Geometry

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1 Algebraic Curves and Surfaces

Our exposure to curves up to now has been with those that result from polynomials in variables x, y and z.

1.1 Straight lines and Planes

The simplest polynomial equation is $y = mx + c$ which is that of a straight line; however it misses out the vertical lines $x = c$. We can include all cases by writing it in the form

$$ax + by + c = 0.$$

The corresponding equation in three variables is

$$ax + by + cz + d = 0,$$

which gives a plane in three dimensions. You may be more familiar with it in the form of $\mathbf{a} \cdot \mathbf{x} = d$ which is an equivalent formulation.

What do you think will be the simplest equation in four variables? and what will it correspond to?

1.2 Conics

This chapter is going to deal with those curves and surfaces that come from quadratic equations. Let's start with quadratic equations in two variables; the most general equation of this sort is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Curves that result from such equations will be called **conics.** You may already be familiar with certain special cases such as the circle with equation $x^2 + y^2 = r^2$ or the parabola $y = ax^2$, and you may even have encountered the ellipse and the hyperbola. Note that you can also say that the equation includes the straight line if we take $a = b = c = 0$. These are all examples of conics; the obvious question now is: are these *all* the conics, or are there others?

To answer this question we will use ideas from vector spaces and we will be using vector notation, denoting $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. We are going to show that by rotating and translating the vectors, every conic is generically (i.e. excluding exceptions) either an ellipse (which includes the case of the circle) or a hyperbola, with the parabola as a limiting case; the exceptional cases are those equations that correspond to a straight line, two parallel lines, two intersecting lines or a single point.

Let's start by rewriting the quadratic equation in vector form as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f = 0,$$

or in short as

$$\boldsymbol{x}^\top A\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x} + f = 0,$$

where $A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ and $\boldsymbol{b}^\top = \begin{pmatrix} d & e \end{pmatrix}$. Note that the matrix A is symmetric, and from the course on vector spaces we know that every symmetric matrix can be diagonalized,

$$D = P^\top AP,$$

where D is a diagonal matrix, and P is an orthogonal matrix. Recall that we do this by finding first the eigenvalues of A with their corresponding eigenvectors and constructing D from the eigenvalues and P from the eigenvectors in the same order. Recall also that P is an orthogonal matrix so that it has the property $P^{-1} = P^\top$ and that geometrically it has the effect of a rotation.

This would be a good thing to do for a general quadratic equation as it gets rid of the mixed term in xy. Let's see how it would work: let $\boldsymbol{X} = P^\top \boldsymbol{x}$, so that $\boldsymbol{x} = P\boldsymbol{X}$; note that the vectors $\boldsymbol{X}$ are the vectors $\boldsymbol{x}$ rotated by the matrix $P^\top$; then

$$\boldsymbol{x}^\top A\boldsymbol{x} = (P\boldsymbol{X})^\top A(P\boldsymbol{X}) = \boldsymbol{X}^\top P^\top AP\boldsymbol{X} = \boldsymbol{X}^\top D\boldsymbol{X} = \lambda X^2 + \mu Y^2.$$

Let's see this part in action.

1.2.1 Example 1

Given the equation

$$3x^2 - 2xy + 3y^2 = 1,$$

let us get rid of the mixed term. Written in vector notation, we get

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

Now find the eigenvalues of the matrix, by first writing out the characteristic equation:

$$0 = \det\begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} = (\lambda-3)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2).$$

The eigenvalues are the roots 2, 4.

Next find the corresponding unit eigenvectors, first for 2:

$$\begin{pmatrix} 3-2 & -1 \\ -1 & 3-2 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0}.$$

Use gaussian elimination to end up with the solution

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

...then for the second eigenvalue 4:

$$\begin{pmatrix} 3-4 & -1 \\ -1 & 3-4 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We build the rotation matrix P from these unit eigenvectors. A quick check assures us that the two eigenvectors are orthogonal.

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Note that this is a rotation by 45°. Apply the rotation to change the variables,

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} X \\ Y \end{pmatrix},$$

so that the original conic equation becomes

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} - 1 = 0,$$

which works out to

$$2X^2 + 4Y^2 = 1.$$

This is the equation of an ellipse in the X, Y co-ordinates. Hence the original conic equation, in the x, y co-ordinates, was that of an ellipse rotated by 45°.

The procedure just outlined always gets rid of the mixed quadratic term in xy. The next step is to get rid of the X and Y terms using the X^2 and Y^2 terms, by completing the square:

$$\lambda X^2 + aX = \lambda(X+b)^2 - c = \lambda\tilde{X}^2 + c,$$

where $b = a/(2\lambda)$ and $c = \lambda b^2$. The second change of variable $\tilde{X} = X + b$ represents a translation in the X direction. This procedure can be repeated for the Y^2 and Y terms.

1.2.2 Example 2

Suppose we're given the conic equation

$$x^2 + y^2 + 6xy + 10\sqrt{2}x + 6\sqrt{2}y + 6 = 0.$$

Then the first procedure of rotating the axes results in the equation (work it out!)

$$2X^2 - Y^2 + 8X + 2Y + 3 = 0.$$

Now let us continue by translating the axes, using completion of the square:

$$2X^2 + 8X = 2(X^2 + 4X) = 2(X + 2)^2 - 8,$$

$$-Y^2 + 2Y = -(Y^2 - 2Y) = -(Y - 1)^2 + 1.$$

Hence if we let

$$\tilde{X} = X + 2, \quad \tilde{Y} = Y - 1,$$

we get

$$2\tilde{X}^2 - 8 - \tilde{Y}^2 + 1 + 3 = 0,$$

or

$$\tilde{X}^2/2 - \tilde{Y}^2/4 = 1,$$

which is a hyperbola in the $\tilde{X}$, $\tilde{Y}$ co-ordinates. Hence the original conic equation is that of a rotated and translated hyperbola.

1.2.3 Note

If you go back carefully to the completion of the square procedure, we divided by λ at one stage to get b. Of course we can only do this when $\lambda \neq 0$. This means that we can perform the translation say of the X co-ordinate as long as there is the term in X^2. If it happens that the very first step results in no X^2 term, then the second step cannot be performed to get rid of the X term, and this will remain in the equation.

However we can use the X term in this case to get rid of the constant at the end. It is imperative that this is done *after* the translations of the other axes, so that no new constants are added.

For example, suppose that applying the first step to a conic gives the equation

$$Y^2 + X + Y + 1 = 0.$$

Notice that there is no X^2 term, so we cannot get rid of the X term. We can only perform completion of the square on the Y variable as

$$Y^2 + Y = (Y + \frac{1}{2})^2 - \frac{1}{4}, \quad \tilde{Y} = Y + \frac{1}{2},$$

to get

$$\tilde{Y}^2 - \frac{1}{4} + X + 1 = 0,$$

simplified to

$$\tilde{Y}^2 + X + \frac{3}{4} = 0.$$

At this stage we can translate the X axis using $\tilde{X} = X + 3/4$ to get the final equation

$$\tilde{Y}^2 = -\tilde{X},$$

which is the equation of a parabola.

1.2.4 Complete Example

A conic is given by the equation

$$6x^2 - 4xy + 9y^2 - 2x + 4y - 2 = 0.$$

Written in vector and matrix notation it becomes

$$(x\ y)\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 2 = 0.$$

Step 1. **Rotation** to diagonalize the quadratic part.
The eigenvalues are found from the characteristic equation

$$0 = \det\begin{pmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{pmatrix} = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10),$$

to be 5 and 10.

The eigenvectors for each eigenvalue are
for $\lambda = 10$,

$$\begin{pmatrix} 6-10 & -2 \\ -2 & 9-10 \end{pmatrix}\boldsymbol{v} = \mathbf{0},$$

which simplifies by Gaussian reduction to

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\boldsymbol{v} = \mathbf{0},$$

giving the solution $\boldsymbol{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ which is normalized to $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$;

for $\lambda = 5$,

$$\begin{pmatrix} 6-5 & -2 \\ -2 & 0-5 \end{pmatrix}\boldsymbol{v} = \mathbf{0},$$

which becomes

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \boldsymbol{v} = \mathbf{0},$$

that has the normalized solution $\boldsymbol{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Notice that the two eigenvectors are perpendicular to each other, as should always be the case for a symmetric matrix.

Hence the rotation matrix ought to be

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Now let $\boldsymbol{x} = P\boldsymbol{X}$, and substitute into the conic equation,

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} -2 & 4 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} - 2 = 0,$$

which evaluates to

$$5X^2 + 10Y^2 - \frac{10}{\sqrt{5}} Y - 2 = 0.$$

Step 2. **Translation** to remove the linear terms.

In this case we only need to translate the Y axis using

$$10Y^2 - \frac{10}{\sqrt{5}} Y = 10(Y^2 - \frac{1}{\sqrt{5}} Y) = 10(Y - \frac{1}{2\sqrt{5}})^2 - \frac{1}{2}.$$

So we let

$$\tilde{X} = X,$$

$$\tilde{Y} = Y - \frac{1}{2\sqrt{5}},$$

and the conic equation becomes

$$5\tilde{X}^2 + 10\tilde{Y}^2 - 5/2 = 0,$$

or

$$2\tilde{X}^2 + 4\tilde{Y}^2 = 1,$$

which is an ellipse.

1.2.5 Classification of Conics

What are all the possible conics that can arise from the conic equation?

