Differential Geometry

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30 March 2021

1 Differentiable Manifolds

A **differentiable manifold** is a topological space M which is locally diffeomorphic to a real Banach space X, i.e., for any $a \in M$, there is a homeomorphism (called a *chart*) from a neighborhood, $u : U \to X$, $u(a) = 0$, which is *differentiable*, i.e., for any two charts u, v centered at a, the maps $f := vu^{-1} : X \to X$ and f^{-1} are locally close to affine maps,

$$f(x+h) = f(x) + f'(x)h + o(h), \qquad f'(x) \in B(X),$$

where $\|o(h)\|/\|h\| \to 0$ as $h \to 0$.

These charts generate the unique *atlas* of all differentiable charts that are compatible with the generating charts.

A single chart u suffices as a chart for all points $b \in U$, using $u_b := u - u(b)$; so a set of charts that cover M is all that is needed. The charts need only map to an open subset $u : U \to V \subseteq X$ since $V \supseteq W \cong X$. The linear map f' shall also be denoted symbolically by $\frac{\partial v}{\partial u}$. If a point is 'doubled' with the same neighborhoods each, then the space remains a manifold; so manifolds are usually assumed to be Hausdorff.

Examples

- Banach spaces, with the identity map as chart. More generally, any open subset of a Banach space.
- Spheres $\mathbb{S}^n := \{\, x \in \mathbb{R}^{n+1} : \|x\| = 1 \,\}$, with the two charts $u(a_1, \ldots, a_{n+1}) := (a_1, \ldots, a_n)/(\pm 1 - a_{n+1})$.
- Graphs of differentiable functions, $\{\, (x, y) \in X \times Y : y = f(x) \,\}$.
- *Grassmann* manifolds: The set of k-dimensional subspaces of X, including *projective* spaces ($k = 1$).
- Open subspaces and products are again differentiable manifolds (by $(\phi, \psi) : U \times V \to X \times Y$) with the same X. For example, Tori

$$\mathbb{T}^n := \mathbb{S} \times \cdots \times \mathbb{S}$$

X can be generalized to a complete locally convex vector space. One can allow X to vary with the neighborhoods, but then it has to be constant on each connected component.

A *manifold with boundary* is a subset of a manifold, which has a non-empty interior and which contains its boundary that is itself a manifold. (Unless specified otherwise, manifolds do not have a boundary.)

The boundary of a manifold of dimension N, has dimension $N-1$. Points on the boundary are locally diffeomorphic to $\mathbb{R}^+ \times \mathbb{R}^{N-1}$.

1. Manifolds have all the local topological properties of Banach spaces, e.g. locally connected, locally metrizable, locally T_2 (hence T_1).

2. The set of points which are linked via intersecting open charts form a path-connected component.

3. A sub-manifold need not be a topological subspace, e.g. a bijective curve in the torus.

4. For manifolds, paracompact $T_2 \Leftrightarrow$ metrizable (by Smirnov's theorem). Hence, their charts have a locally finite refinement.

   For metrizable manifolds, 2nd countable $\Leftrightarrow$ separable $\Leftrightarrow$ Lindelöf. In this case, there is a countable cover of charts.

5. Locally compact T_2 manifolds are finite dimensional. The dimension is constant on components.
   They are metrizable $\Leftrightarrow$ second countable $\Leftrightarrow$ $\sigma-$compact.

   By taking $v^{-1}B_1$ for each chart v, one can form a countable cover of totally bounded open sets, which has a locally finite refinement. These have a countable, locally finite, partition of unity of differentiable functions. A partition of unity can be used to patch local structures into a global one.

6. (Whitney) For a finite dimensional manifold, the differentiable charts give rise to unique smooth charts, so the manifold is smooth.

   Smooth metrizable manifolds of dimension N can be embedded in $\mathbb{R}^{2N}$. (For paracompact manifolds, take smooth non-zero functions $f_i : U_i \to \mathbb{R}$, extended by zero to M, and let $f : p \mapsto (f_i(p))$.)

