Analytical Geometry

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Tutorial 1

1. Suppose a parallelogram $ABCD$ has position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$. Find a formula for $\boldsymbol{d}$ in terms of the other vectors.

2. Show that in general,
$$(\boldsymbol{x}-\boldsymbol{y}) \cdot (\boldsymbol{x}+\boldsymbol{y}) = \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2.$$
Let $\boldsymbol{x}$ and $\boldsymbol{y}$ denote the vectors OA and OB of a rhombus $OACB$; use this identity to deduce that the diagonals are perpendicular.

3. Let O be the origin in the plane, and let A be the point with Cartesian coordinates $\begin{pmatrix}1\\0\end{pmatrix}$. Find
   (a) that point B (with positive y-coordinate) such that OAB is an equilateral triangle;
   (b) the midpoint C between O and A;
   (c) points D and E (with negative y-coordinates) such that OCD and CAE are equilateral;
   (d) the centroids P, Q, R of the triangles OAB, OCD, and CAE, respectively;
   (e) the distances between P, Q, R. What can you say about the triangle PQR?

4. *"Napoleon's theorem"* Let A, B, C have coordinates $\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$. Find points L, M, N which make ABL, BCM and CAN equilateral triangles *outside* ABC (Take $\begin{pmatrix}x\\y\end{pmatrix}$ such that its distance from A and B, say, is the same). Find the centroids P, Q, R of the these triangles and show that PQR is equilateral.

5. An army is situated at point $A = \begin{pmatrix}1\\-1\end{pmatrix}$ and an enemy battalion is at $B = \begin{pmatrix}-1\\2\end{pmatrix}$. There are three fortresses situated at points $\begin{pmatrix}0.5\\0.5\end{pmatrix}$, $\begin{pmatrix}-1\\-2\end{pmatrix}$ and $\begin{pmatrix}3.5\\-2\end{pmatrix}$. B can move at twice the speed of A. Which fortress should A take refuge in (moving in a straight line)?

Tutorial 2

1. Prove
$$[\boldsymbol{a}+\boldsymbol{b},\boldsymbol{x},\boldsymbol{y}] = [\boldsymbol{a},\boldsymbol{x},\boldsymbol{y}] + [\boldsymbol{b},\boldsymbol{x},\boldsymbol{y}]$$
$$[\lambda\boldsymbol{a},\boldsymbol{x},\boldsymbol{y}] = \lambda[\boldsymbol{a},\boldsymbol{x},\boldsymbol{y}]$$

2. Prove the Jacobi identity
$$\boldsymbol{x}\times(\boldsymbol{y}\times\boldsymbol{z}) + \boldsymbol{y}\times(\boldsymbol{z}\times\boldsymbol{x}) + \boldsymbol{z}\times(\boldsymbol{x}\times\boldsymbol{y}) = \boldsymbol{0}$$

3. Prove
$$\begin{aligned}(\boldsymbol{x}\times\boldsymbol{y})\times(\boldsymbol{a}\times\boldsymbol{b}) &= [\boldsymbol{x},\boldsymbol{a},\boldsymbol{b}]\boldsymbol{y} - [\boldsymbol{y},\boldsymbol{a},\boldsymbol{b}]\boldsymbol{x}\\ &= [\boldsymbol{x},\boldsymbol{y},\boldsymbol{b}]\boldsymbol{a} - [\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}]\boldsymbol{b}\end{aligned}$$

4. Find all five possible ways of placing brackets on $\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{c}\times\boldsymbol{d}$ and expand each one out using $(\boldsymbol{x}\times\boldsymbol{y})\times\boldsymbol{z} = (\boldsymbol{x}\cdot\boldsymbol{z})\boldsymbol{y} - (\boldsymbol{y}\cdot\boldsymbol{z})\boldsymbol{x}$.

5. The vectors $\boldsymbol{0}$, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{x}+\boldsymbol{y}$ are the position vectors of the vertices of a parallelogram. Show that the sum of the squares of the diagonals is equal to the sum of the squares of the sides, i.e.,
$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2.$$

6. Prove that if $\boldsymbol{x}\times\boldsymbol{y} = \boldsymbol{0}$ and $\boldsymbol{x}\cdot\boldsymbol{y} = 0$ then $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{y} = \boldsymbol{0}$.