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

When we perform the rotation of the axes, we get the equivalent equation

$$\lambda X^2 + \mu Y^2 + \alpha X + \beta Y + f = 0.$$

We then perform translation of the axes, using the X^2 and Y^2 terms to get rid of the X and Y terms respectively. But this is possible *only* when λ and μ are non-zero.

Case 1. $\lambda \neq 0$, $\mu \neq 0$
We get rid of the X and Y terms, ending with

$$\lambda \tilde{X}^2 + \mu \tilde{Y}^2 = \gamma.$$

Now if $\gamma \neq 0$ then we get separate types of conics, depending on the signs of λ and μ:

Ellipse $\tilde{X}^2/A^2 + \tilde{Y}^2/B^2 = 1$,

Hyperbola $\tilde{X}^2/A^2 - \tilde{Y}^2/B^2 = 1$,

nothing $\tilde{X}^2/A^2 + \tilde{Y}^2/B^2 = -1$.

If $\gamma = 0$ we get, depending on the signs of λ and μ,

Two Intersecting Lines $\tilde{X} = \pm A\tilde{Y}$,

One Point $\tilde{X} = 0 = \tilde{Y}$.

Case 2. $\lambda \neq 0$, $\mu = 0$
In this case we can only get rid of the X term; but we can use the Y term to get rid of the constant, so that the result is

Parabola $\tilde{X}^2 = A\tilde{Y}$.

However it may be the case that there is no Y term either, so that in this case we get either nothing (when $\tilde{X}^2 = -A^2$), or

Two Parallel Lines $\tilde{X} = \pm A$,

One Line $\tilde{X} = 0$.

Of course, if both λ and μ vanish, then we get a linear equation rather than a conic.

1.2.6 Exercises

1. Find what type of conic are the following equations:

   (a) $2x^2 - y^2 - 2y = 5$;
   (b) $3x^2 + 2xy + 3y^2 - 2\sqrt{2}x - 6\sqrt{2}y + 4 = 0$;
   (c) $2x^2 + 2xy + 2y^2 + 3\sqrt{2}x + 3\sqrt{2}y + 4 = 0$;
   (d) $x^2 + 10\sqrt{3}xy + 11y^2 - 8(2 + \sqrt{3})x + 8(1 - 2\sqrt{3})y = 0$;
   (e) $x^2 + y^2 + \sqrt{3}x + y + 1 = 0$;
   (f) $3x^2 + 2\sqrt{3}xy + y^2 + 8\sqrt{3}x + 8y = 0$;
   (g) $x^2 + 2xy + y^2 + 2\sqrt{2}x + 2\sqrt{2}y + 2 = 0$;
   (h) $x^2 + 2xy + y^2 + 5\sqrt{2}x + 3\sqrt{2}y + 6 = 0$.

   (Answers: hyperbola, ellipse, nothing, intersecting lines, single point, parallel lines, single line, parabola)

2. In the above exercise, find the area enclosed by the ellipse, given that the formula for the area is πab where a and b are the semi-major and semi-minor axes.

3. The classical (Greek) definition of a conic is the set (locus) of points in the plane whose distance from a fixed point (called the focus) is a constant multiple e (called the eccentricity) from a fixed straight line (called the directrix). Let the focus be the origin, and the directrix the line $x = d$; show that this definition gives a curve with polar equation $r = ed/(1 + e\cos\theta)$, or in cartesian co-ordinates,

   $$(1 - e^2)x^2 + 2e^2dx + y^2 - e^2d^2 = 0.$$

   Show that the cases $e > 1$, $e = 1$, $e < 1$ and $e = 0$ correspond to a hyperbola, a parabola, an ellipse and a circle respectively (Note: in this last case, you have to take $e \to 0$, $d \to \infty$ such that $ed = R$ constant).

1.3 Quadrics

A **quadric** is a surface described by a quadratic equation in three variables,

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0,$$

which can be written in more compact form as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + j = 0.$$

This is the same form as for conics except that now we are using three variables:

$$\boldsymbol{x}^\top A\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x} + c = 0,$$

where $\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Therefore we can apply the same method as for conics, namely the two steps:

1. **Rotation** by P to diagonalize the matrix A in the quadratic part; for $\boldsymbol{x} = P\boldsymbol{X}$,

$$\boldsymbol{X}^\top D\boldsymbol{X} + \boldsymbol{b}^\top P\boldsymbol{X} + c = 0,$$

2. **Translation** to eliminate the linear part, *if possible*.

1.3.1 Example

Find what type of quadric is given by the equation

$$xy + yz + zx - x + y - 2 = 0.$$

First write it in matrix form, where here we have multiplied the whole equation by 2 to simplify the subsequent working:

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 4 = 0.$$

1. *Rotation*

Eigenvalues: characteristic equation

$$0 = \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 0 & 1-\lambda & -\lambda-1 \end{pmatrix}$$

$$= -\lambda^3 + 3\lambda + 2 = -(\lambda+1)^2(\lambda-2),$$

so that the eigenvalues are the roots $-1, -1, 2$.

Eigenvectors:

for 2,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0}$$

so that the associated eigenvector is $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$;

for -1,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0}$$

Effectively there is only one equation $x+y+z=0$, and we're looking for two perpendicular unit vectors satisfying it; one is easy to get, say, $\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$; the other can be obtained by taking the cross-product of the previous eigenvector with this one

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Hence the required rotation matrix is

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix}.$$

Substituting $\boldsymbol{x} = P\boldsymbol{X}$ we get in general

$$\boldsymbol{X}^\top D\boldsymbol{X} + \boldsymbol{b}^\top P\boldsymbol{X} + c = 0,$$

and in this case,

$$2X^2 - Y^2 - Z^2 + \begin{pmatrix} -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - 4 = 0,$$

$$2X^2 - Y^2 - Z^2 + 2\sqrt{2}Z - 4 = 0.$$

2. *Translation*
In this case we only need to do the translation for the Z variable,

$$-Z^2 + 2\sqrt{2}Z = -(Z^2 - 2\sqrt{2}Z) = -(Z - \sqrt{2})^2 + 2,$$

$$\tilde{X} = X, \tilde{Y} = Y, \tilde{Z} = Z - \sqrt{2},$$

to get

$$2\tilde{X}^2 - \tilde{Y}^2 - \tilde{Z}^2 = 2,$$

$$\tilde{X}^2 - \tilde{Y}^2/2 - \tilde{Z}^2/2 = 1,$$

when written in standard form.

What does this quadric surface look like? We get a good idea by finding its intersections with the $\tilde{X}$-$\tilde{Y}$, and the other planes. For example, in the $\tilde{Y}$-$\tilde{Z}$ plane, when $\tilde{X} = 0$, we get nothing; when $\tilde{Y} = 0$, we get a hyperbola; and similarly when $\tilde{Z} = 0$ we also get a hyperbola. In fact this surface is called a hyperboloid of two sheets (see its shape in the following pages).

1.3.2 Classification of Quadrics

This procedure can always be followed but we'll get various cases as we've already encountered with conics; in this case there are 15 different types, of which only 4 are generic cases.

After the rotation the quadric equation will look like

$$\boldsymbol{X}^\top D\boldsymbol{x} + \boldsymbol{B}^\top \boldsymbol{X} + c = 0,$$

where D is a diagonal matrix with eigenvalues λ, μ, ν.

Case 1. $\lambda, \mu, \nu \neq 0$

In this case we can use the X^2, Y^2, Z^2 variables to eliminate the linear terms, to get

$$\lambda\tilde{X}^2 + \mu\tilde{Y}^2 + \nu\tilde{Z}^2 + \tilde{c} = 0.$$

If $\tilde{c} \neq 0$ then we can take it to the other side of the equation and divide by it to get the following four possible standard forms:

$$\textbf{Ellipsoid} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 + \tilde{Z}^2/C^2 = 1;$$

$$\textbf{Hyperboloid of One Sheet} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 - \tilde{Z}^2/C^2 = 1;$$

$$\textbf{Hyperboloid of Two Sheets} \quad \tilde{X}^2/A^1 - \tilde{Y}^2/B^2 - \tilde{Z}^2/C^2 = 1;$$

$$\textbf{Nothing} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 + \tilde{Z}^2/C^2 = -1.$$

If $\tilde{c} = 0$ then again we get two additional possibilities:

$$\textbf{One Point} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 + \tilde{Z}^2/C^2 = 0;$$
$$\textbf{(Elliptical) Cone} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 = \tilde{Z}^2.$$

Case 2. $\lambda, \mu \neq 0, \nu = 0$

In this case, we can only absorb the X and Y linear terms; if the Z term is present then we can use it to absorb the constant, getting the two possible standard forms:

$$\textbf{Elliptic Parabola} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 = \tilde{Z};$$
$$\textbf{Hyperbolic Parabola} \quad \tilde{X}^2/A^2 - \tilde{Y}^2/B^2 = \tilde{Z}.$$

If however, there is no resulting Z term either, but $\tilde{c} \neq 0$, then we can divide by it to get three possibilities,

$$\textbf{Elliptic Cylinder} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 = 1;$$
$$\textbf{Hyperbolic Cylinder} \quad \tilde{X}^2/A^2 - \tilde{Y}^2/B^2 = 1;$$

and the third possibility $\tilde{X}^2/A^2 + \tilde{Y}^2/B^2 = -1$ leads to no solution (already included in Case 1).

