x = e^{xA}$ and that the Green's function is given by $e^{A(x-s)}$.

   If A is diagonalizable $A = P^{-1}DP$, then $e^{A} = P^{-1}e^{D}P$, where e^{D} is the diagonal matrix with the diagonal components being e^{λ_i}, $\lambda_i = D_{ii}$.

5. Given the equation $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}$, $\boldsymbol{y}(0) = \boldsymbol{Y}$, let $\boldsymbol{v}(x) := \boldsymbol{y}(x) - \boldsymbol{Y}$. Show that $\boldsymbol{v}$ satisfies the equation $\boldsymbol{v}' = A\boldsymbol{v} + \tilde{\boldsymbol{f}}$ where $\tilde{\boldsymbol{f}}(x) = A\boldsymbol{Y} + \boldsymbol{f}(x)$.

6. *Volterra*: The equation $y'' + a_1y' + a_0y = f$, $y(a) = 0 = y'(a)$ has a solution if, and only if, $u + \int_a^x k(x,s)u(s)\,\mathrm{d}s = f$ has a solution. (Hint: take $u := y''$ and $k(x,s) := a_1(x) + a_0(x)(x-s)$.

7. An 'arms race' can be modeled by using two variables u, v denoting the value of arms in two countries. In normal circumstances, obsolete arms are replaced by newer ones, and so $\dot{u} = \alpha - \alpha u$ settling to a stable value of $u(t) = 1$. But under the pressure of a second country, the resulting equations may be

$$\dot{u} = k_1 v + \alpha(1-u)$$

$$\dot{v} = k_2 u + \beta(1-v)$$

   What are the meaning of the terms in these expressions? What is the solution, and why is it related to an *arms race*?

CHAPTER 4

Higher Order Linear O.D.E.'s

4.1 Initial Value Problem

We will consider higher-order linear o.d.e.'s of the type:

$$a_n(x)y^{(n)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x) = f(x).$$

From the second chapter, we know that we can transform this equation into n first order equations by introducing the vector variable $\boldsymbol{y} = \begin{pmatrix} y & y' & \ldots & y^{(n-1)} \end{pmatrix}$. In this case, because of the form of the equation, the resulting first order equation is linear and therefore we can apply the methods that were devised in chapter 2.

$$\boldsymbol{y}' = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ -a_0/a_n & -a_1/a_n & \ldots & -a_{n-1}/a_n \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} 0 \\ \vdots \\ f/a_n \end{pmatrix}$$

First, we find conditions when the equation is well-posed.

Proposition 4.1

(Special case of Picard's theorem)

Given the o.d.e.

$$a_n(x)y^{(n)}(x) + \ldots + a_0(x)y(x) = f(x)$$
$$y(a) = Y_0, \ldots, y^{(n-1)}(a) = Y_{n-1},$$

if $a_i(x)$ for $i = 0, \ldots, n$ and $f(x)$ are continuous functions and $a_n(x) \geqslant c > 0$ on an interval $[\alpha, \beta]$, then there exists a unique solution to the above equation in $[\alpha, \beta]$.

The proof is obvious, since a_i/a_n and f/a_n are continuous functions with the given hypothesis.

Examples

The equation

$$x^2y'' - xy' + (\sin x)y = \tan x, \quad y(1) = 0,\ y'(1) = 1,$$

is well-posed on the interval $(0, \pi/2)$.

For the equation

$$x^2y'' - 3xy' + 3y = 0, \quad y(0) = 0,\ y'(0) = 1,$$

$a_2(x)$ vanishes at the initial point and the proposition does not apply. In fact it is not well-posed and has many solutions $y(x) = Ax^3 + x$ for any A.

Most of the methods that will be developed in this chapter carry over to nth order equations, but we will only consider second-order equations for their simplicity and because they occur very widely in applications.

The initial value problem in this case is of the form,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

$$y(a) = Y_0 \quad y'(a) = Y_1$$

where a_0, a_1, a_2, f are given functions and Y_0, Y_1 given constants, and it is required to find $y(x)$.

From the previous chapter, we know that we need to solve two equations and find

1. A basis for the homogeneous equation $\boldsymbol{y}'(x) = A\boldsymbol{y}(x)$ which in this case translates to:

$$a_2(x)y_H''(x) + a_1(x)y_H'(x) + a_0(x)y_H(x) = 0.$$

2. A particular solution y_P satisfying,

$$a_2(x)y_P''(x) + a_1(x)y_P'(x) + a_0(x)y_P(x) = f(x).$$

4.1.1 Homogeneous Equation

The homogeneous equation can only be solved easily in a few cases, e.g. constant coefficients, or Euler's equations, whose solutions take the form of exponentials, or sines and cosines, together with polynomials and logarithms. If the coefficients are polynomials it may be possible to use a general method called the Frobenius method. The following is a short list of the most common second order o.d.e.'s that are encountered in applications.

Bessel's equation	$x^2y'' + xy' + (x^2 - n^2)y = 0$
Laguerre's equation	$xy'' + (1-x)y' + ny = 0$
Hermite's equation	$y'' - (x^2 - n)y = 0$
Legendre's equation	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$
Chebyshev's equation	$(1-x^2)y'' - xy' + n^2y = 0$
Airy's equation	$y'' - xy = 0$
Euler's equations	$x^2y'' + axy' + by = 0$

Reduction of Degree. If one solution u of the homogeneous equation is found (in some way), the other can be found by letting $w := y/u$, i.e., $y = uw$, so that $y' = uw' + u'w$ and $y'' = uw'' + 2u'w' + u''w$; hence

$$0 = y'' + ay' + by = uw'' + (2u' + au)w'$$

which is a first order equation in w'. (This also works for higher-order linear equations: they are transformed to lower order equations, hence the name.)

Exercise. Solve $uw'' + (2u' + au)w' = 0$ by separation of variables to get the second solution $v(x) = u(x)\int^x (u(s)^2 e^{\int^s a})^{-1}\,\mathrm{d}s$.

The Power Series Method

Given $y'' + ay' + by = 0$ where a and b are polynomials (or more generally, power series). The idea is to suppose $y(x)$ to be a power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$, substitute into the equation to get an infinite number of equations in the unknown coefficients $a_0, a_1, \ldots$.

Examples. *Harmonic equation* $y'' + y = 0$. Suppose

$$\begin{aligned} y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ y'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots \\ y''(x) &= 2a_2 + 3.2a_3x + 4.3a_4x^2 + 5.4a_5x^3 + \cdots \\ y''(x) + y(x) &= (2a_2 + a_0) + (3.2a_3 + a_1)x + (4.3a_4 + a_2)x^2 + \cdots \end{aligned}$$

If this power series is to be zero, then its coefficients must vanish:

$$a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{3.2}, a_4 = -\frac{a_2}{4.3}, \ldots, a_n = -\frac{a_{n-2}}{n(n-1)}.$$

Thus there are two series,

$$a_0, a_2 = -a_0/2, a_4 = a_0/4.3.2, a_6 = -a_0/6.5.4.3.2, \ldots, a_{2n} = \frac{(-1)^n a_0}{(2n)!};$$

$$a_1, a_3 = -a_1/3.2, a_5 = a_1/5.4.3.2, \ldots, a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}.$$

The first few terms of the solution is

$$\begin{aligned} y(x) &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \cdots \\ &= a_0(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) + \\ &\quad a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots) \end{aligned}$$

The two linearly independent solutions are the two series enclosed in brackets; they can be recognized as $\cos x$ and $\sin x$.

Airy's equation $y'' + xy = 0$. This time

$$y''(x) + xy(x) = 2a_2 + (a_0 + 3.2a_3)x + (a_1 + 4.3a_4)x^2 + (a_2 + 5.4a_5)x^3 + \cdots$$

If this series is to vanish, then its coefficients must all be zero, $a_{n-1} + (n+2)(n+1)a_{n+2} = 0$, i.e.,

$$a_2 = 0, a_3 = -a_0/6, a_4 = a_1/12, a_5 = 0, \ldots, a_n = -a_{n-3}/n(n-1),$$

; using this formula, it is clear that there are three sets of coefficients,

$$\begin{aligned} a_2 = a_5 = a_8 = \cdots = 0, a_0, a_3 = -a_0/3.2, a_6 = a_0/6.5.3.2, a_9 = -a_0/9.8.6.5.3.2, \ldots \\ a_1, a_4 = -a_1/4.3, a_7 = a_1/7.6.4.3, \ldots \end{aligned}$$

The first few terms of the solution is

$$\begin{aligned} y(x) &= a_0 + a_1 x - \frac{a_0}{3.2}x^3 - \frac{a_1}{4.3}x^4 + \frac{a_0}{6.5.3.2}x^6 + \cdots \\ &= a_0(1 - \frac{1}{3!}x^3 + \frac{4}{6!}x^6 - \frac{7.4}{9!}x^9 + \cdots) + \\ &\quad a_1(x - \frac{2}{4!}x^4 + \frac{5.2}{7!}x^7 + \cdots) \end{aligned}$$

Again the two linearly independent solutions are the two series enclosed in brackets.

Hermite's equation $y'' - x^2 y = 0$. This time

$$y''(x) - x^2 y(x) = 2a_2 + 3.2a_3 x + (4.3a_4 - a_0)x^2 + (5.4a_5 - a_1)x^3 + \cdots$$

hence $a_2 = 0 = a_3$, $a_4 = a_0/4.3$, $a_5 = a_1/5.4$; in general $a_n = a_{n-4}/n(n-1)$, so there are two independent series solutions

$$a_0(1 + \frac{1}{4.3}x^4 + \frac{1}{8.7.4.3}x^8 + \cdots)$$

$$a_1(x + \frac{1}{5.4}x^5 + \frac{1}{9.8.5.4}x^9 + \cdots)$$

Legendre's equation $(1-x^2)y''-2xy'+2y=0$. This time we get

$$(2a_2+2a_0)+3.2a_3x+(4.3a_4-4a_2)x^2+\cdots$$

so $a_2=-a_0$, $a_3=0$, $a_4=a_2/3$, and in general, $a_{n+2}=a_n(n^2-2)/(n+2)(n+1)$, so $a_3=0=a_5=\ldots$, and $a_2=-a_0$, $a_4=-\frac{4}{4!}a_0$, $a_6=-\frac{14.4}{6!}$, etc. Notice that x is a solution.

The general method is to let

$$y(x)=\sum_{n=0}^{\infty}a_nx^n,\quad y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1},\quad y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

then substitute into the equation $y''+a(x)y'+b(x)y=0$; this gives a *recurrence* equation for the coefficients which can be solved to give the two linearly independent solutions. There is a theorem which states that this method always works as long as $a(x)$ and $b(x)$ are power series with some positive radius of convergence.