7. A finite-dimensional manifold is *orientable* when there is an atlas such that for any two intersecting charts, $(vu^{-1})'$ is orientation-preserving (i.e., have positive determinant). Products and open sub-manifolds of orientable manifolds remain orientable; as is the boundary of a manifold with boundary.

8. Manifolds of dimension 1, 2, or 3, have a 'unique' differentiable structure. But in higher dimensions, the same manifold may have several differentiable structures.

9. Manifolds of the same dimension have a *connected sum* $M\#N$ by removing $B_1(p)$ in M, N and gluing the boundaries; it is associative, commutative, has identity $\mathbb{S}^n$. Every compact manifold can be decomposed into a sum of *prime* manifolds.

   For example, $\mathbb{P}\#\mathbb{T} = \mathbb{P}\#\mathbb{P}\#\mathbb{P}$.

   For compact connected manifolds without boundary, $\chi(M\#N) = \chi(M) + \chi(N) - \chi(\mathbb{S}^n)$;
   also, $\chi(M \sqcup N) = \chi(M) + \chi(N)$, $\chi(M \times N) = \chi(M)\chi(N)$, if M covers N m times, then $\chi(N) = \chi(M)/m$.

10. The 1-dimensional metrizable second countable connected manifolds are diffeomorphic to $\mathbb{R}$ or $\mathbb{S}$.

    The 2-dimensional compact manifolds are diffeomorphic to simplicial complexes and so the prime surfaces are $\mathbb{P}$ and $\mathbb{T}$.

    The 3-D compact prime manifolds can be built up from 8 types.

    The 4-D manifolds cannot be distinguished by any algorithm; each has uncountably many diffeomorphism classes.

    Simply-connected 5-D manifolds (or higher) can be classified up to h-cobordism; (it is not known if compact 5-D manifolds are diffeomorphic to simplicial complexes); in particular, manifolds homotopic to $\mathbb{S}^n$ or $\mathbb{R}^n$ have a unique differentiable atlas.

11. (Jordan-Brouwer-Mazur) If M is a connected finite-dimensional manifold and $A \subseteq M$ is homeomorphic to a compact connected manifold of dimension one less than M, then the exterior of A has two connected components.

12. The global properties of a finite-dimensional manifold can be studied as CW-complexes, i.e., a T_2 space $X = \sum_i E_i$ (cell decomposition − non-unique), where $E_i \cong B_{n(i)}$ ($B_n = B_1(0) \subseteq \mathbb{R}^n$), each $\bar{E} \subseteq X$, ∂E is covered by a finite number of cells, $\bar{A} \cap \bar{E}$ is closed for each $A \subseteq X$. Then its *m-skeleton* is $\mathrm{Sk}_m(X) = \sum_{\dim E_i \leqslant m} E_i$, so $\mathrm{Sk}_1(X) \subseteq \cdots \mathrm{Sk}_n(X) = X$, e.g. the 1-skeleton is a graph with loops.

    Examples: $\mathbb{S}^n \cong B_0 + B_n$, $\mathbb{T}^n \cong B_0 + nB_1 + \cdots \binom{n}{k} B_k + \cdots + B_n$.

13. (Henderson) Every infinite-dimensional, separable, metrizable Banach manifold is embedded as an open subset of ℓ^2.

14. A knot is an embedding $M \to N$.

    In particular, $\mathbb{S}^m \to \mathbb{S}^n$: when $n > \frac{3}{2}(m+1)$, or $n = m+1$, the embedding is unknotted; e.g. $\mathbb{S}^1$ is only knotted in $\mathbb{S}^3$; $\mathbb{S}^2$ in $\mathbb{S}^4$. Knots are the connected sums of prime knots. Jones/Kauffman polynomial (invariant under Radeimeister moves).