When there is no Z term and no constant $\tilde{c} = 0$ either, then we get two possibilities:

$$\textbf{Line} \quad \tilde{X}^2/A^2 + \tilde{Y}^2/B^2 = 0;$$
$$\textbf{Intersecting Planes} \quad \tilde{X}^2 = A^2\tilde{Y}^2.$$

Case 3. $\lambda \neq 0, \mu = \nu = 0$

In this case, we cannot eliminate the Y and Z terms; however in choosing the eigenvectors for P, we can make the choice such that one of these, say Z, vanishes; if a Y term is present, then we can use to absorb the constant, so that we get the case

$$\textbf{Parabolic Cylinder} \quad \tilde{X}^2 = A\tilde{Y}.$$

If however both the Y and Z terms are already zero, then we get three possibilities,

$$\textbf{Two Parallel Planes} \quad \tilde{X}^2 = A^2;$$
$$\textbf{One Plane} \quad \tilde{X}^2 = 0;$$

and the remaining possibility $\tilde{X}^2 = -A^2$ leads to no solutions.

1.3.3 Exercises

1. Describe the following quadric surfaces

   (a) $2x^2 + y^2 + z^2/2 - 2y + 2x + z + 1 = 0$;

   (b) $xy + yz + zx = 0$;

2. Classify the quadric

$$\frac{3}{4}x^2 + y^2 + \frac{3}{4}z^2 + \frac{1}{2}xz - \sqrt{2}x - \sqrt{2}z = 0$$

   Determine whether the plane $x + z = \sqrt{8}$ meets the quadric.

3. Classify the quadric

$$\frac{3}{8}x^2 - \frac{1}{2}y^2 + \frac{1}{8}z^2 + \frac{\sqrt{15}}{4}xz - \sqrt{3}x - y + \sqrt{5}z - \frac{1}{2} = 0$$

   At how many points does the line $\sqrt{3}z + \sqrt{5}x = 0, y + 1 = 0$ meet the quadric?

4. Describe the intersection of the quadric $2x^2+y^2+z^2/2-2y+2x+z+1 = 0$ with the plane $x + 1/2 = 0$.

5. Describe the intersection of the cone $x^2 + y^2 - z^2 = 0$ with the planes (i) $x + y + z = 1$, (ii) $2y + z = 1$ and (iii) $x + z = 1$.

2 Curves

2.1 Introduction

Definition A **path** is a continuous map

$$\boldsymbol{r} : \begin{array}{c} I \to \mathbb{R}^n \\ t \mapsto \boldsymbol{r}(t) \end{array}$$

where $I \subseteq \mathbb{R}$ is an interval of real numbers.

A **curve** is a continuously differentiable path; a **smooth** curve is one which is infinitely differentiable.

When $n = 2$ the curve is called **planar**; when $n = 3$ it is called a "**space**" curve.

A curve is therefore specified by n functions, each specifying how the i-th coordinate varies with a real parameter t.

$$\boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ \vdots \end{pmatrix}$$

Note that the terminology 'path', 'curve', 'arc' mean different things in different books. Some (older) books refer to space curves as *solid* curves; they also refer to a curve by its *parametric* equations $x = x(t), y = y(t), z = z(t)$.

It is obvious that continuous complex functions $\boldsymbol{r} : I \to \mathbb{C}$ are planar paths by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

2.1.1 Examples

1. The equation $\boldsymbol{r}(t) = \boldsymbol{a}$ with parametric equations $x(t) = a, y(t) = b, z(t) = c$, gives the constant "curve", consisting of the single point $\boldsymbol{a}$.

2. A straight line is given by the equation $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{b}$.

3. A parabola is given by the equation $\boldsymbol{r}(t) = (t, at^2)$.

4. An ellipse is given by the curve $\boldsymbol{r}(\theta) = (a\cos\theta, b\sin\theta)$; the circle is the special case when $a = b = r_0$, which can then be written more compactly as $z(t) = r_0 e^{it}$ in the complex plane.

5. The helix is given by the curve $\boldsymbol{r}(\theta) = (a\cos\theta, a\sin\theta, b\theta)$.

Definition A **simple** path is one which is one-one. A **closed** path is one which is periodic i.e. $\boldsymbol{r}(t+T) = \boldsymbol{r}(t)\ \forall t$; it repeats with period T. A *simple closed* path is one which is closed and is one-one for $t \in [0, T)$.

In particular the circle is a simple closed curve.

2.1.2 Differentiation and Integration

The derivative of a vector function (called its *velocity*) is taken by differentiating each co-ordinate function:

$$\dot{\boldsymbol{r}}(t) = \lim_{h\to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h} = \lim_{h\to 0} \begin{pmatrix} \frac{x(t+h)-x(t)}{h} \\ \frac{y(t+h)-y(t)}{h} \\ \frac{z(t+h)-z(t)}{h} \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}.$$

The integral of a vector function is evaluated by integrating each co-ordinate function:

$$\int_{t_0}^{t_1} \boldsymbol{r}(t)\,dt = \begin{pmatrix} \int_{t_0}^{t_1} x(t)\,dt \\ \int_{t_0}^{t_1} y(t)\,dt \\ \int_{t_0}^{t_1} z(t)\,dt \end{pmatrix}$$

2.1.3 Parametrization

Two paths can trace out the same set of points. For example the three curves

$$\boldsymbol{r}_1(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \boldsymbol{r}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad \boldsymbol{r}_3(u) = \begin{pmatrix} \operatorname{sech} u \\ \tanh u \end{pmatrix},$$

all trace out part of the same circle of points but in different ways. They are considered as *different* paths; it is usual, however, to say that they are the *same* path with different parametrizations.

In general, given two parametrizations $\boldsymbol{r}_1(t) = \boldsymbol{r}_2(u)$ we can always write one parameter in terms of the other as $u = u(t)$, whenever they trace out each point on the curve once only.

For example, for $\boldsymbol{r}_1(\theta)$, $\boldsymbol{r}_2(t)$ given above, $t = \pi/2 - \theta$.

2.1.4 Exercises

1. Sketch the following curves (i) $\boldsymbol{r}(t) = (t\cos(t), t\sin(t))$; (ii) $\boldsymbol{r}(t) = (t^2, t^3)$; (iii) $\boldsymbol{r}(t) = (a\sec(t), b\tan(t))$.

2. Find a different parametrization for the straight line and for the circle. Is the curve $\boldsymbol{r}(t) = \begin{pmatrix} a\sin(t) \\ a\cos(t) \\ bt \end{pmatrix}$ a different parametrization of the helix given in the examples above?

2.2 Length of a Curve

Definition The **length** of a curve is

$$L[\boldsymbol{r}] := \int_{t_0}^{t_1} |\dot{\boldsymbol{r}}(t)| \, dt.$$

The idea behind this definition is that the curve can be split up into very small pieces, each of which is approximately straight, so that the total length will be a large sum of parts of type $|\boldsymbol{r}(t_{i+1}) - \boldsymbol{r}(t_i)| \approx |\dot{\boldsymbol{r}}(t_i)|\delta t$.

2.2.1 Example

The length of one turn of a helix is

$$L = \int_0^{2\pi} \left| \begin{pmatrix} -a\sin\theta \\ a\cos\theta \\ b \end{pmatrix} \right| d\theta = \int_0^{2\pi} \sqrt{a^2+b^2} \, d\theta = 2\pi\sqrt{a^2+b^2}.$$

Proposition 2.2.1

The length does not depend on the parametrization.

That is, two parametrizations $\boldsymbol{r}(t) = \boldsymbol{R}(u)$ along the same curve, $t_0 \leqslant t \leqslant t_1$ and $u_0 \leqslant u \leqslant u_1$, should give the same length between corresponding points. In fact, changing parameters $u = u(t)$

$$\begin{aligned} \textstyle\int_{u_0}^{u_1} |\boldsymbol{R}'(u)| \, du &= \textstyle\int_{t_0}^{t_1} |\boldsymbol{R}'(u(t))||\frac{du}{dt}| \, dt \\ &= \textstyle\int_{t_0}^{t_1} |\frac{d\boldsymbol{R}}{du} \cdot \frac{du}{dt}| \, dt \\ &= \textstyle\int_{t_0}^{t_1} |\dot{\boldsymbol{r}}(t)| \, dt. \end{aligned}$$

□

2.2.2 Arclength Parametrization

Define the arclength function from an initial point $\boldsymbol{r}(t_0)$ by

$$s(t) := \int_{t_0}^{t} |\dot{\boldsymbol{r}}(t)|\,dt.$$

We can use this arclength s itself as a standard parameter. Whatever parameter t we start off with to describe the curve, the arclength parameter s will be unique (from the same initial point). Note that

$$\frac{ds}{dt} = |\dot{\boldsymbol{r}}|.$$

In particular, when taking $t = s$,

$$\boxed{|\boldsymbol{r}'(s)| = 1.}$$

We often write ds instead of $|\dot{\boldsymbol{r}}|\,dt$ in integrals, and $d\boldsymbol{r}$ instead of $\dot{\boldsymbol{r}}\,dt$, so that

$$\int f(s)\,ds = \int f(s(t))|\dot{\boldsymbol{r}}(t)|\,dt,$$

$$\int f(s)\,d\boldsymbol{r} = \int f(s(t))\dot{\boldsymbol{r}}\,dt.$$

2.2.3 Example

For the helix, as worked above, $s(t) = \sqrt{a^2+b^2}\,t$ so that the unique arclength parametrization of the helix, starting from $(a,0,0)$ is

$$\boldsymbol{r}(s) = \begin{pmatrix} a\cos(s/\sqrt{a^2+b^2}) \\ a\sin(s/\sqrt{a^2+b^2}) \\ b(s/\sqrt{a^2+b^2}) \end{pmatrix}.$$

2.2.4 Exercise

1. Verify that the length of (i) a straight line segment from $\boldsymbol{a}$ to $\boldsymbol{b}$ is $|\boldsymbol{a}-\boldsymbol{b}|$; (ii) an arc of a circle is given by $r\theta$.