Bessel's equation $y''+\frac{1}{x}y'+y=0$ (There is no guarantee that there is a power series solution, but in this case there is one.) Substituting gives

$$y''+\frac{y'}{x}+y=a_1x^{-1}+(4a_2+a_0)+((3.2+3)a_3+a_1)x+\cdots$$

Hence $a_1=0$, $a_2=-a_0/4$, $a_3=0$, etc.

Euler's equation $x^2y''+xy'-y=0$. Trying a power series we find $\sum_{n=0}^{\infty}(n^2-1)a_nx^n=0$. Thus $a_n=0$ unless $n=1$; so the only power series solution is x. This is to be expected if the second linearly independent solution is not a power series, such as $\sqrt{x}$ or $1/x$. In fact, using reduction of degree, we find the other solution to be $x\log x$. Other equations may have no power series solutions at all.

Try some other problems of this type: *Laguerre's equation* $y''+\frac{1-x}{x}y'+\frac{1}{x}y=0$; $x^2y''+xy'+y=0$ (has no power series solutions).

Regular Singular Points: If $a(x)$ has a simple pole and $b(x)$ has a double pole (at most), then we can try a series of the type

$$\begin{aligned}y(x)&=a_0x^{-r}+\cdots+a_r+a_{r+1}x+a_{r+2}x^2+a_{r+3}x^3+\cdots\\y'(x)&=-ra_0x^{-r-1}+\cdots+a_{r+1}+2a_{r+2}x+3a_{r+3}x^2+4a_{r+4}x^3+\cdots\\y''(x)&=r(r+1)a_0x^{-r-2}+\cdots+2a_{r+2}+3.2a_{r+3}x+4.3a_{r+4}x^2+\cdots\end{aligned}$$

Example. $y''+\frac{2}{x}y'+y=0$. Let $y(x):=\sum_{n=0}^{\infty}a_nx^{n-r}$. Then

$$y''+\frac{2}{x}y'+y=\sum_{n=0}^{\infty}((n-r)(n-r-1)a_n+2(n-r)a_n+a_{n-2})x^{n-r-2}$$

The first term of this series determines r and is called the *indicial equation*: $r(r+1)-2r=0$, i.e., $r(r-1)=0$, i.e., $r=0,1$. Choosing $r=1$, we find $n(n-1)a_n+a_{n-2}=0$, i.e., $a_n=-a_{n-2}/n(n-1)$ $(n\neq 0,1)$; thus $a_2=-a_0/2!$, $a_4=a_0/4!$, …, and $a_3=-a_1/3!$, $a_5=a_1/5!$, …, so the solution is

$$y(x)=a_0\frac{\cos x}{x}+a_1\frac{\sin x}{x}$$

The general *Frobenius* method is to let

$$y(x):=x^r\sum_{n=0}^{\infty}a_nx^n$$

where $a_0\neq 0$, and r can be fractional or negative, so that

$$y'(x)=\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1},\qquad y''(x)=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2},$$

and then substitute. The coefficient in front of the lowest order term must be equal to zero and this gives the indicial equation for r.

Examples. $x^2y''+xy'-\frac{1}{4}y=0$ (a_1 is a simple pole and a_0 is a double pole). Let $y=x^r(a_0+a_1x+a_2x^2+\cdots)$, then we get

$$\sum_{n=0}^{\infty}((n+r)(n+r-1)+(n+r)-\frac{1}{4})a_nx^{n+r}=0.$$

The first coefficient, corresponding to $n=0$, gives the indicial equation $r(r-1)+r-\frac{1}{4}=0$, i.e., $r=\pm\frac{1}{2}$. For $r=\frac{1}{2}$, we get $((n+\frac{1}{2})^2-\frac{1}{4})a_n=0$, so $a_n=0$ unless $n=0,-1$. This means that the solution is $y(x)=a_0\sqrt{x}$. For $r=-\frac{1}{2}$, we get $((n-\frac{1}{2})^2-\frac{1}{4})a_n=0$, so $a_n=0$ unless $n=0,1$, hence $y(x)=x^{-\frac{1}{2}}(a_0+a_1x)$. The two independent solutions are $\sqrt{x}$ and $1/\sqrt{x}$. Note that in general, the smaller r will include the larger value of r if the difference between them is a positive integer.

$4xy''+2y'+y=0$ (both a_1 and a_0 are simple poles). Trying $y=x^r(a_0+a_1x+a_2x^2+\cdots)$ gives

$$\sum_{n=0}^{\infty}(4(n+r)(n+r-1)a_n+2(n+r)a_n+a_{n-1})x^{n+r-1}=0$$

(take $a_{-1}=0$). The indicial equation is $r(2r-2+1)=0$, giving two possible values of $r=0,\frac{1}{2}$.

For $r=0$, we get $a_n=\frac{a_{n-1}}{2n(2n-1)}$ and yielding

$$y(x)=a_0(1-\frac{x}{2}+\frac{x^2}{4!}-\frac{x^3}{6!}+\cdots)=a_0\cos\sqrt{x},$$

while for $r = \frac{1}{2}$, we get $a_n = -\frac{a_{n-1}}{2n(2n+1)}$ and

$$y(x) = a_0\sqrt{x}(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots) = a_0 \sin\sqrt{x}.$$

$x^2y'' + xy' + y = 0$. The power series is now $\sum_{n=0}^{\infty}((n+r)(n+r-1)a_n + (n+r)a_n + a_n)x^{n+r} = 0$, so the indicial equation is $r^2+1 = 0$. This seems to indicate that there are no solutions, but in fact power series still make sense for complex numbers, indeed have a nicer theory in that context. So we should try $r = \pm i$. For $r = i$, we find (show!) $y(x) = a_0x^i$, while for $r = -i$, we get $y(x) = a_0x^{-i}$. Note that $x^{\pm i} = e^{\pm i \log x} = \cos\log x \pm i \sin\log x$, and further that if $y(x) = \alpha x^i + \beta x^{-i}$ is to be real, then $\beta = \overline{\alpha}$ and $y(x) = A\cos\log x + B\sin\log x$.

There are special cases when this method may not work: when r is a double root, or when r has roots which differ by an integer, the method may give one solution only, in which case use reduction of degree to find the other. Another failure case is when a_1 is not a simple pole and a_0 a double pole: e.g. $x^2y'' + (3x-1)y' + y = 0$ gives the "solution" $y(x) = \sum_{n=0}^{\infty} n!x^n$ which has zero radius of convergence; $x^4y'' + 2x^3y' - y = 0$ gives no solutions at all. In such cases one can also consider power series about a different point.

Exercises 4.2

1. Solve $y' = y$ using the power series method.

2. $x^2y''-2xy'+2y = 0$, $y''+xy'+y = 0$, $y''-xy'+y = 0$, $y''-3y'+2xy = 0$, $y'' + (1-x^2)y' - xy = 0$, $y'' = x^2y$.

3. In these problems, a power series solution is not guaranteed:

   (a) $xy'' + 2y' + xy = 0$ (has one power series solution $\sin x/x$)

   (b) $xy'' - (1-x)y' - 2y = 0$ (gives x^2)

   (c) $xy'' + (1-2x)y' - (1-x)y = 0$ (gives e^x)

   (d) $4x^2y'' + 4xy' - y = 0$

4. Solve the problems above using Frobenius' method.

5. $4x^2y'' + 4x^2y' + y = 0$, $x^3y'' - xy' + y = 0$ (show that x is a solution, hence find the other solution by reduction of degree; can you see why the second solution cannot be written as a power series?), *Bessel's equation* $x^2y'' + xy' + (x^2 - m^2)y = 0$.

6. Solve the equation $x^4y'' + 2x^3y' - y = 0$ by using the transformation $t = 1/x$.

Proposition 4.3

(Sturm separation theorem)

> **The zeros of the two independent solutions of**
>
> $$y'' + a_1(x)y' + a_0(x)y = 0$$
>
> **(with a_1, a_0 continuous) are simple and interlace.**

PROOF. Let u, v be two linearly independent solutions of the equation, and let a, b be two successive zeros of u. As u is continuous, it must remain, say, positive between the two zeros, by the Intermediate Value Theorem (if u is negative, replacing u by $-u$ makes it positive without altering the zeros). Hence $u'(a) \geqslant 0$ and $u'(b) \leqslant 0$.

But the Wronskian $W = uv' - u'v$ remains of the same sign throughout the interval of existence of u and v. At a, $W = u'(a)v(a) \neq 0$, so $u'(a) > 0$ (a must be a simple zero) and $v(a)$ has the same sign as W. At b however, $W = u'(b)v(b)$, so $u'(b) < 0$ and $v(b)$ must have opposite sign to W. It follows by the intermediate value theorem that there is a zero of v in between a and b. This zero is in fact the only one, otherwise the same argument with u and v in reverse roles would imply there is another zero of u in between the two zeros of v.

□

Corollary: If u oscillates, then so does v.

Proposition 4.4

> **In the equation $y'' + ay = 0$ (a continuous),**
>
> **If a increases then the distance between the zeros decreases,**
>
> **If $a \geqslant \omega^2 > 0$ $\omega \in \mathbb{R}$, then the solutions are oscillatory; more generally, if $\int_\alpha^\infty a = \infty$ then the solutions are oscillatory on $[\alpha, \infty[$.**

PROOF. (a) Suppose $a_2 \geqslant a_1$ and let y_1, y_2 be the two solutions of the two corresponding equations $y'' + ay = 0$. Translate y_2 so y_1 and y_2 have same zero at 0. Without loss of generality, y_1 is positive up to its next zero at $x = \alpha$. So $y_1'(0) > 0$ and $y_1'(\alpha) < 0$. Suppose $y_2'(0) > 0$, then $-y_2y_1'' - a_1y_1y_2 = 0$ and $y_1y_2'' + a_2y_1y_2 = 0$. Hence $(y_1y_2' - y_2y_1')' =$

$(a_1 - a_2)y_1y_2$, so $y_2(\alpha)y_1'(\alpha) = W = \int_0^\alpha (a_1 - a_2)y_1y_2$, so $y_2(\alpha) < 0$ and y_2 has a zero before α.

(b) Since $y'' + \omega^2 y = 0$ has oscillatory solutions, the solutions of $y'' + ay = 0$ must remain oscillatory by (a).

More generally, if y is not oscillatory beyond β, let $v := y'/y$, so the equation becomes $v' + v^2 = -a$, which gives $v(x) = -\int_\beta^x v^2 + a \to -\infty$. So $v(x) \leqslant -\int_\beta^x v^2$, so $v(x)^2 \geqslant (\int v^2)^2$. Let $R(x) := \int^x v^2$, then $R^2 \geqslant R'$, so $R(x) \leqslant -\frac{1}{x+c}$, i.e., $x \leqslant x_0 + \frac{1}{R(x_0)} \leqslant A$.