Tangent Vectors

The morphisms are the **differentiable** maps, $f : M \to N$ such that $g := vfu^{-1} : X \to Y$ satisfy

$$g(x+h) = g(x) + A(x)h + o(h)$$

with $A(x) \in B(X,Y)$ (well-defined since if A and B are both candidates, then $\|(A-B)e\| := \|(A-B)h\|/\|h\| \leqslant (o_A(h)+o_B(h))/\|h\| \to 0$ as $h \to 0$ for all e). In local coordinates, a map $f : M \to N$ can be thought of as mapping $X \to Y$ (that depends on p); the morphisms are the ones for which this is approximately linear locally.

A *local* diffeomorphism is a map f such that for any $p \in M$, there is an open neighborhood $p \in U \subseteq M$ with $f : U \to fU$ a diffeomorphism.

Morphisms $f : M \to N$ with $f(p) = q$, where M, N are locally X, Y give rise to linear maps $A \in B(X,Y)$ based at p; A depends on the charts (modulo o)

$$B = v_2 f u_2^{-1} = v_2 v_1^{-1} v_1 f u_1^{-1} u_1 u_2^{-1} = \frac{\partial v_2}{\partial v_1} A \frac{\partial u_1}{\partial u_2}$$

But the following is an equivalence relation: $f \sim g \Leftrightarrow vfu^{-1} - vgu^{-1} = o$ for some charts $u : M \to X$, $u(p) = 0$, and $v : N \to Y$, $v(q) = 0$; transitivity: if $f \sim g \sim h$ via charts u_1, v_1 and u_2, v_2, then $v_1 h u_1^{-1} = v_1 v_2^{-1} v_2 h u_2^{-1} u_2 u_1^{-1} = v_1 v_2^{-1} v_2 g u_2^{-1} u_2 u_1^{-1} + o = v_1 g u_1^{-1} + o = v_1 f u_1^{-1} + o$. Thus A (at p) is associated with the equivalence class $[f]$. Each point in M and N has a vector space of operators $B(X,Y)$ associated with it.

In particular, the *tangent* Banach space $X \cong B(\mathbb{R}, X)$ at p is denoted T_pM. In coordinates, the vector x may be denoted by X^i in the chart u, and Y^j in the chart v, with $Y^j = \frac{\partial y^j}{\partial x^i} X^i$. The *cotangent* dual space $B(X, \mathbb{R}) = X^*$ at p is denoted T_p^*M; in coordinates, $Y_j = \frac{\partial x^i}{\partial y^j} X_i$.

Hence a differentiable function induces a map $f' : TM \to TN$, also denoted Df, and also called *push-forward*, defined by $f' : [x]_p \mapsto [f \circ x]_{f(p)}$, that is, the operator A; in local coordinates, $(f')^a_i = \partial_i f^a$. It also acts on co-vectors $f'^* : T_q^*N \to T_p^*M$, $[\alpha]_q \mapsto [\alpha \circ f]_p$ (a pull-back), i.e., by $(f'^*\alpha)x = \alpha(f'x)$; $(f')^*$ is the dual operator of f' and is often written as f^* for short. For example, $v : \mathbb{S} \to M$, $\alpha \in T^*M$, give $(v^*\alpha)_t = \alpha_{v(t)} \cdot v'_t = \alpha_i v^i$ in coordinates.

f is called an *immersion* when f' is 1-1 at any point, and a *submersion* when it is onto.

The tangent space $TM := \bigcup_{p \in M} T_pM$ is given that topology generated from all TU, which makes it locally like $U \times X$ in the sense that for each $p \in M$ with chart $u : U \to X$, there is a bijection $\phi_U : TU \to U \times X$, $x_a \mapsto (a, x)$, which is a homeomorphism in a and a Banach isomorphism in x, and such that $\phi_V \circ \phi_U^{-1}$ is differentiable in a and a Banach isomorphism in x. Then

- TM is a manifold locally isomorphic to $X \times X$, via the homeomorphism $\phi : x_p \mapsto (u(p), x)$;
- $T(M \times N) = TM \times TN$.