2. Find the arclength parametrization of the circle of radius r (starting from $(r,0,0)$). Verify that you get the same parametrization starting from $\boldsymbol{r}_3$ (Hint: you need $\int_0^u \operatorname{sech} u\,du = 2\tan^{-1}\tanh(u/2)$.)

3. * Let $\boldsymbol{r}_1 \sim \boldsymbol{r}_2$ be true when they trace out the same curve with possibly different parametrizations. Show that this is an equivalence relation.

2.3 Tangent, Normal and Binormal

Suppose we take the Taylor expansion of the co-ordinate functions $x(t), y(t), z(t)$ near to a point $t = t_0$; we get

$$x(t_0 + h) = x(t_0) + h\dot{x}(t_0) + \ldots,$$

and similarly for y and z, which can be grouped together into a vector equation

$$\boldsymbol{r}(t_0 + h) = \boldsymbol{r}(t_0) + h\dot{\boldsymbol{r}}(t_0) + \ldots.$$

These first two terms define a **tangent line** to the curve at the point $\boldsymbol{r}(t_0)$, with equation

$$\boldsymbol{r}_{\text{tangent}}(t) = \boldsymbol{r}(t_0) + (t - t_0)\dot{\boldsymbol{r}}(t_0),$$

assuming that $\dot{\boldsymbol{r}}(t_0) \neq \mathbf{0}$.

A point $\boldsymbol{r}(t_0)$ whose velocity $\dot{\boldsymbol{r}}(t_0) = 0$ is called a singular point; there is no tangent line defined at such a point.

The tangent line remains the same when we change parametrization, say $t = t(u)$:

$$\frac{d}{du}\boldsymbol{r}(t(u)) = \frac{d}{dt}\boldsymbol{r}(t)\frac{dt}{du}.$$

The direction of the velocity remains the same, but its length (the speed) is not.

The tangent vector is then defined as the unit vector in the direction of the tangent line; in particular if we take the arc-length parametrization, then the tangent vector is defined as follows

Definition The **tangent** vector at a point $\boldsymbol{r}(s)$ of a curve, in arc-length parametrization, is given by

$$\boldsymbol{t}(s) := \boldsymbol{r}'(s) = \frac{d\boldsymbol{r}}{ds}(s).$$

Note that $\boldsymbol{t}$ is a unit vector, and that given a curve in any parametrization $\boldsymbol{r}(t)$, then $\boldsymbol{t} = \dot{\boldsymbol{r}}/|\dot{\boldsymbol{r}}|$.

Example. Suppose that a curve is such that its tangent vector is constant $\boldsymbol{t} = \boldsymbol{a}$. Then we have that $\boldsymbol{r}'(s) = \boldsymbol{a}$, and integrating with respect to s gives

$$\boldsymbol{r}(s) = s\boldsymbol{a} + \boldsymbol{b},$$

which is the equation of a straight line.

If we repeat the Taylor series approximation using second-order terms we get the following:

$$x(s_0+h) = x(s_0) + hx'(s_0) + \frac{1}{2}h^2 x''(s_0) + \ldots,$$

and similarly for y and z; when grouped together in vector form,

$$\boldsymbol{r}(s_0+h) = \boldsymbol{r}(s_0) + h\boldsymbol{r}'(s_0) + \frac{1}{2}h^2\boldsymbol{r}''(s_0) + \ldots.$$

We have already defined the unit vector $\boldsymbol{t} = \boldsymbol{r}'$, so we need to define a unit vector in the direction of $\boldsymbol{r}'' = \boldsymbol{t}'$, called the normal of the curve.

Definition The **normal** vector at a point of a curve is defined as the unit vector in the direction of $\boldsymbol{t}'$. The **curvature** is defined as $\kappa = |\boldsymbol{t}'|$. The inverse of the curvature $1/\kappa$ is called the *radius of curvature*.

The normal is defined only when $\boldsymbol{t}' \neq \boldsymbol{0}$; otherwise when $\boldsymbol{t}' = \boldsymbol{0}$ we can only say that $\kappa = 0$ and the normal direction remains unspecified (e.g. the straight line).

When it is defined, the normal is perpendicular to the tangent vector. This is because $\boldsymbol{t}\cdot\boldsymbol{t} = 1$, so differentiating with respect to s gives $\boldsymbol{t}'\cdot\boldsymbol{t}+\boldsymbol{t}\cdot\boldsymbol{t}' = 0$ which implies $\kappa\boldsymbol{t}\cdot\boldsymbol{n} = 0$.

In this case, we get

$$\boldsymbol{r}(s_0+h) = \boldsymbol{r}(s_0) + h\boldsymbol{t} + \frac{\kappa h^2}{2}\boldsymbol{n} + \ldots$$

This means that if we take the origin to be $\boldsymbol{r}(s_0)$ and the axes to be along the directions $\boldsymbol{t}$ and $\boldsymbol{n}$, then the curve is given up to order 2 by the $\boldsymbol{r}(s_0+h) = \begin{pmatrix} h \\ \kappa h^2/2 \end{pmatrix}$ which is a parabola. That is, at any point on a curve, we can approximate the curve up to second order by a parabola, just as we can approximate up to first order by a tangent line.

2.3.1 Examples

(i) Find the curvature and normal vector at any point of the circle $\boldsymbol{r}(t) = \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}$.

First we find the arc-length parameter $s = at$; then we find the tangent and normal by differentiation with respect to s:

$$\boldsymbol{t} = \boldsymbol{r}' = \dot{\boldsymbol{r}}/\dot{s} = \frac{a}{a}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$

$$\kappa \boldsymbol{n} = \boldsymbol{t}' = \dot{\boldsymbol{t}}/\dot{s} = \frac{1}{a}\begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix},$$

hence $\kappa = 1/a$ and $\boldsymbol{n} = -\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

(ii) The log spiral has equation $\boldsymbol{r}(\theta) = e^{k\theta}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Find the length of one turn from $\boldsymbol{i}$, and the curvature, tangent and normal vectors.

First, the arc-length

$$\begin{aligned} \dot{s} = |\dot{\boldsymbol{r}}| &= \left|\begin{pmatrix} ke^{k\theta}\cos\theta - e^{k\theta}\sin\theta \\ ke^{k\theta}\sin\theta + e^{k\theta}\cos\theta \end{pmatrix}\right| \\ &= e^{k\theta}\left|\begin{pmatrix} k\cos\theta - \sin\theta \\ k\sin\theta + \cos\theta \end{pmatrix}\right| \\ &= e^{k\theta}\sqrt{k^2+1}. \end{aligned}$$

$$\therefore \quad s(\theta) = \int |\dot{\boldsymbol{r}}|\, d\theta = \sqrt{k^2+1}\int e^{k\theta}\, d\theta = \frac{\sqrt{k^2+1}}{k}e^{k\theta}.$$

In particular, the length of one turn of the log spiral is

$$s(2\pi) = \int_0^{2\pi} |\dot{\boldsymbol{r}}|\, d\theta = \frac{\sqrt{k^2+1}}{k}e^{2k\pi}.$$

Now for the tangent,

$$\boldsymbol{t} = \boldsymbol{r}' = \dot{\boldsymbol{r}}/\dot{s} = \frac{1}{\sqrt{k^2+1}}\begin{pmatrix} k\cos\theta - \sin\theta \\ k\sin\theta + \cos\theta \end{pmatrix}.$$

The normal,

$$\kappa\boldsymbol{n} = \boldsymbol{t}' = \dot{\boldsymbol{t}}/\dot{s} = \frac{e^{-k\theta}}{k^2+1}\begin{pmatrix} -k\sin\theta - \cos\theta \\ k\cos\theta - \sin\theta \end{pmatrix};$$

$$\kappa = \frac{e^{-k\theta}}{k^2+1}\left|\begin{pmatrix} -k\sin\theta - \cos\theta \\ k\cos\theta - \sin\theta \end{pmatrix}\right| = \frac{e^{-k\theta}}{\sqrt{k^2+1}} = \frac{1}{ks}.$$

$$\boldsymbol{n} = \boldsymbol{t}'/\kappa = \frac{1}{\sqrt{k^2+1}}\begin{pmatrix} -k\sin\theta - \cos\theta \\ k\cos\theta - \sin\theta \end{pmatrix}.$$

Definition The **binormal** vector at a point of a curve is defined as $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$.

This means that the vectors $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ form a set of three perpendicular vectors just like $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ except that instead of being fixed, they change their direction as they move along the curve.