□

4.2 Green's Function

We now substitute into the formula found in Chapter 3 to get a concise formula for the particular solution.

Theorem 4.5

The equation, $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$ **has a particular solution,**

$$y_P(x) = \int^x G(x,s)f(s)\,\mathrm{d}s$$

where $G(x,s)$**, the Green's function for the initial value problem is defined by**

$$G(x,s) = \frac{u(s)v(x) - v(s)u(x)}{W(s)}$$

where u **and** v **form a basis for the solutions of the homogeneous equation and**

$$W(s) = u(s)v'(s) - v(s)u'(s).$$

Proof. Following the method in chapter 2, we define the matrix $U_x = \begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix}$, and its determinant $W(x) = \det U_x$. We can now substitute into the formula

$$\begin{aligned}\boldsymbol{y}'(x) &= U_x \int^x U_s^{-1}\boldsymbol{f}(x)\,\mathrm{d}s + U_x\boldsymbol{c}\\ \begin{pmatrix} y(x)\\ y'(x)\end{pmatrix} &= \begin{pmatrix} u(x) & v(x)\\ u'(x) & v'(x)\end{pmatrix}\int^x \frac{1}{W(s)}\begin{pmatrix} v'(s) & -u(s)\\ -u'(s) & v(s)\end{pmatrix}\begin{pmatrix}0\\ f(s)\end{pmatrix}\mathrm{d}s\\ &= \int^x \begin{pmatrix} -u(x)v(s)+v(x)u(s)\\ -u'(x)v(s)+v'(x)u(s)\end{pmatrix}\frac{f(s)}{W(s)}\,\mathrm{d}s + \begin{pmatrix} Au(x)+Bv(x)\\ Au'(x)+Bv'(x)\end{pmatrix}.\end{aligned}$$

Reading off the first component we get

$$y_P(x) = \int^x \frac{u(s)v(x)-v(s)u(x)}{W(s)}f(s)\,\mathrm{d}s.$$

The first part is the particular solution, which is of the required form, while the remaining part is the homogeneous solution.

□

Examples

1. Solve the equation $y''(x) + y(x) = \tan x \qquad y(0) = 1,\ y'(0) = 0.$

   Solution. There is a unique solution because the coefficients are constant and $\tan x$ is continuous in x around the initial point $x = 0$. $y'' + y = 0$ has solutions $\cos(x)$ and $\sin(x)$, which we can take to be the basis for $\ker L$. Their Wronskian is

   $$W(s) = \det\begin{pmatrix} \cos(s) & \sin(s)\\ -\sin(s) & \cos(s)\end{pmatrix} = 1$$

   Therefore the Green's function for this problem is given by,

   $$G(x,s) = \cos(s)\sin(x) - \sin(s)\cos(x)$$

   and the particular solution by,

   $$\begin{aligned} y_P(x) &= \int_0^x [\sin(x)\cos(s)\tan(s) - \cos(x)\sin(s)\tan(s)]\,\mathrm{d}s\\ &= \sin(x)\int_0^x \sin(s)\,\mathrm{d}s - \cos(x)\int_0^x \sec(s) - \cos(s)\,\mathrm{d}s\\ &= -\sin(x)(\cos(x)-1) - \cos(x)(\log(\sec(x)+\tan(x)) - \sin(x))\\ &= \sin(x) - \cos(x)\log(\sec(x)+\tan(x))\end{aligned}$$

   The general solution is,

   $$y(x) = A\cos(x) + B\sin(x) + \sin(x) - \cos(x)\log(\sec(x)+\tan(x))$$

Substituting the initial conditions,

$$1 = y(0) = A, \qquad 0 = y'(0) = B + 1 - 1$$

gives the solution

$$y(x) = \cos x + \sin x - \cos x \log(\sec(x) + \tan(x))$$

2. Solve $xy''(x) - 2(x+1)y'(x) + (x+2)y(x) = e^x$, $y(1) = 0 = y'(1)$, given that the solutions of the homogeneous equation are of the type $e^x g(x)$.

Solution. There is a unique solution because the coefficients are continuous in x around the initial point $x = 1$. Substituting $y = e^x g$, $y' = e^x(g' + g)$, $y'' = e^x(g'' + 2g' + g)$ into the homogeneous equation gives, after simplifying and dividing by e^x,

$$xg''(x) - 2g'(x) = 0$$

This can be solved: if necessary, substitute $h(x) := g'(x)$ to get

$$\frac{h'}{h} = \frac{2}{x} \;\Rightarrow\; g'(x) = h(x) = cx^2 \;\Rightarrow\; g(x) = Ax^3 + B$$

$$y_H(x) = Ax^3 e^x + Be^x$$

So, $u(x) = x^3 e^x$, $v(x) = e^x$.

The Wronskian is then

$$W(x) = \det \begin{pmatrix} x^3 & 1 \\ 3x^2 + x^3 & 1 \end{pmatrix} e^x = -3x^2 e^{2x}$$

and the Green's function is

$$G(x, s) = \frac{s^3 e^s e^x - x^3 e^x e^s}{-3s^3 e^{2s}}.$$

A particular solution is then

$$y_P(x) = \int^x G(x, s) f(s)\,\mathrm{d}s = \frac{x^3 e^x}{3} \int^x \frac{1}{s^3}\,\mathrm{d}s - \frac{e^x}{3} \int^x 1\,\mathrm{d}s = -\frac{e^x x}{2}$$

Combined with u and v, we get the general solution

$$y(x) = Ax^3 e^x + Be^x - \frac{1}{2} x e^x$$

Substituting the initial conditions,

$$\begin{array}{l} 0 = y(1) = Ae + Be - \frac{1}{2}e \\ 0 = y'(1) = 4Ae + Be - e \end{array} \;\Rightarrow\; A = 1/6, B = 1/3$$

and the solution is

$$y(x) = \frac{e^x}{6}(x^3 - 3x + 2)$$

Notice that once $G(x, s)$ is found for L, it can be used for another f to give the corresponding particular solution.

Trial solutions The method outlined above need not be the best method to apply when f is a simple function such as polynomials, exponentials or trigonometric functions; in this case, one can often guess the answer or try solutions of the same type. For example, $y' + y = e^x$ gives $y(x) = Ae^{-x} + \frac{1}{2}e^x$ by inspection.

Linearity of the Particular Solutions Of course one can split f into simpler parts $f = f_1 + \cdots + f_N$ and find a particular solution y_i for each f_i; in this case $y_1 + \cdots + y_N$ is a particular solution for f, by linearity.

4.3 Boundary Value Problem

Second order differential equations require two conditions to determine the constants A and B that appear in the homogeneous solution. Up to now we have considered initial conditions, but there can also be **boundary conditions** of the type:

$$y(a) = Y_a \qquad y(b) = Y_b$$

where Y_a, Y_b are given constants. Such equations are called *boundary value problems.*

Note that boundary value problems are not of the type treated by Picard's theorem, and in fact we are not guaranteed that solutions always exist in this case. For example, there are no solutions of $y'' - y = 0$ which satisfy the conditions $y(0) = 0 = y(1)$; there are an infinite number of solutions of $y'' + y = 0$ satisfying $y(0) = 0 = y(\pi)$.

However a solution, if it exists, will be of the type found above.

$$y(x) = y_P(x) + Au(x) + Bv(x)$$

One can proceed to find A and B via

$$\begin{aligned} Y_a = y(a) &= y_P(a) + Au(a) + Bv(a) \\ Y_b = y(b) &= y_P(b) + Au(b) + Bv(b), \end{aligned}$$

which is equivalent to solving

$$\begin{pmatrix} u(a) & v(a) \\ u(b) & v(b) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Y_a - y_P(a) \\ Y_b - y_P(b) \end{pmatrix}.$$

It can happen however that the matrix on the left is not invertible, and hence either no A and B exist or an infinite number of them do, depending on whether the equations are consistent or not.

Exercises 4.6

1. Determine ranges where the following equations have unique solutions:

   (a) $y'' + y = \tan x$, $y(0) = 1$, $y'(0) = 0$,

   (b) $x^2 y'' - xy' + y = x$, $y(1) = 1$, $y'(1) = 1$,

   (c) $xy'' - 2(x+1)y' + (x+2)y = e^x/x$, $y(1) = 0 = y'(1)$

   (d) $xy'' - (\sin x)y = 0$, $y(1) = 0 = y'(1)$.

2. By first solving the homogeneous equation, calculate the Green's function of the following differential equation:

   $$y'' - 3y' + 2y = f(x) \qquad y(0) = 1 \;\; y'(0) = 2$$

   Solve the equation for the case $f(x) = e^x$.

   (Answer: $2e^{2x} - e^x(1+x)$)

3. Find the general solutions of the equations

   $$\begin{aligned} \text{(i) } y'' + 4y &= \sin 2x, \\ \text{(ii) } y'' + 2y' + (1+\alpha^2)y &= f, \quad \alpha \neq 0. \end{aligned}$$

   (Answers: (i) $\frac{1}{16}\sin 2x - \frac{1}{4}x\cos 2x + A\cos 2x + B\sin 2x$, (ii) $G(x,s) = (e^{x+s}\sin\alpha(x-s)/\alpha e^s$.

4. Find the general solutions of

   (a) $x^2y'' - 2xy' + 2y = x^5$ given that the solutions of the homogeneous equation are of the type x^n. (Answer: $\frac{1}{12}x^5 + Ax^2 + Bx$)

   (b) $xy'' - 2(x+1)y' + (x+2)y = e^x/x$ given that $y(x) = e^x g(x)$. (Answer: $e^x(\log x - x)/2 + Ax^2e^x + Be^x$)

   (c) $x^2y'' - xy' + y = x$ given that $y(x) = g(\log x)$. (Answer: $-\frac{1}{2}x(\log x)^2 + Ax + Bx\log x$)

5. Solve the equation,

   $$y'' - (2 + \frac{1}{x})y' + (1 + \frac{1}{x})y = xe^{x-1}$$

   with initial conditions $y(1) = 0$, $y'(1) = 1$.

   (Hint: the solutions of the homogeneous equation are of the form $e^x g(x)$. Answer: $\frac{1}{3}(x^3 - 1)e^{x-1}$)

6. Solve

   $$y'' + (\tan x - 2)y' + (1 - \tan x)y = \cos x$$

   given that the homogeneous solutions are of the type $e^x g(x)$ and that the initial conditions are $y(0) = 1$, $y'(0) = 0$.