Similarly for the cotangent space $T^*M := \bigcup_{p\in M} T_p^*M$. More generally, *tensors* form a manifold $T_l^kM := \bigcup_{p\in M} T_l^k(T_pM)$; each tensor has coordinates of the form $T^{i_1\ldots i_k}_{j_1\ldots j_l}$. They act on vector and covector fields pointwise, independently of their neighborhoods,

$$T(x,\ldots,\alpha,\ldots)_p := T_p(x_p,\ldots,\alpha_p,\ldots).$$

The pull-back of covariant tensors is $f^*T(x_1,\ldots,x_n) = T(f'x_1,\ldots,f'x_n)$; the push-forward of contra-variant tensors is $f'T(\alpha_1,\ldots,\alpha_n) = T(f^*\alpha_1,\ldots,f^*\alpha_n)$. In finite dimensions, T_l^kM has dimension $\dim(M)^{k+l}$.

A *tensor field* is a continuously differentiable choice of tensors: $p \mapsto X_p$, $M \to T_l^kM$ (i.e., $\pi\circ X = I$). They are not only multi-linear on their arguments, but *tensorial*, $T_p(\lambda_p x_p,\ldots) = \lambda_p T_p(x_p,\ldots)$ (λ is a scalar field). Tensor fields form an algebra over the differentiable scalar functions.

A k-form is a field of totally anti-symmetric tensors of type T_0^k.

1. The composition of differentiable maps is differentiable,

   $$(f\circ g)'(x) = f'(g(x))g'(x)$$

   Proof: Locally, $f(g(x+h)) = f(g(x)+g'(x)h+o(h)) = f(g(x)) + f'(g(x))(g'(x)h+o(h)) + o(g'(x)h+o(h))$.

2. Differentiable maps are locally Lipschitz, hence continuous:

   $$\|f(x+h)-f(x)\| \leqslant (\|f'(x)\|+1)\|h\|$$

3. $f'(p)$ is an operator $T_pM \to T_{f(p)}N$,

   $$f'(p)(x+y) = f'(p)(x)+f'(p)(y), \quad f'(p)(\lambda x) = \lambda f'(p)(x), \quad I'(p) = I, \quad c' = 0$$

   When the manifolds are Banach spaces themselves, and $A \in B(X,Y)$, then $A'(p) = A$.

   In local coordinates f' is the *Jacobian* $f' = \partial_i f^j$, i.e., the derivative along the ith coordinate, keeping the others fixed.

   For example, the derivative of a curve is its *tangent vector* $\dot{x}$.

4. For functions $M \to Y$,

   $$(f+g)' = f'+g', \qquad (\lambda f)' = \lambda f'$$

5. f' preserves tensor products (including wedge products); in general given a bilinear map $\cdot$, then $(f\cdot g)'(p) = f'(p)\cdot g(p) + f(p)\cdot g'(p)$ (for example $(\phi f)' = \phi' f + \phi f'$).

   $$f^*(A\otimes B) = (f^*A)\otimes(f^*B), \quad f^*A_i^i = (f^*A)_i^i, \quad f^*(A\cdot X) = (f^*A)\cdot(f^*X)$$

For the reciprocal of a function to a Banach algebra, $f^{inv} : x \mapsto f(x)^{-1}$,

$$(f^{inv})' = -f^{inv} f' f^{inv}$$

(where only f' acts on tangent vectors.)

Proof: $f(x+h)^{-1} = (1 + f(x)^{-1} f'(x)h + f(x)^{-1} o(h))^{-1} f(x)^{-1}$.

6. $f' = 0$ iff f is constant on components of M.

   Proof: Let $g := \phi \circ v \circ f \circ u^{-1} \circ \psi$, where $\phi \in Y^*$, $\psi(t) = tv$; $g' = 0$. If the difference between $g(t)$ and $tg(a)/a$ has a max/min at $t_0 \in]0, a[$ then $g(t_0 + h) - (t_0 + h)g(a)/a \leqslant g(t_0) - t_0 g(a)/a$ (say), so $o(h) \leqslant hg(a)/a$, i.e., $g(a) = 0$. Thus g, and f, are locally constant.