2.4 Serret-Frenet Formulæ

Proposition 2.4.1

There is a scalar function $\tau(s)$ called the torsion at a point on the curve, such that

$$\boldsymbol{t}' = \kappa\boldsymbol{n},$$

$$\boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b},$$

$$\boldsymbol{b}' = -\tau\boldsymbol{n}.$$

Symbolically,

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

Proof. The first equation is really the definition of the curvature and normal vectors. For the second equation, differentiate the equation $\boldsymbol{n}\cdot\boldsymbol{n} = 1$ with respect to s to get $2\boldsymbol{n}'\cdot\boldsymbol{n} = 0$; similarly differentiate $\boldsymbol{n}\cdot\boldsymbol{t} = 0$ to get $\boldsymbol{n}'\cdot\boldsymbol{t} + \boldsymbol{n}\cdot\boldsymbol{t}' = 0$ which gives $\boldsymbol{n}'\cdot\boldsymbol{t} = -\kappa$, since $\boldsymbol{t}' = \kappa\boldsymbol{n}$. But $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ form a basis of vectors at any point on the curve; hence we get $\boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}$ where $\tau = \boldsymbol{n}'\cdot\boldsymbol{b}$.

For the last equation, differentiate $\boldsymbol{b}\cdot\boldsymbol{t} = 0$ to get $\boldsymbol{b}'\cdot\boldsymbol{t} + \boldsymbol{b}\cdot\kappa\boldsymbol{n} = 0$ which gives $\boldsymbol{b}'\cdot\boldsymbol{t} = 0$; similarly differentiating $\boldsymbol{b}\cdot\boldsymbol{n} = 0$ gives $\boldsymbol{b}'\cdot\boldsymbol{n} + \boldsymbol{b}\cdot(-\kappa\boldsymbol{t} + \tau\boldsymbol{b}) = 0$ and hence $\boldsymbol{b}'\cdot\boldsymbol{n} = -\tau$; finally differentiating $\boldsymbol{b}\cdot\boldsymbol{b} = 1$ gives $2\boldsymbol{b}'\cdot\boldsymbol{b} = 0$. The result is that $\boldsymbol{b}' = -\tau\boldsymbol{n}$.

$\square$

2.4.1 Example

Find the curvature and torsion, and the tangent, normal and binormal vectors at any point of the helix $\boldsymbol{r}(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix}$. We find the arc-length $s(t) = t\sqrt{a^2 + b^2}$, then differentiate:

$$\boldsymbol{t} = \boldsymbol{r}' = \dot{\boldsymbol{r}}/\dot{s} = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} -a\sin t \\ a\cos t \\ b \end{pmatrix},$$

$$\kappa\boldsymbol{n} = \boldsymbol{t}' = \dot{\boldsymbol{t}}/\dot{s} = \frac{1}{a^2+b^2}\begin{pmatrix}-a\cos t\\ -a\sin t\\ 0\end{pmatrix},$$

so that $\kappa = a/(a^2+b^2)$ (constant) and $\boldsymbol{n} = -\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix}$. Notice that the radius of curvature is larger than a, which is the radius of the helix when seen along its axis.

$$\boldsymbol{b} = \boldsymbol{t}\times\boldsymbol{n} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}-a\sin t\\ a\cos t\\ b\end{pmatrix} \times - \begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}-b\sin t\\ b\cos t\\ a\end{pmatrix}.$$

Hence

$$-\tau\boldsymbol{n} = \boldsymbol{b}' = \dot{\boldsymbol{b}}/\dot{s} = \frac{1}{a^2+b^2}\begin{pmatrix}-b\cos t\\ -b\sin t\\ 0\end{pmatrix},$$

so that $\tau = b/(a^2+b^2)$, constant.

Proposition 2.4.2

When defined,

$$\kappa(t) = \frac{|\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}|}{|\dot{\boldsymbol{r}}|^3}, \qquad \tau(t) = \frac{\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}\cdot\dddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}|^2}.$$

Proof. We know that

$$\dot{\boldsymbol{r}} = \dot{s}\boldsymbol{t}.$$

Differentiating with respect to t we get

$$\ddot{\boldsymbol{r}} = \ddot{s}\boldsymbol{t} + \dot{s}^2\boldsymbol{t}' = \ddot{s}\boldsymbol{t} + \dot{s}^2\kappa\boldsymbol{n}.$$

Again,

$$\dddot{\boldsymbol{r}} = A\boldsymbol{t} + B\boldsymbol{n} + \dot{s}^3\kappa\tau\boldsymbol{b}.$$

Hence,

$$\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}} = \dot{s}^3\kappa\boldsymbol{b},$$

and

$$\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}\cdot\dddot{\boldsymbol{r}} = \dot{s}^6\kappa^2\tau.$$

This gives the required formulæ for κ and τ.

$\square$

There is a theorem that states that there is a unique curve (up to rotations and translations) with specified $\kappa(s)$ and $\tau(s)$. That is, if we are given $\kappa(s)$ and $\tau(s)$, both continuous functions, then we ought to, in principle, be able to find only one curve with the curvature and torsion equal to $\kappa(s)$ and $\tau(s)$. In practice this is easy to show directly only in a few cases. The proof of this theorem relies on a theorem (*Picard's*) from differential equations that states that a system of equations, such as the Serret-Frenet equations, has a unique solution when its matrix is continuous (in this case when κ and τ are continuous).

2.4.2 Example

1. $\kappa = 0$. Then $\boldsymbol{t}' = \mathbf{0}$, and integrating gives $\boldsymbol{r}' = \boldsymbol{t} = \boldsymbol{a}$, integrating again $\boldsymbol{r} = s\boldsymbol{a} + \boldsymbol{b}$, which is the equation of a straight line.
2. When the torsion is zero, the curve is planar. $\tau = 0 \Rightarrow \boldsymbol{b}' = \mathbf{0}$, hence $\boldsymbol{b}$ is a constant vector and $(\boldsymbol{r}\cdot\boldsymbol{b})' = \boldsymbol{r}'\cdot\boldsymbol{b} = 0$, which implies that $\boldsymbol{r}\cdot\boldsymbol{b} = c$, which is the equation of a plane; i.e. the curve lies on a plane.
3. $\kappa = \kappa_0$ constant and $\tau = 0$. By the second example, we know that the curve is planar, and we get from $\boldsymbol{n}' = -\kappa\boldsymbol{t}$ that $\boldsymbol{r}' = \boldsymbol{t} = -\boldsymbol{n}'/\kappa_0$; integrating gives $\boldsymbol{r} = -\boldsymbol{n}/\kappa_0 + \boldsymbol{a}$ i.e. $\boldsymbol{r} - \boldsymbol{a} = -\boldsymbol{n}/\kappa_0$ or $|\boldsymbol{r} - \boldsymbol{a}| = 1/\kappa_0$ which means that the distance from a fixed point is constant i.e. the planar curve is part of a circle.
4. It is left as an exercise to show that when $\kappa = \kappa_0$ and $\tau = \tau_0$ both constants, then the curve must be a helix.

2.4.3 Exercises

1. Consider the curve given by the graph of a real-valued function: $\boldsymbol{r}(t) = (t, f(t))$. Show that the curvature is given by

   $$\kappa(t) = \frac{f''(t)}{(1 + f'(t)^2)^{\frac{3}{2}}}$$

   In particular, show that for the catenary for which $f(t) = a\cosh(t/a)$, the radius of curvature is given by

   $$\rho = a\cosh^2(t/a) = a + s^2/a$$

   where s is the arclength from $t = 0$.

2. Calculate $\int_0^{2\pi r_0} xy\,ds$ for the circle with equation $\boldsymbol{r}(\theta) = \begin{pmatrix} r_0\cos\theta \\ r_0\sin\theta \end{pmatrix}$.

3. The log spiral is the planar curve $\boldsymbol{r}(\theta) = e^{k\theta}(\cos\theta, \sin\theta)$. Show that the arclength between the point $(1, 0)$ and the origin is finite. Show also that the angle between the tangent vector and $\boldsymbol{r}$ is constant and equal to $\cot^{-1} k$.

4. The tractrix is the planar curve $\boldsymbol{r}(t) = (\log(\tan\frac{1}{2}t) + \cos t, \sin t)$ for $0 < t < \pi$. Show that $\boldsymbol{t} - \boldsymbol{r}$ lies on the x-axis. Find the curvature of the curve in terms of t and deduce that it satisfies the differential equation
$$\frac{d\kappa}{ds} = \kappa(1 + \kappa^2)$$

5. Show that
$$\boldsymbol{t} = \frac{\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|}, \qquad \boldsymbol{n} = \frac{\ddot{\boldsymbol{r}} - \ddot{\boldsymbol{r}}\cdot\boldsymbol{t}\,\boldsymbol{t}}{|\ddot{\boldsymbol{r}} - \ddot{\boldsymbol{r}}\cdot\boldsymbol{t}\,\boldsymbol{t}|}, \qquad \boldsymbol{b} = \frac{\boldsymbol{t}\times\ddot{\boldsymbol{r}}}{|\boldsymbol{t}\times\ddot{\boldsymbol{r}}|}.$$

6. Show that $\boldsymbol{r}''' = -\kappa^2\boldsymbol{t} + \kappa'\boldsymbol{n} + \kappa\tau\boldsymbol{b}$. Hence show that if we take the Taylor series at a point of a curve, up to third order in h, and place the axes along the vectors $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ we get the approximating curve
$$\boldsymbol{r}(s_0 + h) = \begin{pmatrix} h - \kappa^2 h^3/3! \\ \kappa h^2/2 + \kappa' h^3/3! \\ \kappa\tau h^3/3! \end{pmatrix} \approx \begin{pmatrix} h \\ \kappa h^2/2 \\ \kappa\tau h^3/3! \end{pmatrix}.$$

7. Now suppose that $\boldsymbol{t} = 0$ (ie at a singular point); show that if we place the axes along the vectors $\boldsymbol{n}, \boldsymbol{b}$ we get the curve
$$\boldsymbol{r}(s_0) \approx \begin{pmatrix} \kappa h^2/2 \\ \kappa\tau h^3/3! \end{pmatrix}.$$

8. Show that if the curvature $\kappa = \kappa_0$ and the torsion $\tau = \tau_0$ of a curve are constant, then the curve must be part of a helix.