   (Answer: $\frac{1}{2}(e^x + \cos x - \sin x)$)

7. By making the substitution $x = e^X$, find $y'(x)$ and $y''(x)$ in terms of dy/dX and d^2y/dX^2. Find two linearly independent solutions of the equation,
$$x^2y'' + xy' + y = 0$$
and calculate the Green's function.

   Hence find the solution of the equation,
$$x^2y'' + xy' + y = 1 \qquad y(1) = 0\ y'(1) = 2$$
(Answer: $2\sin\log x - \cos\log x - 1$)

8. Solve
$$x^2y'' - xy' + y = 1$$
given the initial conditions $y(1) = 1$, $y'(1) = 1$.

   (Answer: $x\log x + 1$)

9. Assume that two linearly independent solutions of the equation,
$$x^2y'' + xy' + (x^2 - 1)y = 0$$
are $J(x)$ and $Y(x)$ *(Do not attempt to find them).*

   Calculate the Green's function for the equation, and use it to find a formula (involving integrals) for the general solution of the equation,
$$x^2y'' + xy' + (x^2 - 1)y = f(x) \qquad y(1) = A \quad y'(1) = B$$

10. Let u, v be two solutions of the linear second order o.d.e.
$$a_2y'' + a_1y' + a_0y = 0,$$
where a_0, a_1, a_2 are continuous, $a_2 \neq 0$.

    Show that the Wronskian of u and v satisfies the differential equation $a_2W' + a_1W = 0$.

    Solve this differential equation to get an explicit formula for $W(x)$.

    Deduce that W vanishes if it vanishes at one point.

11. (*) Show that the Green's function $G(x, s)$ has the properties (a) $G(s, s) = 0$, (b) $\partial_x G(s, s) = 1/a_2(s)$, (c) $LG = 0$, and that these properties determine G.

CHAPTER 5

Dynamical Systems

Let us return to the general non-linear o.d.e. for a function $\boldsymbol{y}(x)$,

$$\boldsymbol{y}'(x) = \boldsymbol{F}(\boldsymbol{y}(x)).$$

Such differential equations are usually impossible to solve in closed form i.e., written in terms of some standard functions. Only a handful of such equations can in fact be so analyzed.

But such equations are very common in nature or in applications.

Examples

1. Consider a model for rabbit reproduction, where $u(t)$ denotes the number of rabbits at time t,

   $$\dot{u} = au(1 - u/b),\ \ u(0) = u_0.$$

   The parameter a is the natural fertility rate minus the natural death rate, while b is a term that gives the stable population given the size of the habitat etc. This equation can in fact be solved explicitly (exercise) to give

   $$u(t) = b/(1 + (b/u_0 - 1)e^{-at}).$$

   The rabbits increase or decrease in number until they reach to the fixed value b, independent of the initial number $u(0)$.

   Now consider a model representing the interaction between rabbits u and foxes v, where the latter prey upon the former. The model now consists of two interdependent differential equations.

   $$\dot{u} = au(1 - u/b) - cuv$$

   $$\dot{v} = duv - ev$$

   This time, although there is a unique solution for every initial condition (by Picard's theorem), we cannot write it in closed form.

2. The pendulum satisfies the differential equation,

$$\ddot{\theta} = -\sin\theta - c\dot{\theta}$$

This is not first order, but we can change it to one, by introducing the variables $v = \dot{\theta}$ and $u = \theta$,

$$\dot{u} = v$$
$$\dot{v} = -\sin u - cv$$

3. A model for how the body's immunization proceeds is to consider two types of immunity cells, called lymphocytes, denoted here by $u(t)$ and $v(t)$, which depend on each other via

$$\dot{u} = a - bu + \frac{cuv}{1+duv}$$
$$\dot{v} = a' - b'v + \frac{c'uv}{1+d'uv}$$

Different persons have different values for the constants $a, b, \ldots$.

4. A van der Pol electrical circuit is such that the current and voltage satisfy the equations,

$$\dot{u} = v + u - p(u,v)$$
$$\dot{v} = -u$$

where $p(u,v)$ is a cubic polynomial of u and v. This electrical circuit behaves abnormally in that it keeps oscillating with a strange waveform.

5. A very simple model of the atmosphere, using the barest minimum of variables, consists of the equations,

$$\dot{u} = -10(u-v)$$
$$\dot{v} = 30u - v - uw$$
$$\dot{w} = uv - 3w$$

6. The onslaught of a contagious disease can be modeled using three variables u, v, w denoting respectively the number of persons who have not yet had the disease, the number who are currently ill, and the number who have had the disease and are now immune to it. One model is the following:

$$\dot{u} = -auv + \lambda - \lambda u,$$

$$\dot{v} = auv - bv - \mu v,$$
$$\dot{w} = bv - \lambda w,$$

where a is the rate at which infections occur, λ is the birth rate (equal to the normal death rate in this case), μ is the rate at which the ill die, and b is the rate at which they recover (and become immune). Try values $a = 4, b = 2, \lambda = 0.02$ and varying values of μ.

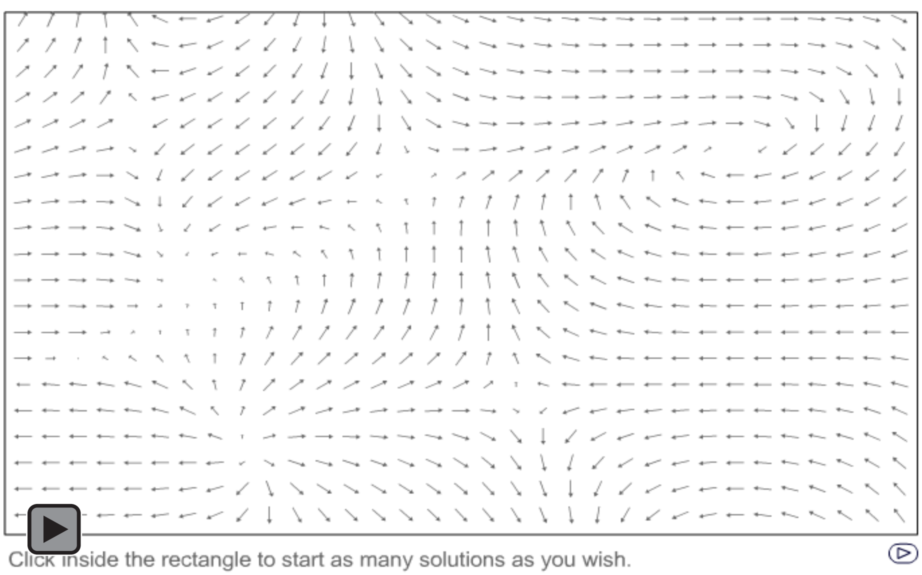
7. Applications to economics involve the dynamic evolution of income, consumption, investment, inflation and unemployment. To chemistry includes the variation of chemical concentrations while they are reacting or mixing. See the section on Non-linear equations in the Introduction.

5.1 State Space

In this chapter we analyze, in a general way, **autonomous** equations of the form

$$\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{y}).$$

More general differential equations of the type $\boldsymbol{y}' = \boldsymbol{F}(x, \boldsymbol{y})$, can be reduced to this form by letting $\boldsymbol{v} := (x, \boldsymbol{y})$ so that $\boldsymbol{v}' = (1, \boldsymbol{F}(\boldsymbol{v})) = \boldsymbol{G}(\boldsymbol{v})$ (at the cost of incrementing the number of variables).



We call the space $\mathbb{R}^n$ of variables $u, v, \ldots$, the **state space** of the system.

For simplicity we restrict ourselves to the case of two variables u, v. The presence of more variables makes the analysis progressively harder with ever more possibilities for the behavior of the solutions. With three or more variables, the solutions may behave *chaotically* without ever converging yet not diverging to infinity, or they may "converge" and trace out a surface that may be as simple as a torus or more complicated shapes, called *strange attractors*. The weather example above "converges" to the Lorenz attractor.

At each point (u, v) in the state space there is a corresponding vector $\boldsymbol{y}' = (u', v') = \boldsymbol{F}(u, v)$ which indicates the derivative of a solution $\boldsymbol{y}(x) = (u(x), v(x))$ passing through that point. Solutions must be such that they follow the vectors of the points that they traverse, starting from any initial point. It follows that if we plot these vectors in the state space we ought to get a good idea of how the solutions behave. Such a plot of the *vector field* $\boldsymbol{F}(u, v)$ is called a **phase portrait** and is usually done using a computer. The commands are usually variants of `VectorPlot` and `StreamPlot`.

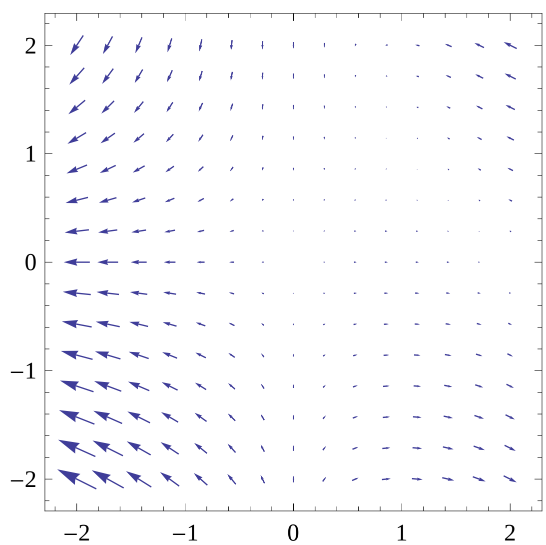


Figure 5.1: The vector field of the predator-prey equations $\dot{u} = 2u - u^2 - uv$, $\dot{v} = uv - v$.

The problem with such simple computer plots is that one cannot be sure that all the important information has been captured within the range of the plot. Moreover, we would need to substitute particular values for the parameters and initial conditions. In this section we develop a more comprehensive treatment.

A good way to analyze the vector field is to divide it into separate regions, each region having corresponding vectors in the four quadrant directions i.e., N/E, S/E, S/W and N/W. The boundaries of such regions will have vectors that point purely to the "north", "east", "south" or "west".

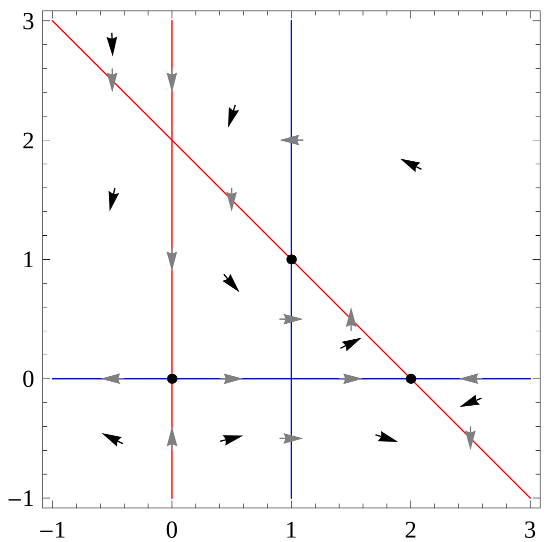


Figure 5.2: A schematic plot of the vector field of the same predator-prey equation.