7. The derivative of products is $(f, g)' = \begin{pmatrix} \partial_x f & \partial_y g \\ \partial_x g & \partial_y g \end{pmatrix}$.

   Proof: $(f, g)(x + h) = (f(x + h), g(x + h)) = (f(x) + f'(x)h + \ldots, g(x) + g'(x)h + \ldots) = (f(x), g(x)) + (f'(x), g'(x))h + \ldots$.

8. If $f : M \to N$ is differentiable, then f is locally invertible at $x \Leftrightarrow f'(x)$ is invertible. Then $(f^{-1})'(y) = f'(x)^{-1}$ when $y = f(x)$, and f' can act as both a push-forward and a pull-back on tensors: $f'^* Y = (f^{-1})' Y$.

   Proof: $o'(h) = f'(x + h) - f'(x)$, so $\|o(h_1) - o(h_2)\| \leqslant c\|h_1 - h_2\|$ with $c < 1$ for h small enough. So $F(h) := f'(x)^{-1}(v - o(h))$ is a contraction map; its fixed point solves $f(x+h) = y+v$. Then $f^{-1}(y+v) = f^{-1}(y)+h = f^{-1}(y) + f'(x)^{-1}v + o(v)$.

   Hence an invertible morphism is an isomorphism.

   If $f : M \to P \subseteq N$ and $\iota \circ f : M \to N$ is differentiable do not imply $f : M \to P$ differentiable (e.g. curves).

9. For a differentiable map f, the push-forward of a vector field need not be a vector field, but pull-backs of 1-forms remain so.

   For example, $f : \mathbb{S} \to \mathbb{R}^2$, $\theta \mapsto (\cos\theta, \frac{1}{2}\sin 2\theta)$ takes the tangent vectors to $(\pm 1, -1)$ at $(0, 0)$.

10. If $f : M \to N$ is a *submersion* at $y \in N$ (i.e., f' is onto $T_y N$), then the 'level surface' $f^{-1}(y) = \{ x \in M : f(x) = y \}$ is a differentiable submanifold of M. Its tangent space is $T_p f^{-1}(y) = \ker f'(x)$. (Proof: For $u\gamma(t) = Xt+o(t)$, $0 = v(q) = vfu^{-1}u\gamma(t) = f'(p)Xt+o$, so $X \in \ker f'(p)$.)

    A point $y \in N$ at which f is not a submersion is called a *critical* point. For example, $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) := (x^2 - 1)^2 + y^2$ has a critical point at $f = 1$.

    For example, for $f : M \to \mathbb{R}$, the covector $f'(p)$ annihilates tangents to the level surface $f^{-1}(a)$.

11. If $f : M \to N$ is an *immersion* at $x \in M$ (i.e., f' is 1-1 $T_{f(x)}N$), then f is a local diffeomorphism. Its tangent space is $T_{f(x)}fU = \operatorname{im} f'(x)$. A topological embedding which is an immersion is an embedding, i.e., M is diffeomorphic to fM, which is thus a differentiable sub-manifold of N.

    A point $x \in M$ at which f' is not 1-1 is called a *singular* point. f' need not be 1-1 even if f is. For example, the curve $t \mapsto (t^2, t^3)$, $\mathbb{R} \to \mathbb{R}^2$ does not map to a sub-manifold (at $(0,0)$). A 1-1 immersion need not be an embedding, e.g. $\theta \mapsto (\sin 2\theta, \sin 3\theta)$, $-\frac{7\pi}{12} < \theta < \frac{7\pi}{12}$; or $\mathbb{R} \to \mathbb{T}^2$, $(e^{2\pi it}, e^{2\pi i\alpha t})$ $(\alpha \in \mathbb{Q}^{\mathsf{c}})$.