   (Hint: find $\boldsymbol{n}''$ and integrate twice.)

9. Let $\boldsymbol{r}(t)$ trace out a conic curve given by equation
$$\boldsymbol{r}^\top A\boldsymbol{r} + \boldsymbol{b}^\top\boldsymbol{r} + c = 0.$$
   By differentiating with respect to t, show that the tangent line at a point $\boldsymbol{r}_0$ is the line $\boldsymbol{x} = \boldsymbol{r}_0 + \lambda\dot{\boldsymbol{r}}$ satisfying the equation
$$\boldsymbol{r}_0^\top A\boldsymbol{x} + \boldsymbol{b}^\top\frac{\boldsymbol{x} + \boldsymbol{r}_0}{2} + c = 0.$$

10. Given a curve $\boldsymbol{r}(t)$ define the curve $\boldsymbol{R}(t) = P\boldsymbol{r}(t) + \boldsymbol{a}$ where P is a rotation (orthogonal) matrix and $\boldsymbol{a}$ is a constant vector (in other words rotate and translate the original curve). Show that the corresponding tangent, normal and binormal vectors for the new curve are $\boldsymbol{T} = P\boldsymbol{t}$, $\boldsymbol{N} = P\boldsymbol{n}$ and $\boldsymbol{B} = P\boldsymbol{b}$. Deduce that the curvature and torsion of the new curve are unchanged.

11. The evolute of a planar curve $\boldsymbol{r}(t)$, is defined to be the curve $\boldsymbol{R} = \boldsymbol{r} + \frac{1}{\kappa}\boldsymbol{n}$ where $\boldsymbol{n}$ is the normal vector and κ the curvature.

    Differentiate $\boldsymbol{R}$ twice with respect to s, the arclength of $\boldsymbol{r}$, and deduce that the curvature of $\boldsymbol{R}$ is equal to κ^3/κ'.

    Show that the evolute of a helix is another helix.

    (*)Show that the evolute of the log spiral defined above is another log spiral.

12. A space curve $\boldsymbol{r}(t)$ satisfies the following equation $\ddot{\boldsymbol{r}} = e\boldsymbol{B} \times \dot{\boldsymbol{r}}$ (eg a charged particle moving in a magnetic field), where $e\boldsymbol{B}$ is a constant vector.

    Write the equation in terms of $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ and deduce that $\dot{s}$ is constant and that $\dot{\kappa} = 0 = \dot{\tau}$. Deduce that the curve is a helix.

13. A curve $\boldsymbol{r}(t)$ satisfies the equations $\boldsymbol{r} \cdot \boldsymbol{n} = c$ and $\boldsymbol{b} = \alpha\boldsymbol{t} + \boldsymbol{a}$ where $c, \alpha, \boldsymbol{a}$ are constants. Show that the curve must be a helix.

14. Let $\boldsymbol{r}(s)$ be a closed planar curve. Then its tangent vector at a point is given by $\boldsymbol{t}(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix}$ where $\theta(s)$ is the angle that the tangent makes with the horizontal. Show that $\boldsymbol{r}(s) = \int \boldsymbol{t}(s)\,ds$ and that $\boldsymbol{n}(s) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. By integrating $\boldsymbol{t}' \cdot \boldsymbol{n}$ show that

    $$\int \kappa(s)\,ds = \Delta\theta = 2\pi.$$

    You already know this theorem for polygons: the sum of the external angles of any polygon is equal to 360°; for triangles it can be modified to read that the sum of the internal angles is 180°.

15. Let $\boldsymbol{r}(t)$ be a curve with tangent vector $\boldsymbol{t}(s)$. Show that

    $$\frac{d}{dt}\boldsymbol{r}(t) = \boldsymbol{t}(s)\dot{s},$$

$$\frac{d^2}{dt^2}\boldsymbol{r}(t) = \dot{s}^2\kappa\boldsymbol{n}(s) + \ddot{s}\boldsymbol{t}(s),$$

and hence show that the plane generated by the vectors $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ is the same as the plane generated by $\boldsymbol{t}$ and $\boldsymbol{n}$ (assuming $\kappa \neq 0$); hence this plane is independent of parametrization and is called the *osculating* plane at the point $\boldsymbol{r}(t)$.

16. * More generally, for a curve in $\mathbb{R}^N$, prove by induction that the vector $d^n\boldsymbol{r}/dt^n$ can be written as a linear combination of the vectors $\boldsymbol{t}, \boldsymbol{t}', \ldots, \boldsymbol{t}^{(n-1)}$. These subspaces are called the *osculating* subspaces of the curve.

17. * For a curve in $\mathbb{R}^N$, generalize the Serret-Frenet formulæ by showing that there is an orthogonal set of vectors at each point, $\boldsymbol{t}_1 = \boldsymbol{t}, \boldsymbol{t}_2 = \boldsymbol{n}, \boldsymbol{t}_3, \ldots, \boldsymbol{t}_N$ and 'curvatures' $\kappa_1 = \kappa, \kappa_2 = \tau, \kappa_3, \ldots, \kappa_{N-1}$ such that $\boldsymbol{t}_i' = -\kappa_{i-1}\boldsymbol{t}_{i-1} + \kappa_i\boldsymbol{t}_{i+1}$.

    You may have noticed that when $\kappa = 0 = \tau$ then the curve is a straight line (a one-dimensional subspace); when $\kappa \neq 0, \tau = 0$ the curve is planar (in a two-dimensional subspace); when $\kappa, \tau \neq 0$ the curve is a 'space' curve (three-dimensional). Generalize this, and show that if $\kappa_1, \ldots \kappa_i \neq 0, \kappa_{i+1} = 0$ then the curve is in an $(i+1)$-dimensional subspace.

3 Surfaces

3.1 Introduction

Definition A **surface** is a continuous map

$$\begin{aligned} \boldsymbol{r}: \quad \mathbb{R}^2 &\to \mathbb{R}^3 \\ (u,v) &\mapsto \boldsymbol{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \end{aligned}$$

A map from a subset of $\mathbb{R}^2$ is also accepted.

3.1.1 Examples

The *plane* is given by $\boldsymbol{r}(u,v) = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$, $u, v \in \mathbb{R}$.

The *sphere* is given by the (longitude-latitude) map

$$\boldsymbol{r}(u,v) = \begin{pmatrix} \cos u \, \cos v \\ \cos u \, \sin v \\ \sin u \end{pmatrix}, \quad -\frac{\pi}{2} < u < \frac{\pi}{2}, 0 \leqslant v < 2\pi.$$

The *helicoid* is given by the map

$$\boldsymbol{r}(u,v) = \begin{pmatrix} u\cos v \\ u \sin v \\ v \end{pmatrix}, \quad 0 < u, v \in \mathbb{R}.$$

The *torus* is given by the map

$$\boldsymbol{r}(u,v) = \begin{pmatrix} (a + b\cos u)\cos v \\ (a + b\cos u)\sin v \\ b \sin u \end{pmatrix}, \quad 0 \leqslant u < 2\pi, 0 \leqslant v < 2\pi.$$

Graphs of Functions, $z = f(x,y)$

For any given function of two variables $f(x,y)$, we can form the surface, called its *graph*, using the map

$$\boldsymbol{r}(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}.$$

For example, the *cone* has a map

$$\boldsymbol{r}(u,v) = \begin{pmatrix} u \\ v \\ \sqrt{u^2+v^2} \end{pmatrix},$$

while the *elliptic paraboloid* has a map with $f(u,v) = u^2/a^2 + v^2/b^2$.

Surfaces of Revolution

When we start with a function $f(u)$ and rotate it completely around the u-axis, we get a surface in three-dimensions. The map in this case is given by

$$\boldsymbol{r}(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ u \end{pmatrix}.$$

For example, the circular *cylinder* is a surface of revolution with $f(u) = 1$; the circular *hyperboloid of one sheet* has $f(u) = \sqrt{1+u^2}$; the *catenoid* has $f(u) = \cosh u$.

What are the following surfaces?

$$\boldsymbol{r}(u,v) = \begin{pmatrix} u \\ v \\ \sqrt{u^2+v^2-1} \end{pmatrix}, \quad \begin{pmatrix} a\cos u\cos v \\ b\cos u\sin v \\ c\sin u \end{pmatrix}, \quad \begin{pmatrix} u\cos u(a+\cos v) \\ u\sin u(a+\cos v) \\ u\sin v \end{pmatrix}.$$

3.1.2 Parametrizations

As with curves, there may be several parametrizations $\boldsymbol{r}(u,v)$ giving the same surface; although the maps themselves may be different, the end-result is the same.