To do this we can plot the set of curves resulting from $u' = 0$ (vertical vectors) and $v' = 0$ (horizontal vectors). Moreover we can decide whether vertical vectors are pointing "north" or "south", and similarly for horizontal vectors, by taking sample points on them and checking whether v' or u' is positive or negative.

5.1.1 Equilibrium Points

The path that a solution $\boldsymbol{y}(x)$ takes as x ranges in $\mathbb{R}$ is called a *trajectory* (depending on the initial point). One of the aims of Dynamics is to describe, at least qualitatively, the possible trajectories of an equation. There are three basic types: those that remain fixed at one point, those that close on themselves (periodic), and those that never repeat. Note carefully that trajectories *cannot* intersect (otherwise there would be two tangent vectors at an intersection point).

Definition An **equilibrium point** or **fixed point** is a point $\boldsymbol{y}_0$ in the state space with zero tangent, i.e., $\boldsymbol{y}' = \boldsymbol{F}(\boldsymbol{y}_0) = \boldsymbol{0}$.

If a solution $\boldsymbol{y}(x)$ is started at an equilibrium point $\boldsymbol{y}(0) = \boldsymbol{y}_0$ it remains there. They can be found at the intersection of the boundaries of the different regions.

Exercise: Show that the equilibrium points for the examples given above are, respectively, 2) $(0, n\pi)$, 4) $(0, 0)$, 5) $(0, 0, 0)$.

The equilibrium points are important because they are the points to which the solution may converge to. Each such equilibrium point has its own *basin of attraction* set: a solution that starts from inside a basin converges to the associated equilibrium point. However, solutions may diverge to infinity, or even "converge" to a limit cycle, as happens with the van der Pol equation. So knowing the fixed points only gives us partial answers to the behavior of the variables.

We can see what goes on near to an equilibrium point by taking points $\boldsymbol{y} = \boldsymbol{y}_0 + \boldsymbol{h}$ with $\boldsymbol{h} = (h, k)$ small in magnitude. Substituting into the differential equation leads to a linear differential equation, approximately correct in the vicinity of the equilibrium point $\boldsymbol{y_0}$. This is achieved by taking a first order Taylor expansion of $\boldsymbol{F}(\boldsymbol{y})$ around the fixed point:

$$\boldsymbol{y}' = \boldsymbol{F}(\boldsymbol{y}) = \boldsymbol{F}(\boldsymbol{y_0} + \boldsymbol{h}) = D_{\boldsymbol{y}}\boldsymbol{F}(\boldsymbol{y_0})\boldsymbol{h} + O(2).$$

Notice that for equilibrium points, $\boldsymbol{F}(\boldsymbol{y_0}) = \boldsymbol{0}$. Also, the second term $D_{\boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x_0})$ is simply a matrix A with constant coefficients.

Therefore we get

$$\boldsymbol{h}' = A\boldsymbol{h} + O(2).$$

We have already treated such equations in Chapter 2, where we found that the eigenvalues of A are, in almost all cases, enough to specify the type of solution. In our case these solutions will only be accurate *near* to the equilibrium points, as the further we go from them, the larger $\boldsymbol{h}$ becomes and the $O(2)$ terms cannot be neglected any more.

Not all equilibrium points are similar. Some are **stable**, meaning that slight changes to $\boldsymbol{y}$ lead to solution curves $\boldsymbol{y}(x)$ that converge back to the equilibrium point. Others are **unstable**: any slight change in $\boldsymbol{y}$ from $\boldsymbol{y}_0$ increases in magnitude, with $\boldsymbol{y}(x)$ moving away from the equilibrium point.

We have already classified all the possible generic types of linear equations in two variables. Each equilibrium point will, in general, be one of these types:

Node/Saddle: Both eigenvalues real (positive/negative). Of these only a node sink is stable (both eigenvalues negative).

Spirals: Eigenvalues are complex conjugates. The equilibrium point is stable if $\mathrm{Re}(\lambda)$ is strictly negative and unstable if it is strictly positive. It is neutral if it is zero.

Deficient node: Eigenvalues are equal, and non-diagonalizable. A deficient node is stable or unstable depending on whether the eigenvalue is negative or positive respectively.

In short, an equilibrium point is stable if all its eigenvalues have negative real part. If at least one eigenvalue has a positive real part then it is unstable. In the special case that one eigenvalue has negative real part and the other zero, or both have zero real part, then the first-order term in the Taylor expansion is not enough to decide stability.

Examples

1. *Resistive Pendulum* Find and analyze the fixed points of the "resistive" pendulum:
$$\ddot{\theta} = -b\dot{\theta} - a\sin\theta,$$
where a and b are positive constants.

   Defining $v = \dot{\theta}$ and $u = \theta$ we get the system of equations,
$$\begin{aligned}\dot{u} &= v\\ \dot{v} &= -bv - a\sin u\end{aligned}$$

   To find the fixed points, we set both equations equal to 0, and solve simultaneously to get the points $(n\pi, 0)$.

   Let us fix n, and linearize the equations about this fixed point by taking $u = n\pi + h$ and $v = 0 + k$. The Taylor expansion gives, up to first order, $\dot{h} = 0 + k$ and $\dot{k} = -b(0+k) - a\sin(n\pi + h) = -bk - a\cos(n\pi)h$, that is
$$\dot{\boldsymbol{h}} = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}a & -b \end{pmatrix} \boldsymbol{h}$$
The eigenvalues of the matrix are
$$\frac{1}{2}(-b \pm \sqrt{b^2 - 4(-1)^n a})$$
For n odd, there are two real eigenvalues, with one of them certainly positive, making the fixed point unstable;

   For n even, the discriminant may be positive or negative: if $b^2 - 4a \geqslant 0$ (i.e., the resistive forces are high) then the eigenvalues are both real and negative, making the fixed point stable; if the discriminant is negative, then the eigenvalues are complex, but with negative real part $-b/2$; the fixed point is stable in this case also.

2. *Species Habitat* What happens to a species if its habitat is destroyed slowly? Will it reduce in number proportionally, or will something more catastrophic happen?

   A simple model that includes two predator/prey species and a habitat parameter D is the following:
$$\begin{aligned}u' &= u(D-u) - uv,\\ v' &= uv - v.\end{aligned}$$
The curves in state space with $u' = 0$ are given by $u = 0$ or $v = D - u$. Similarly $v' = 0$ gives $v = 0$ or $u = 1$. The equilibrium points are

therefore $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}D\\0\end{pmatrix}$ and $\begin{pmatrix}1\\D-1\end{pmatrix}$. The last two points are only feasible when $D > 0$ and $D > 1$ respectively. Let us analyze each in turn:

At $\begin{pmatrix}0\\0\end{pmatrix}$, we let $u = 0 + h$, $v = 0 + k$ to get

$$\begin{pmatrix}h'\\k'\end{pmatrix} = \begin{pmatrix}D & 0\\0 & -1\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix} + O(2).$$

The point is therefore a saddle point when $D > 0$, a sink otherwise.

At $\begin{pmatrix}D\\0\end{pmatrix}$, we let $u = D + h$, $v = 0 + k$ to get

$$\begin{pmatrix}h'\\k'\end{pmatrix} = \begin{pmatrix}-D & -D\\0 & D-1\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix} + O(2).$$

When $D > 1$ the point is a saddle point again, but when $0 < D < 1$ it is a sink.

At $\begin{pmatrix}1\\D-1\end{pmatrix}$, we let $u = 1 + h$, $v = (D-1) + k$ to get

$$\begin{pmatrix}h'\\k'\end{pmatrix} = \begin{pmatrix}-1 & -1\\D-1 & -1\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix} + O(2).$$

Its eigenvalues are $-1 \pm \sqrt{1-D}$. Hence we get an inward spiral if $D > 1$.

We get the following picture: when the habitat is still large, $D > 1$, both species coexist spiralling about the third fixed point. As the habitat is reduced beyond a critical value, $D < 1$, the predator species becomes extinct and the prey stabilize to the second fixed point. With a further reduction in habitat to $D < 0$, the solution converges to the origin, and even the prey species becomes extinct.

Other Cases

Degenerate Equilibrium Points It is possible for an equilibrium point to consist of two or more superimposed simple equilibrium points as described above, in which case it is called *non-hyperbolic* or *degenerate*. The picture below shows a spiral coinciding with a saddle point. The analysis of such an equilibrium point yields a number of zero eigenvalues (or zero real parts) which cannot specify the type of point: higher-order terms are needed.

Limit Sets In some differential equations, the solutions converge to a set other than an equilibrium point, such as a periodic orbit. The van der Pol equation, and the equation governing the heart, are two such examples.

These special points are often said to be *non-wandering*: the trajectory of such a point $\boldsymbol{y}_0$ returns infinitely often to its vicinity,

$$\forall \epsilon > 0, \forall x_0, \exists x > x_0, \|\boldsymbol{y}(x) - \boldsymbol{y_0}\| < \epsilon.$$

Chaos theory essentially started when it was discovered that in 3 dimensions or higher, there may be non-wandering points other than equilibrium or periodic points (forming *strange attractors* or *repellers*).

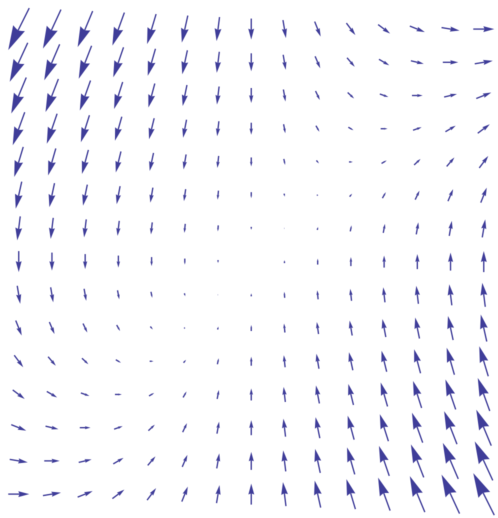


Figure 5.3: An example of a degenerate equilibrium point $\dot{u} = uv$, $\dot{v} = u - v$.