7. Write $\begin{pmatrix}1\\2\\3\end{pmatrix}$ in terms of the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

8. Find the area of

   (a) the triangle with coordinates $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}$,

   (b) the pentagon with coordinates $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$, $\begin{pmatrix}4\\0\end{pmatrix}$, $\begin{pmatrix}2\\-2\end{pmatrix}$, $\begin{pmatrix}1\\-4\end{pmatrix}$.

9. Show that the three vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\3\end{pmatrix}$ are coplanar.

Tutorial 3

1. Find the distance between the two lines

$$\boldsymbol{x}_1(t) = \begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}, \quad \boldsymbol{x}_2(s) = \begin{pmatrix}0\\1\\1\end{pmatrix} + s\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

   (Answer:$5/\sqrt{14}$)

2. Consider the two circles with equations

$$x^2 + y^2 = 1, \quad (x-1)^2 + (y-1)^2 = 1$$

   (a) Find their two points of intersection, A, B.

   (b) Find the equation of the line which passes through these two points.

   (c) Find the equation of the line which passes through the centers of the circles, P, Q.

   (d) Hence show that the two lines AB and PQ are perpendicular and intersect at their midpoint.

3. A broken fragment of an ancient Greek plate is found by an archaeologist, who now wants to find its original size. You plot it out on a grid paper and determine three points on the outer circumference as $A = \begin{pmatrix}0\\2.3\end{pmatrix}$, $B = \begin{pmatrix}3.0\\3.0\end{pmatrix}$, and $C = \begin{pmatrix}6.0\\1.6\end{pmatrix}$.

   (a) Find the midpoint M of AB and the midpoint N of BC.

   (b) Find the equation of the line passing through M and perpendicular to AB, and of the line through N perpendicular to BC.

   (c) Find the point of intersection O of these two lines; this is the center of the original plate. Hence find its diameter.

4. A triangle has vertices with positions $\boldsymbol{a} = \begin{pmatrix}1\\2\end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix}0\\1\end{pmatrix}$ in the xy-plane.

   (a) Find the equations of the three altitudes (each altitude passes through a vertex and is perpendicular to the opposite side). Find the point of intersection of the three altitudes (the *orthocenter*).

(b) Find the equations of the three medians (the line which passes through a vertex and the midpoint of the opposite side) and find their common point of intersection (the *centroid*).

(c) Find also the equations of the three lines perpendicular through the midpoints of the sides and their common point of intersection (the *circumcenter*).

(d) Show that the orthocenter, the centroid and the circumcenter are collinear (a line passes through them).

5. A line has equation $\boldsymbol{x} \times \boldsymbol{b} = \boldsymbol{c}$, while a plane has equation $\boldsymbol{x} \cdot \boldsymbol{n} = d$. Expand $(\boldsymbol{x} \times \boldsymbol{b}) \times \boldsymbol{n}$ and then find the point of intersection,

$$\boldsymbol{x} = \frac{d\boldsymbol{b} - \boldsymbol{c} \times \boldsymbol{n}}{\boldsymbol{b} \cdot \boldsymbol{n}}.$$

6. Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be three unit vectors on the unit sphere. Let OBC be the plane formed by $\boldsymbol{b}$ and $\boldsymbol{c}$; similarly OAB and OAC. (Their intersection with the sphere gives a *spherical triangle*.) Let A be the angle between the planes OAB and OAC, similarly for the angles B and C.

(a) Show that $\|\boldsymbol{a} \times \boldsymbol{b}\| = \sin \angle AOB$.

(b) Let $\boldsymbol{n}_1 := \boldsymbol{a} \times \boldsymbol{b}$ be normal to the plane OAB, and $\boldsymbol{n}_2 := \boldsymbol{c} \times \boldsymbol{a}$ normal to OAC. Show that

$$\|\boldsymbol{n}_1 \times \boldsymbol{n}_2\| = |\sin \angle AOC \sin \angle AOB \sin A|$$

(c) Prove (using an identity for the vector triple product)

$$\|(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{a})\| = |[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]|$$

(d) Deduce that

$$\frac{\sin \angle BOC}{\sin A} = \frac{\sin \angle AOC}{\sin B} = \frac{\sin \angle AOB}{\sin C}$$

7. (Challenge question) Find the vertices of a pentagon given the (five) midpoints of the sides. (Hint: first try finding the vertices of a triangle given the three midpoints).