For example, the sphere (which is the most well-studied example historically) has several other parametrizations (historically called projections). The following are a few examples:

$$\text{orthogonal projection} \quad \boldsymbol{r}(u,v) = \begin{pmatrix} u \\ v \\ \sqrt{1-u^2-v^2} \end{pmatrix};$$

$$\text{Mercator projection} \quad \boldsymbol{r}(u,v) = \begin{pmatrix} \mathrm{sech} u\cos v \\ \mathrm{sech} u\sin v \\ \tanh u \end{pmatrix},$$

$$\text{Lambert projection} \quad \begin{pmatrix} \cos U \cos V \\ \cos U \sin V \\ \sin U \end{pmatrix},$$

where $U = \cot^{-1}\sqrt{u^2+v^2}$ and $V = \tan^{-1}(v/u)$;

$$\text{stereographic projection} \quad \boldsymbol{r}(u,v) = \frac{1}{u^2+v^2+4}\begin{pmatrix} 2u \\ 2v \\ u^2+v^2+2 \end{pmatrix}.$$

Notice moreover, that in some cases, the map leaves out a few points, or even whole areas out. For example, the longitude-latitude map leaves out the north and south poles; the orthogonal projection leaves out a whole hemisphere; the stereographic projection leaves out just the north pole. To remedy this, we can use more than one map to cover the whole surface e.g. in actual atlases, a longitude-latitude map may be used to cover most of the world, and two orthogonal projections are used to cover Antarctica and the Arctic.

3.2 Tangents and Normal

Consider the curves $\boldsymbol{r}(u, v_0)$ and $\boldsymbol{r}(u_0, v)$. On the u-v map they appear as straight horizontal and vertical lines. On the surface, however, they become curves lying in the surface itself; they form what is called the *wire-frame* model of the surface, and is used by computers as a first step to draw the surfaces. For example, the wire-frame model for the latitude-longitude sphere, giving the latitudes and longitudes on the sphere, is a very familiar picture.

Let us investigate how we can represent vectors on a surface. Notice that if we draw a curve on the u-v map, it will be mapped to a curve on the surface; and hence we expect the tangent vectors to be mapped to tangent vectors on the corresponding surface curve. In particular, for the horizontal and vertical lines mentioned above, we get that the tangents on the u-v map straight lines, namely,

$$\boldsymbol{i} = \frac{d}{du}\begin{pmatrix} u \\ v_0 \end{pmatrix}, \quad \boldsymbol{j} = \frac{d}{dv}\begin{pmatrix} u_0 \\ v \end{pmatrix},$$

will be mapped to the tangents on the surface curve,

$$\boldsymbol{g}_u = \frac{d}{du}\boldsymbol{r}(u, v_0) = \frac{\partial}{\partial u}\boldsymbol{r}(u, v_0),$$

$$\boldsymbol{g}_v = \frac{d}{dv}\boldsymbol{r}(u_0, v) = \frac{\partial}{\partial v}\boldsymbol{r}(u_0, v).$$

This way we get two tangent vectors at each point $\boldsymbol{r}(u_0, v_0)$ on the surface.

Definition The canonical **tangent** vectors at a point $\boldsymbol{r}(u, v)$ on the surface are defined by $\boldsymbol{g}_u = \frac{\partial \boldsymbol{r}}{\partial u}$, $\boldsymbol{g}_v = \frac{\partial \boldsymbol{r}}{\partial v}$.

A vector $\boldsymbol{x} = \big(A, B\big) = A\boldsymbol{i} + B\boldsymbol{j}$ on the u-v map, will then be mapped to the vector $\boldsymbol{X} = A\boldsymbol{g}_u + B\boldsymbol{g}_v$.

To complete the picture, we define the normal vector at a point on the surface as that unit vector perpendicular to the canonical tangent vectors;

Definition The **normal** vector at a point $\boldsymbol{r}(u, v)$ on the surface is given by

$$\boldsymbol{n} = \frac{\boldsymbol{g}_u \times \boldsymbol{g}_v}{|\boldsymbol{g}_u \times \boldsymbol{g}_v|}.$$

Notice that for curves, which are 1-dimensional, there is one tangent and two normals at each point, while for surfaces, which are 2-dimensional, there are two tangents and one normal at each point.

Notice also, that the normal is really defined only up to a sign; if we switch the u-v axis we get the opposite normal.

3.3 Scalar Product of Vectors

To be able to measure the length of a vector or the angle between vectors at a point, we need to be able to find the scalar product between any two vectors. Of course we know how to find such a product for any 3-dimensional vectors, but in this case, we would like to find the product *in terms of* the vectors as written in the u-v map. In other words, if $\boldsymbol{x} = \begin{pmatrix} A \\ B \end{pmatrix}$ and $\boldsymbol{y} = \begin{pmatrix} C \\ D \end{pmatrix}$ are two vectors based at the point $\begin{pmatrix} u \\ v \end{pmatrix}$ on the u-v map, then

$$\begin{aligned}
\boldsymbol{X} \cdot \boldsymbol{Y} &= (A\boldsymbol{g}_u + B\boldsymbol{g}_v) \cdot (C\boldsymbol{g}_u + D\boldsymbol{g}_v) \\
&= AC\boldsymbol{g}_u \cdot \boldsymbol{g}_u + AD\boldsymbol{g}_u \cdot \boldsymbol{g}_v + BC\boldsymbol{g}_v \cdot \boldsymbol{g}_u + BD\boldsymbol{g}_v \cdot \boldsymbol{g}_v \\
&= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \\
&= \boldsymbol{x}^\top g \boldsymbol{y},
\end{aligned}$$

where we have taken the following definitions,

Definition The **first fundamental form** of a surface is the matrix

$$g := \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

where

$$E := \boldsymbol{g}_u \cdot \boldsymbol{g}_u, \quad F := \boldsymbol{g}_u \cdot \boldsymbol{g}_v, \quad G := \boldsymbol{g}_v \cdot \boldsymbol{g}_v.$$

Notice that the actual product $\boldsymbol{X} \cdot \boldsymbol{Y}$ is not equal to $\boldsymbol{x} \cdot \boldsymbol{y}$ because the first fundamental form g is not usually equal to the identity matrix. Hence the length of a vector $\boldsymbol{X}$ and the angle between two vectors $\boldsymbol{X}, \boldsymbol{Y}$ on the surface, are not usually the same as the length of $\boldsymbol{x}$ and the angle between $\boldsymbol{x}, \boldsymbol{y}$ on the u-v map.

In particular, the vectors $\boldsymbol{g}_u$ and $\boldsymbol{g}_v$ may not be unit length or perpendicular to each other (check this out by finding $\boldsymbol{g}_i \cdot \boldsymbol{g}_j$ for $i, j = u, v$, for any surface.)

3.3.1 Example

For the sphere with the latitude-longitude map, the tangent vectors are given by

$$\boldsymbol{g}_u = \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix}, \quad \boldsymbol{g}_v = \begin{pmatrix} -\cos u \sin v \\ \cos u \cos v \\ 0 \end{pmatrix}.$$

Hence the cross product $\boldsymbol{g}_u \times \boldsymbol{g}_v = \begin{pmatrix} -\cos^2 u \cos v \\ -\cos^2 u \sin v \\ -\sin u \cos u \end{pmatrix}$, and dividing by its length gives the normal vector

$$\boldsymbol{n} = \begin{pmatrix} -\cos u \cos v \\ -\cos u \sin v \\ -\sin u \end{pmatrix}.$$

Taking the various scalar products of $\boldsymbol{g}_u$ and $\boldsymbol{g}_v$ gives the first fundamental form

$$E = \boldsymbol{g}_u \cdot \boldsymbol{g}_u = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = 1,$$
$$F = \boldsymbol{g}_u \cdot \boldsymbol{g}_v = \sin u \cos v \cos u \sin v - \sin u \sin v \cos u \cos v = 0,$$
$$G = \boldsymbol{g}_v \cdot \boldsymbol{g}_v = \cos^2 u \sin^2 v + \cos^2 u \cos^2 v = \cos^2 u,$$
$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}.$$

Given vectors on the map, say $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ at the point $(u, v) = (\pi/4, 0)$, then their true scalar product (on the surface) is

$$\boldsymbol{X} \cdot \boldsymbol{Y} = \boldsymbol{x}^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} \boldsymbol{y} = (1\ 1) \begin{pmatrix} 1 & 0 \\ 0 & \cos^2(\pi/4) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 - \frac{1}{2} = \frac{1}{2};$$

similarly we find that $|\boldsymbol{X}| = \sqrt{\boldsymbol{X}\cdot\boldsymbol{X}} = \sqrt{3/2}$ and $|\boldsymbol{Y}| = \sqrt{3/2}$, so that the true angle between the two vectors on the surface is given by $\cos\theta = (\boldsymbol{X}\cdot\boldsymbol{Y})/(|\boldsymbol{X}||\boldsymbol{Y}|) = 1/3$ and $\theta = 70.5°$. Whereas the angle as measured on the u-v map would be $\cos\alpha = (\boldsymbol{x}\cdot\boldsymbol{y})/(|\boldsymbol{x}||\boldsymbol{y}) = 0$ and $\alpha = 90°$.