Exercises

Describe the fixed points of the following differential equations and make a sketch of the phase portrait (for the 2-D problems):

1.
$$\begin{aligned} u' &= uv - v^2 \\ v' &= 1 - u^2 \end{aligned}$$

2.
$$\begin{aligned} u' &= v \\ v' &= 2 - u^2 v \end{aligned}$$

3.
$$\begin{aligned} rlu' &= uv \\ v' &= -4u + \sin 2v \end{aligned}$$

4.
$$\begin{aligned} rl\dot{x} &= x + y + zy \\ \dot{y} &= -2y + z \\ \dot{z} &= 2z + \sin y \end{aligned}$$

5.
$$\ddot{x} + \dot{x} + x = 0$$

6.
$$\ddot{x} - \dot{x} + x = 0$$

7. For which values of a, b in $\ddot{x} + a\dot{x}^2 + bx = 0$ does $x(t)$ have non-trivial periodic solutions?

8. Duffing's equation $u'' = u - u^3 - u'$.

9. Show that in $\mathbb{R}^2$, an equilibrium point is stable if $\det A > 0$ and $\mathrm{tr}A < 0$, and unstable if $\det A < 0$ or $\det A > 0$ but $\mathrm{tr}A > 0$. Note that the stability of the equilibrium point cannot be decided if $\det A = 0$.

5.1.2 Gradient Flows

A gradient flow is a set of equations of the type $\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x})$ where V is a differentiable function (the sign is negative by convention). The dynamics of these equations are special:

The equilibrium points are the maxima/minima/critical points of V. In general, $V(\boldsymbol{x}(t))$ is decreasing (because $\frac{\mathrm{d}}{\mathrm{d}t}V(\boldsymbol{x}(t)) = \nabla V \cdot \dot{\boldsymbol{x}} = -\|\nabla V\|^2$) so the trajectories $\boldsymbol{x}(t)$ move away from the maxima and towards the minima, and there can be no periodic solutions.

5.1.3 Hamiltonian Flows

$u' = \frac{\partial H}{\partial v}, v' = -\frac{\partial H}{\partial u}$. The function $H(u, v)$ is called the *Hamiltonian* of the system.

Conservation of *Energy*: The Hamiltonian function is preserved along a trajectory $H(u(x), v(x))$ is constant, independent of x. That is, the trajectories belong to the level curves of H.

PROOF.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}H(u(x), v(x)) &= \frac{\partial H}{\partial u}u'(x) + \frac{\partial H}{\partial v}v'(x) \\ &= v'(x)u'(x) - u'(x)v'(x) = 0 \end{aligned}$$

□

Example, $u' = av$, $v' = -bu$ (a, b constant), is a Hamiltonian flow with $H := \frac{1}{2}(bu^2 + av^2)$. More generally, the system $u' = f(v)$, $v' = g(u)$ is Hamiltonian with $H = F(v) - G(u)$, where F and G are the integrals of f and g. In particular, equations of the type $u'' = -V'(u)$ (so the Hamiltonian is $\frac{1}{2}p^2 + V(x)$.

The equilibrium points of a Hamiltonian flow are also the maxima/minima/critical points of $H(u, v)$, but this time the trajectories move "around" such points.

At an equilibrium point, (u_0, v_0), let $u = u_0 + h$, $v = v_0 + k$, then

$$\begin{aligned} h' &= \frac{\partial^2 H}{\partial u \partial v} h + \frac{\partial^2 H}{\partial v^2} k \\ k' &= -\frac{\partial^2 H}{\partial u^2} h - \frac{\partial^2 H}{\partial u \partial v} k \end{aligned}$$

This gives a matrix of the type $K := \begin{pmatrix} a & b \\ -c & -a \end{pmatrix}$, which has eigenvalues $\lambda = \pm\sqrt{-\det K}$. There are thus two possibilities, either a center or a saddle point (unless degenerate).

A remarkable property of Hamiltonian flows is that areas are preserved: consider an infinitesimal rectangle with one corner (u, v) and sides δu, δv. After x changes to $x + \delta x$, the corners move to new positions; (u, v) moves to

$$\begin{aligned} u(x + \delta x) &= u(x) + \frac{\partial H}{\partial v} \delta x \\ v(x + \delta x) &= v(x) - \frac{\partial H}{\partial u} \delta x \end{aligned}$$

$(u + \delta u, v)$ moves to

$$\begin{aligned} (u + \delta u)(x + \delta x) &= u(x) + \delta u(x) + \frac{\partial H}{\partial v} \delta x + \frac{\partial^2 H}{\partial u \partial v} \delta u \delta x \\ v(x + \delta x) &= v(x) - \frac{\partial H}{\partial u} \delta x - \frac{\partial^2 H}{\partial u^2} \delta u \delta x \end{aligned}$$

$(u, v + \delta v)$ moves to

$$\begin{aligned} u(x + \delta x) &= u(x) + \frac{\partial H}{\partial v} \delta x + \frac{\partial^2 H}{\partial v^2} \delta v \delta x \\ (v + \delta v)(x + \delta x) &= v(x) + \delta v(x) - \frac{\partial H}{\partial u} \delta x - \frac{\partial^2 H}{\partial u \partial v} \delta v \delta x \end{aligned}$$

So the sides of the infinitesimal rectangle become the vectors $\begin{pmatrix} 1 + \frac{\partial^2 H}{\partial u \partial v} \delta x \\ -\frac{\partial^2 H}{\partial u^2} \delta x \end{pmatrix} \delta u$ and $\begin{pmatrix} \frac{\partial^2 H}{\partial v^2} \delta x \\ 1 - \frac{\partial^2 H}{\partial u \partial v} \delta x \end{pmatrix} \delta v$. Its area, which initially was $\delta u \delta v$, becomes $|side_1 \times side_2| = \delta u \delta v |1 + \det K \delta x^2|$. To first order, there is no change in the area, i.e., $\frac{\mathrm{d}Area}{\mathrm{d}x} = 0$.

CHAPTER 6

Numerical Methods

Since most non-linear differential equations are impossible to solve analytically, it becomes imperative to have algorithms for such cases. We seek methods to get an approximate solution for $y(x)$ that satisfies $y'(x) = F(x, y)$, $y(0) = Y$, where F is continuous in x and Lipschitz in y. The methods that we will describe here work by dividing the range $[0, x]$ into small steps of size h, so that $x_n = nh$, and finding approximations $y_n \approx y(x_n)$ for the solution.

The algorithm must have two important properties:

1. The approximations y_n must *converge* to the real solution $y(x)$ when we take smaller step sizes i.e., as $h \to 0$;

2. The algorithm must be *numerically stable*, meaning that slight errors in round-off do not swamp the results.

In the following sections we will denote $y'_n := F(x_n, y_n)$. This is *not* in general equal to $y'(x_n) = F(x_n, y(x_n))$ because y_n may not be equal to the solution $y(x)$. In fact the difference $e_n := y(x_n) - y_n$ is the total error of the algorithm. Recall from the first chapter that when a solution is started at a slightly different point $y(0) = Y + \delta$, the difference tends to increase exponentially; this means that the errors e_n may, and often will, become magnified.

6.1 Euler's Method

The simplest iteration is the following:

$$y_{n+1} := y_n + hy'_n, \qquad y_0 := Y.$$

The idea is that $y(x_{n+1}) = y(x_n + h) \approx y(x_n) + hy'(x_n)$ and we use the available information to substitute y_n instead of $y(x_n)$ and y'_n instead of $y'(x_n)$.

Example

Let us try this method on $y' = y$, $y(0) = 1$, whose exact solution we know. The iteration becomes

$$y_{n+1} := y_n + hy_n, \qquad y_0 := 1.$$

Try it out with some value of h, say 0.1, and see whether the values are close to the actual solution $y(x) = e^x$. For this simple example we can actually get an exact formula for y_n by solving the recursive equation to get

$$y_n = (1+h)^n.$$

Since $x_n = nh$ we have $y_n = (1+h)^{x_n/h} \to e^{x_n}$ as $h \to 0$. So Euler's method does work for this simple problem.

6.1.1 Error Analysis

Let us see how the errors increase in size in general. Let $e_n := y(x_n) - y_n$ be the total error at the point x_n. Then, by Taylor's theorem (assuming y is twice differentiable),

$$\begin{aligned} y(x_{n+1}) &= y(x_n + h) = y(x_n) + hy'(x_n) + O(h^2), \\ y_{n+1} &= y_n + hy'_n. \end{aligned}$$

Subtracting gives,

$$e_{n+1} = e_n + h(y'(x_n) - y'_n) + O(h^2).$$

But from the Lipschitz property of $F(x, y)$,

$$|y'(x_n) - y'_n| = |F(x_n, y(x_n)) - F(x_n, y_n)| \leqslant k|y(x_n) - y_n| = k|e_n|.$$

So

$$|e_{n+1}| \leqslant |e_n| + hk|e_n| + ch^2 = \alpha|e_n| + ch^2$$

where c is the maximum of the residual $O(h^2)$ function, given approximately by $|y''(x)/2|$. Taking the worst case for the error at each stage, we get (for h sufficiently small)

$$\begin{aligned} |e_n| &\leqslant \alpha^n|e_0| + ch^2(1 + \alpha + \ldots + \alpha^{n-1} \\ &= (1+hk)^n|e_0| + ch^2[(1+kh)^n - 1]/hk \\ &= (1 + x_nk/n)^n|e_0| + \frac{ch}{k}[(1 + kx_n/n)^n - 1] \\ &\leqslant e^{kx_n}|e_0| + \frac{ch}{k}(e^{kx_n} - 1) \\ &\leqslant e^{kx_n}(|e_0| + \frac{ch}{k}) \end{aligned}$$

We can conclude various results here:

1. An error is introduced at each step, of order h^2 at most; and the total error added by the algorithm is proportional to h. Hence halving the step size reduces this algorithmic error e_n by half.

2. There is a second error term, called the *propagation error*, originating from $|e_0|$. This is the same error we had analyzed in Chapter 1, where we called $e_0 = \delta$. But it can also represent the computer number accuracy. Both errors increase exponentially as we get further away from the initial point. The worst error can be expected to be at the furthest point from 0.

3. The step size should be chosen so that $\frac{ch}{k}(e^{kx}-1) < \epsilon$, the target error; but it does not make sense to make it smaller in value than $|e_0|e^{kx}$.

4. Decreasing the step size h, increases the number of steps n to get to the same point x. There is therefore a trade-off between accuracy and speed. In fact we can get a relation between the total number of calculations and the target accuracy ϵ, as follows:

$$\epsilon \approx ch = cx/n$$

   implying that the number of calculations needed is inversely proportional to target accuracy. In practice there is a limit to how small we can take h to be due to round-off error.

5. Large values of k or c lead to larger errors. In particular, corners in the solution (large y'') and large powers of y in F (large k) must be treated with more care. Note further that usually we will not get the worst errors as we have supposed in the error analysis above, so that k and c should be average values not maxima.