12. Example: Let $f : \mathbb{S}^2 \to \mathbb{R}^3$, $(\theta, \phi) \mapsto \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix}$. Then $f' = \frac{\partial(x,y,z)}{\partial(\theta,\phi)} = \begin{pmatrix} -\sin\theta\cos\phi & -\cos\theta\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta & 0 \end{pmatrix}$ maps tangent 2-vectors of $\mathbb{S}^2$ to tangent 3-vectors in $\mathbb{R}^3$. The covector field $(-y, x, 0)$ is pulled back to the covector field on $\mathbb{S}^2$, $(-\cos\theta\sin\phi, \cos\theta\cos\phi, 0)\frac{\partial(x,y,z)}{\partial(\theta,\phi)} = (0, \cos^2\theta)$.

13. A local maximum/minimum point of $f : M \to \mathbb{R}$ occurs at a critical point of f, $f'(p) = 0$.

    Proof: Take $g := f \circ u^{-1} \circ \psi : \mathbb{R} \to \mathbb{R}$, $\psi(t) := tv$; then $g(0 + h) = g(0) + g'(0)h + o(h) \geqslant g(0)$, so $g'(0) = 0$ and $f'(p) = 0$.

14. *Lagrange multiplier*: A local maximum/minimum point of $f : M \to \mathbb{R}$ constrained on the sub-manifold $g(x) = c$, satisfies $f'(p) = \lambda g'(p)$.

    Proof: $c = g \circ u^{-1}(x+h) = c + g'(p)h + o(h) \ \Rightarrow\ f(p) \leqslant f \circ u^{-1}(x+h) = f(p) + f'(p)h + o(h)$, so $\ker g'(p) \subseteq \ker f'(p)$.

    More generally, for g_i constraints, $f'(p) = \sum_i \lambda_i g_i'(p)$ (using $\cap_i \ker\phi_i \subseteq \ker\psi \ \Rightarrow\ \psi \in [\![\phi_i]\!]$).

15. If $f : M \to N$ is differentiable, and $L \subsetneq N$ and $\forall x \in f^{-1}L, \operatorname{Im} f'(x) + T_{f(x)}L = T_{f(x)}N$, then $f^{-1}L$ is an embedded manifold in M; in particular if $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is differentiable, then $f(x, y) = 0$ gives a local mapping $y = g(x)$.

16. If $f : M \times N \to X$ is continuously differentiable, $f(a_0, b_0) = c$, and $f'(a_0, b_0)|_N, N \to X$ is an isomorphism, then there is a local diffeomorphism $g : U \to V$ with $a_0 \in U$, $b_0 \in V$ such that for $x \in U$, $y \in V$,
$$y = g(x) \Leftrightarrow f(x, y) = c$$

17. Locally, $TU \cong U \times X$, so a vector field can be given coordinates (x, X).

    $TM \cong M \times X$ globally iff there is a basis field (i.e., vector fields that are a basis at each point).

18. The scalar differentiable fields form a commutative C^*-algebra $C^1(M)$.

19. Since TM and TN are themselves manifolds, can take $f''(p)$ which is a tensor from X^2 to Y (more generally $f^{(r)}(x)$).

20. Vectors at the boundary of a manifold are of three types: tangent (in $T_p\partial M$), inward, or outward.

21. (Poincaré-Hopf) For a vector field on a compact manifold,

$$\sum_i \mathrm{index}_i(v) = \chi(M).$$

For example, a vector field on $\mathbb{S}^{2n}$ must vanish somewhere (hairy ball problem); so any two vector fields are linearly dependent somewhere.

22. The automorphisms of a smooth compact manifold form a Frechet-Lie group.

23. Orientable metrizable finite-dimensional manifolds admit a *volume* form, i.e., nowhere-degenerate n-form (by patching the signed Lebesgue volume form on charts). All volume forms then partition into two *orientation* types by the equivalence relation $\mu \sim \nu \Leftrightarrow \exists f > 0, \nu = f^*\mu$.