Tutorial 4

1. Show

   (a) $\frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}$ is a reflection, and find its line of reflection,

   (b) $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$ is a reflection and find its plane of reflection,

   (c) $\frac{1}{4}\begin{pmatrix}2\sqrt{3} & 0 & -2\\ \sqrt{3} & 2 & 3\\ 1 & -2\sqrt{3} & \sqrt{3}\end{pmatrix}$ is a rotation; how would you go about finding its axis of rotation?

2. Find the eigenvectors and eigenvalues of the following matrices:

$$\text{(i)}\ \begin{pmatrix}1&1\\0&2\end{pmatrix},\ \text{(ii)}\ \begin{pmatrix}1&2\\1&2\end{pmatrix},\ \text{(iii)}\ \begin{pmatrix}1&2\\2&4\end{pmatrix},\ \text{(iv)}\ \begin{pmatrix}1&-1\\0&-1\end{pmatrix}$$

   (Answers: (i) 1, $\begin{pmatrix}1\\0\end{pmatrix}$ and 2, $\begin{pmatrix}1\\1\end{pmatrix}$, (ii) 2, $\begin{pmatrix}1\\1\end{pmatrix}$ and 0, $\begin{pmatrix}-2\\1\end{pmatrix}$, (iii) 5, $\begin{pmatrix}1\\2\end{pmatrix}$ and 0, $\begin{pmatrix}-2\\1\end{pmatrix}$, (iv) 1, $\begin{pmatrix}1\\0\end{pmatrix}$ and -1, $\begin{pmatrix}1\\2\end{pmatrix}$)

3. Find what type of conic are represented by the following equations:

   (a) $2x^2 - y^2 - 2y = 5$;

   (b) $3x^2 + 2xy + 3y^2 - 2\sqrt{2}x - 6\sqrt{2}y + 4 = 0$;

   (c) $2x^2 + 2xy + 2y^2 + 3\sqrt{2}x + 3\sqrt{2}y + 4 = 0$;

   (d) $x^2 + 10\sqrt{3}xy + 11y^2 - 8(2+\sqrt{3})x + 8(1-2\sqrt{3})y = 0$;

   (e) $x^2 + y^2 + \sqrt{3}x + y + 1 = 0$;

   (f) $3x^2 + 2\sqrt{3}xy + y^2 + 8\sqrt{3}x + 8y = 0$;

   (g) $x^2 + 2xy + y^2 + 2\sqrt{2}x + 2\sqrt{2}y + 2 = 0$;

   (h) $x^2 + 2xy + y^2 + 5\sqrt{2}x + 3\sqrt{2}y + 6 = 0$.

   (Answers: hyperbola, ellipse, nothing, intersecting lines, single point, parallel lines, single line, parabola)

4. In the above exercise, find the area enclosed by the ellipse, given that the formula for the area is πab where a and b are the semi-major and semi-minor axes.

5. The classical (Greek) definition of a conic is the set (*locus*) of points in the plane whose distance from a fixed point (called the *focus*) is a constant multiple e (called the *eccentricity*) of the distance from a fixed straight line (called the *directrix*). Let the focus be the origin and the directrix the line $x = d$;

   (a) Find the distance between a point $\begin{pmatrix} x \\ y \end{pmatrix}$ and the directrix, and the distance between the point and the origin;

   (b) Show that this definition of a conic gives a curve with Cartesian equation
   $$(1 - e^2)x^2 + 2e^2dx + y^2 - e^2d^2 = 0.$$

   (c) Show that in polar coordinates, the equation is $r = ed/(1+e\cos\theta)$.

   (d) Show that the cases $e > 1$, $e = 1$, $e < 1$ and $e = 0$ correspond to a hyperbola, a parabola, an ellipse and a circle respectively (Note: in this last case, you have to take $e \to 0$, $d \to \infty$ such that $ed = R$ constant).