6.2 Improving the Euler method – Taylor series

An obvious way to improve upon the Euler method, is to take further terms in the Taylor series:

$$y(x_n + h) = y(x_n) + hy'(x_n) + \frac{1}{2}h^2 y''(x_n) + O(h^3)$$

where we substitute $y' = F(x, y(x))$ and so $y'' = F_1(x, y) := \frac{d}{dx}F(x, y(x)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}F$. This gives the following algorithm

$$y_{n+1} := y_n + hy'_n + \frac{1}{2}h^2 y''_n, \qquad y_0 = Y$$

where $y'_n := F(x_n, y_n), y''_n := F_1(x_n, y_n)$.

For example, if we want to solve the equation $y' = 1 + xy^2$, $y(0) = 0$, then $y'' = y^2 + x(2y)y' = y^2 + 2xy(1 + xy^2) = y(y + 2x + 2x^2y^2)$. So we let $y_{n+1} = y_n + h(1 + x_n y_n^2) = \frac{1}{2}h^2 y_n(y_n + 2x_n + 2x_n^2 y_n^2)$, starting from $y_0 = 0$.

6.2.1 Error analysis

Taking the difference between the equations

$$\begin{aligned} y(x_n+h) &= y(x_n) + hy'(x_n) + \tfrac{1}{2}h^2y''(x_n) + O(h^3) \\ y_{n+1} &= y_n + hy'_n + \tfrac{1}{2}h^2y''_n \end{aligned}$$

we get

$$e_{n+1} = e_n + h(F(x_n, y(x_n)) - F(x_n, y_n)) + \frac{1}{2}h^2(F_1(x_n, y(x_n)) - F_1(x_n, y_n)) + O(h^3)$$

so that, assuming that both F and F_1 are Lipschitz, we get

$$\begin{aligned} |e_{n+1}| &\leqslant |e_n| + hk|y(x_n) - y_n| + \frac{1}{2}h^2k_1|y(x_n) - y_n| + O(h^3) \\ &\leqslant |e_n|(1 + hk + \frac{1}{2}h^2k_1) + ch^3 \\ &\leqslant \alpha^n|e_0| + ch^3(1 + \alpha + \ldots + \alpha^{n-1}) \\ &\leqslant \alpha^n|e_0| + ch^2(\alpha^n - 1)/(\alpha - 1) \\ &\approx e^{kx}(|e_0| + ch^2/k) \end{aligned}$$

The algorithmic error is guaranteed to be of order h^2. This is a huge improvement over the Euler method.

For example, let us suppose that we wish the maximum error ϵ, at $x = 10$, to be at most 0.01, and that $c = 1, k = 1$; then for the Euler method, we need $ch/k \leqslant \epsilon$, so $h \leqslant \epsilon k/c = 0.01$; thus we need $n = x/h = 10/0.01 = 1000$ steps. For the improved Euler-Taylor method we need $ch^2/k \leqslant \epsilon$; so $h \leqslant \sqrt{\epsilon k/c} = 0.1$, and we need $n = x/h = 100$ steps, an improvement of 10.

Note, in practice the improvement is not this large, because we have slightly more to calculate at each step. Moreover this method assumes that $F(x, y)$ is differentiable in x, which it need not be the case in general.

6.3 Improving the Euler method – Interpolation

Let us rewrite each step for the solution as

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + \int_{x_n}^{x_{n+1}} y'(s)\,\mathrm{d}s = y(x_n) + h\langle y'\rangle,$$

where $\langle y'\rangle$ is the average value of y' in the range $[x_n, x_{n+1}]$. Hence Euler's method is equivalent to making the approximation $\langle y'\rangle = y'(x_n)$ i.e., the average over the range is taken to be the value at the first endpoint. This rough estimate works as we have seen, but it is not difficult to improve upon it — in most cases, at least for small steps, the value of y' is either increasing or decreasing, so the value $y'(x_n)$ is an extreme one.

We should get better results if we find better ways of approximating the average value of y' over each step. There are two obvious improvements (i) take $\langle y' \rangle$ to be approximately the value of $y'(x_n + \frac{1}{2}h)$, or (ii) take $\langle y' \rangle$ to be approximately the average of the two slopes at the extreme ends $\langle y' \rangle \approx (y(x_n) + y(x_{n+1}))/2$. The problem is that we do not know the value of y at these other points! But we can use the Euler estimate to get approximate values for them, as in the following sections.

6.3.1 Midpoint Rule

Start with $y_0 = Y$, and at each step, let

$$\begin{aligned} y'_n &:= F(x_n, y_n), \\ u &:= y_n + \frac{1}{2} y'_n, \\ u' &:= F(x_{n+1/2}, u), \\ y_{n+1} &:= y_n + hu'. \end{aligned}$$

where $x_{n+1/2} = x_n + \frac{1}{2}h$.

Error Analysis: The algorithm has errors of order $O(h^2)$.
Proof. Consider

$$\begin{aligned} y(x_{n+1}) = y(x_n + h) &= y(x_n) + hy'(x_n) + \frac{1}{2}h^2 y''(x_n) + O(h^3) \\ y'(x_{n+1/2}) = y'(x_n + \frac{1}{2}h) &= y'(x_n) + \frac{1}{2}hy''(x_n) + O(h^2) \\ \text{so }, y(x_{n+1}) &= y(x_n) + hy'(x_{n+\frac{1}{2}}) + O(h^3) \end{aligned}$$

Now,

$$|y'_n - y'(x_n)| = |F(x_n, y_n) - F(x_n, y(x_n))| \leqslant k|y_n - y(x_n)| = ke_n.$$

$$\begin{aligned} |u - y(x_{n+1/2})| &= |y_n + \frac{1}{2}hy'_n - y(x_n) - \frac{1}{2}hy'(x_n) + O(h^2)| \\ &\leqslant |y_n - y(x_n)| + \frac{1}{2}h|y'_n - y'(x_n)| + O(h^2) \\ &\leqslant e_n + \frac{1}{2}hke_n + O(h^2). \end{aligned}$$

So,

$$\begin{aligned} |u' - y'(x_{n+1/2})| &= |F(x_{n+\frac{1}{2}}, u) - F(x_{n+1/2}, y(x_{n+1/2}))| \\ &\leqslant k|u - y(x_{n+1/2})| \\ &\leqslant k(1 + \frac{1}{2}hk)e_n + O(h^2) \end{aligned}$$

And finally,

$$\begin{aligned}|y_{n+1}-y(x_{n+1})| &= |y_n+hu'-y(x_n)-hy'(x_{n+1/2})+O(h^3)|\\ &\leqslant |y_n-y(x_n)|+h|u'-y'(x_{n+1/2})|+O(h^3)\\ e_{n+1} &\leqslant e_n+hk(1+\frac{1}{2}kh)e_n+ch^3\end{aligned}$$

Let $\alpha=1+hk+\frac{1}{2}h^2k^2$, so that by iterating we get

$$\begin{aligned}e_n &\leqslant \alpha^n e_0+ch^3(1+\alpha+\ldots+\alpha^{n-1})\\ &\leqslant \alpha^n e_0+ch^3\frac{\alpha^n-1}{\alpha-1}\\ &\leqslant \alpha^n(e_0+ch^2/k)\\ &\leqslant e^{kx}(e_0+ch^2/k).\end{aligned}$$

□

Trapezoidal Rule

At each step, let

$$\begin{aligned}y'_n &:= F(x_n,y_n),\\ u &:= y_n+hy'_n,\\ u' &:= F(x_{n+1},u),\\ y_{n+1} &:= y_n+h(y'_n+u')/2,\\ y_0 &:= Y.\end{aligned}$$

Error Analysis: the algorithm has errors of order $O(h^2)$.
Proof.

$$\begin{aligned}y(x_{n+1})=y(x_n+h) &= y(x_n)+hy'(x_n)+\frac{1}{2}h^2y''(x_n)+O(h^3)\\ y'(x_{n+1})=y'(x_n+h) &= y'(x_n)+hy''(x_n)+O(h^2)\\ \text{so } (y'(x_{n+1})+y'(x_n))/2 &= y(x_n)+\frac{1}{2}h(y'(x_{n+1})+y'(x_n))+O(h^2)\\ \therefore y(x_{n+1}) &= y(x_n)+\frac{1}{2}h(y'(x_n)+y'(x_{n+1}))+O(h^3)\end{aligned}$$

As before, we have

$$|y'_n-y'(x_n)|\leqslant ke_n$$

$$\begin{aligned}|u-y(x_{n+1})| &= |y_n+hy'_n-y(x_n)-hy'(x_n)+O(h^2)|\\ &\leqslant |y_n-y(x_n)|+h|y'_n+y'(x_n)|+O(h^2)\\ &\leqslant e_n+hke_n+O(h^2)\end{aligned}$$

So,

$$\begin{aligned}|u' - y'(x_{n+1})| &= |F(x_{n+1}, u) - F(x_{n+1}, y(x_{n+1}))| \\ &\leqslant k|u - y(x_{n+1})| \\ &\leqslant k(1 + hk)e_n + O(h^2)\end{aligned}$$

So,

$$\begin{aligned}|y_{n+1} - y(x_{n+1})| &= |y_n + h(y_n' + u')/2 - y(x_n) - h(y'(x_{n+1}) + y'(x_n))/2 + O(h^3)| \\ &\leqslant |y_n - y(x_n)| + \frac{1}{2}hke_n + \frac{1}{2}hk(1 + hk)e_n + O(h^3)\end{aligned}$$

Hence we get the same error at each step of $e_{n+1} \leqslant (1 + hk + \frac{1}{2}h^2k^2)e_n + ch^3$ as in the midpoint algorithm. The total error is therefore at most $e_n \leqslant e^{kx}(e_0 + ch^2/k)$.

□

Note that now the error introduced in each stage is proportional to h^3, and hence the total error is proportional to h^2. This is called *quadratic* convergence which is much faster than the linear convergence of the simple Euler method. The total number of calculations needed is now inversely proportional to the *square root* of the accuracy $N = c/\sqrt{\epsilon}$.

In practice, the method can be improved even further by inserting more averaging steps as follows:

$$\begin{aligned}u &= y_n + hy_n' \\ v &= y_n + \frac{1}{2}h(y_n' + u') \\ y_{n+1} &= y_n + \frac{1}{2}h(y_n' + v').\end{aligned}$$

6.3.2 Error Stability

It is not enough to have an algorithm that converges to the required solution, using exact calculations. In practice every computer (including working by hand) introduces round-off errors at some stage. The algorithm must be immune to these errors i.e., these errors ought not to increase to a value larger than the inherent errors of the algorithm. Here is an example of a method that appears better than the simple Euler method but is not numerically stable.