3.4 Length of a Curve

We can now use the first fundamental form to find the length of curves on a surface.

A curve on a surface is given by a curve $(u(t), v(t))$ on the u-v map via $\boldsymbol{r}(u(t), v(t))$. That is, as we trace out the curve on the surface, we trace out a corresponding curve on the u-v map. Taking the derivative with respect to t gives

$$\frac{d}{dt}\boldsymbol{r}(u(t), v(t)) = \frac{\partial \boldsymbol{r}}{\partial u}\frac{du}{dt} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{dv}{dt} = \dot{u}\boldsymbol{g}_u + \dot{v}\boldsymbol{g}_v.$$

That is the vector $\dot{\boldsymbol{r}}$ on the surface corresponds to the vector $(\dot{u}, \dot{v})$ on the u-v map.

Now the actual length of a curve was found in Chapter 2 to be given by the formula:

$$\begin{aligned} L &= \textstyle\int_{t_0}^{t_1} |\dot{\boldsymbol{r}}(t)|dt = \int_{t_0}^{t_1} \sqrt{\dot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t)}\,dt \\ &= \textstyle\int_{t_0}^{t_1} \sqrt{\begin{pmatrix}\dot{u} & \dot{v}\end{pmatrix}\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}}\,dt \\ &= \textstyle\int_{t_0}^{t_1} \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\,dt. \end{aligned}$$

3.4.1 Example

For example, let us find the length of the latitude at $\pi/4$ on the unit sphere, between longitudes from 0 to π: First we find the curve on the u-v map that corresponds to this curve, namely $u(t) = \pi/4$ and $v(t) = t$, with $0 \leqslant t \leqslant \pi$, then we substitute into the above formula to get

$$L = \int_0^\pi \sqrt{0 + 0 + \cos^2(\pi/4)1}\,dt = \int_0^\pi \frac{1}{\sqrt{2}}dt = \frac{\pi}{\sqrt{2}}.$$

Note that this is completely different from the apparent length as found on the u-v map, where all the latitudes have the same length.

3.5 Area of a Region

A region Ω on the u-v map will be mapped to a region $\boldsymbol{r}\Omega$ on the surface. To find the surface area of $\boldsymbol{r}\Omega$ we will divide the corresponding region Ω on the u-v map into very small rectangles of sides $\delta u\,\boldsymbol{i}$ and $\delta v\,\boldsymbol{j}$. These small rectangles will be mapped to a very good approximation to a parallelogram on the surface with sides $\delta u\,\boldsymbol{g}_u$ and $\delta v\,\boldsymbol{g}_v$. Now, in general, any parallelogram with sides $\boldsymbol{a}$ and $\boldsymbol{b}$ has area given by $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$; this area can be written as $|\boldsymbol{a}\times\boldsymbol{b}|$. So, in our case of the small parallelograms, we get each small area to be equal to $\delta u\,\delta v\,|\boldsymbol{g}_u\times\boldsymbol{g}_v|$. Summing up all the parallelogram areas that are in the region $\boldsymbol{r}\Omega$ gives the total area $A\sim\sum_{u,v}|\boldsymbol{g}_u\times\boldsymbol{g}_v|\delta u\delta v$, which motivates the following definition: **Definition** The area of a region $\boldsymbol{r}\Omega$ on the surface is defined to be

$$A=\iint_\Omega|\boldsymbol{g}_u\times\boldsymbol{g}_v|dudv.$$

Furthermore, we denote

$$dS=|\boldsymbol{g}_u\times\boldsymbol{g}_v|dudv,$$

$$d\boldsymbol{S}=(\boldsymbol{g}_u\times\boldsymbol{g}_v)dudv=\boldsymbol{n}dS,$$

so that the area of a region is just $\iint_\Omega dS$. We also use these symbols to define the integrals

$$\int f(\boldsymbol{r})dS,\int \boldsymbol{f}(\boldsymbol{r})dS,\int f(\boldsymbol{r})d\boldsymbol{S},\int \boldsymbol{f}(\boldsymbol{r})\cdot d\boldsymbol{S},\int \boldsymbol{f}(\boldsymbol{r})\times d\boldsymbol{S}.$$

Before we proceed with an example, let us derive an equivalent formula. One of the vector identities reads

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a},$$

from which follows

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}\times\boldsymbol{b})=(\boldsymbol{a}\cdot\boldsymbol{a})\,(\boldsymbol{b}\cdot\boldsymbol{b})-(\boldsymbol{a}\cdot\boldsymbol{b})^2.$$

Applying this to $\boldsymbol{a}=\boldsymbol{g}_u$ and $\boldsymbol{b}=\boldsymbol{g}_v$ we get

$$|\boldsymbol{g}_u\times\boldsymbol{g}_v|^2=EG-F^2=\det g,$$

and

$$A=\iint_\Omega\sqrt{\det g}\,dudv.$$

3.5.1 Example

Find the area of a region on the sphere, which appears as the rectangle $0 \leqslant u \leqslant \pi/4$, $0 \leqslant v \leqslant \pi/2$ on the latitude-longitude map.

From the first fundamental form of the sphere, we find that $\det g = EG - F^2 = \cos^2 u$. Hence the required area is

$$A = \int_0^{\pi/2} \int_0^{\pi/4} |\cos u| du dv = \frac{\pi}{2} \sin \frac{\pi}{4} = \frac{\pi}{2\sqrt{2}}.$$

Proposition 3.5.1

The Area formula is well-defined, in the sense, that if the same region on the surface is given by different parametrizations, the resulting area is still the same.

Proof. Suppose that a region on the surface is given by both Ω on a u-v map, and also by Ω' on a U-V map, where $\boldsymbol{r}(u,v)$ and $\boldsymbol{R}(U,V)$ are two parametrizations of the same surface.

Then

$$\begin{aligned} \boldsymbol{g}_U \times \boldsymbol{g}_V &= \tfrac{\partial \boldsymbol{R}}{\partial U} \times \tfrac{\partial \boldsymbol{R}}{\partial V} \\ &= \left(\tfrac{\partial \boldsymbol{r}}{\partial u}\tfrac{\partial u}{\partial U} + \tfrac{\partial \boldsymbol{r}}{\partial v}\tfrac{\partial v}{\partial U}\right) \times \left(\tfrac{\partial \boldsymbol{r}}{\partial u}\tfrac{\partial u}{\partial V} + \tfrac{\partial \boldsymbol{r}}{\partial v}\tfrac{\partial v}{\partial V}\right) \\ &= \left(\tfrac{\partial u}{\partial U}\boldsymbol{g}_u + \tfrac{dv}{dU}\boldsymbol{g}_v\right) \times \left(\tfrac{du}{dV}\boldsymbol{g}_u + \tfrac{dv}{dV}\boldsymbol{g}_v\right) \\ &= \left(\tfrac{du}{dU}\tfrac{dv}{dV} - \tfrac{dv}{dU}\tfrac{du}{dV}\right) \boldsymbol{g}_u \times \boldsymbol{g}_v \end{aligned}$$

Hence $|\boldsymbol{g}_U \times \boldsymbol{g}_V| = J|\boldsymbol{g}_u \times \boldsymbol{g}_v|$, where J is the Jacobian determinant for a change of basis from U, V to u, v. So,

$$\begin{aligned} A &= \textstyle\iint_{\Omega'} |\boldsymbol{g}_U \times \boldsymbol{g}_V| dU dV \\ &= \textstyle\iint_{\Omega'} |\boldsymbol{g}_u \times \boldsymbol{g}_v| J \, dU dV \\ &= \textstyle\iint_{\Omega} |\boldsymbol{g}_u \times \boldsymbol{g}_v| \, du dv, \end{aligned}$$

where we applied a change of variable from U, V to u, v.

□

3.6 Exercises

1. For the sphere with latitude-longitude parametrization, find $\iint_\Omega \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (y,z,x)$ and Ω is the region corresponding to $0 \leqslant u \leqslant \pi/2$, $0 \leqslant v \leqslant \pi$.

2. Find the first fundamental form of the catenoid surface given by the map
$$\boldsymbol{r}(u,v) = \begin{pmatrix} \cosh u \cos v \\ \cosh u \sin v \\ u \end{pmatrix}$$
where $-u_0 < u < u_0, 0 < v < 2\pi$ and hence find the length of the closed curve, with $u = 0, 0 \leqslant v < 2\pi$, on the surface.

3. Find the first fundamental form for the helicoid given parametrically by $(u\cos v, u\sin v, v)$.

4. Find the length of the curve $u(t) = t$, $v(t) = \log(\sec t + \tan t)$ on the sphere parametrized by latitude/longitude u, v.

5. Find the first fundamental form of the tractroid surface given by the map
$$\boldsymbol{r}(u,v) = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \log(\sec u + \tan u) - \sin u \end{pmatrix}$$
where $-\frac{\pi}{2} < u, v < \frac{\pi}{2}$ and hence find its area.

6. Show that if $E = G$ and $F = 0$ then the angles between two vectors on a map and the corresponding vectors on the surface parametrized by the map are equal. We call such maps *conformal*. Find examples of conformal maps.

7. * Show conversely that a conformal map must satisfy $E = G$ and $F = 0$.

8. Show that if $EG - F^2$ is constant, independent of u and v, then the area of a region on a surface is a constant multiple of the corresponding region on the map. We call such maps *area-preserving*. Find examples of area-preserving maps.