Let

$$y_{n+1} := y_{n-1} + 2hy_n', \qquad y_0 := Y,$$

and use some method (Euler's or the trapezoid rule) to estimate y_1.

It is not too difficult (exercise) to show that this method has quadratic convergence with

$$|e_n| \leqslant A\lambda^n + B\mu^n + \frac{h^2M}{2k}$$

where $\lambda, \mu = hk \pm \sqrt{1+h^2k^2} \approx \pm 1 + hk$. If we take $|e_0| = 0$ and $|e_1| = O(h^2)$ we would get $A = B = O(h^2)$, so that the total error $|e_n| \leqslant ch^2$.

However let us apply this algorithm to an example $y' = -y, y(0) = 1$ (for which we know the solution $y(x) = e^{-x}$.) The algorithm becomes

$$y_{n+1} = y_{n-1} - 2hy_n, \qquad y_0 = 1.$$

This is a recurrence equation that can be solved (exercise) to get

$$\begin{aligned} y_n &= A\lambda^n + B\mu^n, \\ &= A(1-h+\ldots)^{x/h} + B(-1)^n(1+h+\ldots)^{x/h} \\ &\approx Ae^{-x} + (-1)^n Be^x \end{aligned}$$

where $\lambda, \mu = -h \pm \sqrt{1+h^2} \approx \pm 1 - h + O(h^2)$. Even if we take $|e_0| = 0 = |e_1|$, that is $y_0 = 1$ and $y_1 = e^{-h}$, we get (exercise) that $A = 1 + O(h^2)$, $B = O(h^2)$. Hence B, although small, is non-zero and the term $B\mu^n$ increases exponentially with each step.

6.4 Runge-Kutta Method

One can improve further on the trapezoid rule by using Simpson's rule,

$$\frac{1}{h}\int_{a_1}^{a_2} f \approx \frac{1}{6}(f(a_1) + 4f(a_{1.5}) + f(a_2)),$$

which is exact on quadratic functions. Adapting it to our case we get,

$$\frac{1}{h}\int_{x_n}^{x_{n+1}} y' \approx \frac{1}{6}(y'(x_n) + 4y'(x_n + h/2) + y'(x_n + h)).$$

This is a weighted average of the function y'. However we do not know this function yet, so we must devise some method to estimate the terms in the bracket. The Runge-Kutta method does this in the following way, for each step:

$$\begin{aligned} u_1 &:= y_n, \\ u_2 &:= y_n + \frac{1}{2}hu_1', \\ u_3 &:= y_n + \frac{1}{2}hu_2', \\ u_4 &:= y_n + h\frac{u_2' + u_3'}{3}, \end{aligned}$$

and then let

$$y_{n+1} := y_n + h(u'_1 + 2u'_2 + 2u'_3 + u'_4)/6.$$

Note that in fact, taking $u_4 := y_n + hu'_3$ is just as good.

It can be shown that the error for this method is of order 4 ie. $|e_n| \leqslant ch^4$, making the Runge-Kutta one of the most popular algorithms for solving differential equations numerically.

Of course, there are yet other formulas of higher order (cubic) that are more accurate than Simpson's. However these need not necessarily be more accurate when applied to differential equations. Each step may be more accurate but more calculations would be needed, introducing more round-off errors. For example a fourth-order method with step-size h need not be more accurate than a second-order method (eg Runge-Kutta) with step-size of $h/2$.

6.5 Adams-Bashfort Method

The formula

$$\frac{1}{h}\int_{x_n}^{x_{n+1}} y' = \frac{1}{2}(-y'(x_{n-1}) + 3y'(x_n))$$

is another approximation that is accurate to first order. This gives rise to the following method of calculating y_{n+1} from y_n and y_{n-1}.

$$\begin{aligned} u &:= y_{n-1} + h\frac{-y'_{n-1} + 3y'_n}{2}, \\ y_{n+1} &:= y_n + \frac{1}{2}h\frac{y'_n + u'}{2}. \end{aligned}$$

The steps involve a *predictor* u of $y(x_{n+1})$ using the above approximation, and a *corrector* step using the trapezoid rule and the predicted value u. To start the iteration we need $y_0 = Y$ and y_1 estimated by using some accurate method, say the trapezoid rule iterated several times.

These *predictor-corrector* methods, as they are called, can have different methods for the predictor/corrector stages. In fact the simplest is to use the trapezoid method for both, but the Adams-Bashfort is more accurate (of order $O(h^4)$). Such methods are in general faster than the interpolation methods used in the previous sections.

Bulirsch-Stoer Method

In fact the unstable midpoint method described previously can be salvaged in this manner. At each step calculate

$$u := y_{n-1} + 2hy'_n,$$

$$v := y_n + hu',$$

$$y_{n+1} := u + v/2.$$

Because of the changing sign of B, the average of u and v almost cancels it out, giving a stable algorithm with error of order $O(h^2)$.

6.6 Improvements

All of the above methods can be improved further in accuracy. The Achilles' heel, so to speak, of the methods is to choose a good value of h that is good enough but that does not make the algorithm too slow.

Adaptive Step Size

The best solution is to find some method how to estimate the step size, making it smaller when y is changing rapidly, and larger when a small value is not needed. We want the total error for the final point to be at most ϵ. To achieve this we need to make sure that each step introduces an error smaller than ϵ/N i.e.,

$$|e_{n+1} - e_n| \leqslant h\epsilon/x,$$

so that

$$e_N \leqslant e_0 + Nh\epsilon/x = e_0 + \epsilon.$$

Suppose we are working with an order $O(h^4)$ method (say Runge-Kutta), which has an error *at each step* of order $O(h^5)$. This means that $|e_{n+1} - e_n| \approx c_n h^5$, except that we would not know c_n precisely. If we are able to estimate c_n then we can estimate h. To do so we can adopt the following method:

1. Evaluate $y(x_n + h)$ using the algorithm with step size of h. This gives an error from the true value of about $c_n h^5$.

2. Evaluate $y(x_n + h)$ again using *two* steps of size $h/2$. This gives an error from the true value of about $2c_n(h/2)^5 = c_n h^5/16$, which is much smaller than the error in 1.

3. Hence we can say that the difference between the two estimates, call it δ, is approximately equal to $e_{n+1} - e_n$ and is equal to $c_n h^5$. Hence $c_n \approx \delta/h^5$.

4. The ideal step size would therefore be h_{new} for which $c_n h_{new}^5 \approx h_{new}\epsilon/x$. Now that we have an estimate for c_n, we get $h_{new}^4 \approx \epsilon h^5/(\delta x)$. Hence we modify the step size to

$$h_{new} = h\sqrt[4]{\epsilon/\delta x}$$

   Note that the exact power of h ought to be $h^{5/4}$ but the proposed formula is even better.

Half Step Trick

Suppose we are working with an algorithm which is accurate to order $O(h^2)$. Let us get approximate solutions for $y(x)$, once using a step size of h and again using a step size of $h/2$. Call the two approximations u_n and v_n, so that $u_n \approx y(x) \approx v_{2n}$. Since the error is of order $O(h^2)$ we get

$$\begin{aligned} u_n &= y(x) + ch^2 + O(h^3) \\ v_{2n} &= y(x) + c(h/2)^2 + O(h^3). \end{aligned}$$

We can get rid of the ch^2 terms by taking

$$y_n = \frac{4v_{2n} - u_n}{3},$$

making the error better by one order i.e., $y_n = y(x) + O(h^3)$.

This trick can be applied to any other algorithm already described. In practice it is always used when adaptive steps are taken because it involves a very few extra calculations.

Extrapolation with $h \to 0$

For very accurate results, one can evaluate the approximate values of y_n with smaller values of h. As $h \to 0$, the worked-out values become closer to the true value $y(x)$. For each x one can plot these approximate values against h and extrapolate by finding a statistical best-fit to $h = 0$.

Exercises 6.1

1. Apply the Euler method to get an approximate solution for the equation $y' = 1 + xy^2, y(0) = 0$ for the region $x \in [0, 1]$, using a step size of (i) $h = 0.1$, (ii) $h = 0.05$, (iii) $h = 0.1$ with $y(0) = 0.05$.

2. Repeat for the equation $y' = 12y, y(0) = 1$. Compare with the exact solution.

3. Repeat the first exercise using the Trapezoid rule, and the Runge-Kutta method, with a step size of 0.1.

4. In practice, algorithms are run by computers, which make round-off errors with each calculation. Suppose that at each step of the Euler method (or any other method), an additional error of δ is introduced so that we have
$$e_{n+1} \leqslant (1 + hk)e_n + ch^2 + \delta.$$
Show that the final error $e_n \leqslant e^{kx}(e_0 + ch/k + \delta/hk)$, and deduce that making h too small actually increases this error.

5. In the section on Error Stability, show the two unproved assertions about $|e_n|$ and y_n. (Hint: use Taylor's theorem for $y(x_n \pm h)$.)

6. In the Euler method, we took $y_{n+1} = y_n + hy'_n$. Suppose now that we can easily evaluate $\partial F/\partial y$. Suppose that we are at the nth step.

   (a) Let
   $$u = y_n, \quad u' = F(x_{n+1}, y_n), \quad k = \frac{\partial F}{\partial y}(x_{n+1}, y_n).$$
   Show that y'_{n+1} is approximately equal to $u' + k(y_{n+1} - y_n)$.

   (b) By taking $y_{n+1} = y_n + hy'_{n+1}$ and the above approximation, show that $y_{n+1} = y_n + h(1 - hk)^{-1}u'$.

   (c) Show further that the total error in this case is equal to $e_n = (1 - hK)^{-n}e_0 + \ldots$, where K is the largest value of k, and deduce that the propagation error remains small even when the step size is large.

7. Find general conditions on A, B, C which make
   $$\frac{1}{h}\int_{x_n}^{x_{n+1}} f \approx Af(x_n) + Bf(x_{n+\frac{1}{2}}) + Cf(x_{n+1})$$
   exact on quadratic expressions i.e., on $f(x) = ax^2 + bx + c$. Solve the resulting simultaneous equations to get Simpson's coefficients. (Note: you need only consider the case $x_n = 0$.)

8. Show that if the formula
   $$\frac{1}{h}\int_{x_n}^{x_{n+1}} f \approx Af(x_{n-1}) + Bf(x_n)$$
   is correct on linear expressions i.e., on $f(x) = ax + b$, then we get the Adams-Bashfort coefficients.

9. Devise a predictor-corrector method using the Trapezoid rule for both steps.

10. Apply the adaptive step analysis to an algorithm that is accurate to order $O(h^2)$. Show that the step size should be taken to be $h_{new} = constant \times h\sqrt{\epsilon/\delta x}$.

11. Apply the half-step trick to an algorithm of order $O(h^4)$.