Conversely, if $0 < \mu(X_1, \ldots, X_n) = \mu(\frac{\partial u}{\partial v}Y, \ldots) = \det \frac{\partial u}{\partial v}\mu(Y, \ldots)$, then $\det \frac{\partial u}{\partial v} > 0$.

$f^*\mu = \det(f')\mu$
(since $(f^*\mu)(v_1, \ldots, v_n) = \mu(f'v_1, \ldots, f'v_n) = \det(f')\mu(v_1, \ldots, v_n)$).

24. For an orientable manifold with volume form μ, there is a dual correspondence $* : \Lambda^n M \to \Lambda_{N-n}M$, $A \mapsto \mu \cdot A$; in local coordinates, $(*A)_{i\cdots j} = \mu_{i\cdots j\bullet\cdots\bullet}A^{\bullet\cdots\bullet}$. Then $\alpha \wedge *A = (\alpha \cdot A)\mu$, $*(X \wedge Y) = (*X) \cdot Y = X \cdot (*Y)$.

μ has an inverse, such that $\mu_{i\cdots j\bullet\cdots\bullet}\mu^{i'\cdots j'\bullet\cdots\bullet} = m! \det[\delta_k^{k'}]$ where m is the number of summed variables.

Vector Field Derivatives

Flow of a vector field: the equation $\dot{x} = X_x$ has a unique local solution for $X \in C^{0,1}$, called the local flow of X, here denoted by $x(t) = \Phi_t(x(0))$. Φ_t is a diffeomorphism, with

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_t = X, \quad \Phi_{t+s} = \Phi_t\Phi_s, \quad \Phi_0 = I$$

Morphisms preserve the flow: $F(\Phi_t^X x) = \Phi_t^{F'X}F(x)$. More generally, for "time"-dependent vector fields, $\dot{x} = X(t, x)$ defines a flow $\Phi_{t,s}x(s) = x(t)$ with $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$, $\Phi_{t,t} = I$.

Lie derivative of a tensor field along the flow of a vector field:

$$\pounds_X A = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi_t^X)'^* A := \lim_{t\to 0} \frac{(\Phi_t^X)^* A_{x(t)} - A_{x(0)}}{t}$$

(i.e., $\frac{\mathrm{d}}{\mathrm{d}t}(A\circ\phi_t) = \phi_t^*\pounds_X A$). It is a tensor of the same type as A, measuring how A changes relative to X. For a t-dependent tensor, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(A_t\circ\phi_t) = \partial_t A_t + \pounds_X A_t$.

The Lie derivative of a function on a Banach space simplifies to the directional derivative

$$\pounds_X f = \lim_{t\to 0}\frac{f(p+tX)-f(p)}{t} = f'(p)X$$

The Lie derivative along a coordinate direction is denoted ∂_i.

1. The Lie derivative is a derivation on tensor fields, characterized by

$$\begin{gathered}\pounds_X(A+B) = \pounds_X A + \pounds_X B, \quad \pounds_X \lambda A = \lambda \pounds_X A,\\ \pounds_X(A\otimes B) = (\pounds_X A)\otimes B + A\otimes(\pounds_X B),\\ \pounds_X(A(Y,\alpha)) = \pounds_X A(Y,\alpha) + A(\pounds_X Y,\alpha) + A(Y,\pounds_X\alpha),\\ \pounds_X f = f'X,\end{gathered}$$

   In particular, it is preserved by differential maps F:

$$\begin{gathered}\pounds_X(fA) = (\pounds_X f)A + f\pounds_X A,\\ \pounds_X A_i^i = (\pounds_X A)_i^i\\ \pounds_X(A|_V) = \pounds_X A|_V\\ F'\pounds_X A = \pounds_{F'X}F'A\end{gathered}$$

2. $\pounds_{X_1+X_2}Y = \pounds_{X_1}Y + \pounds_{X_2}Y, \quad \pounds_{\lambda X}Y = \lambda\pounds_X Y$.

   But it is not tensorial, $\pounds_{fX}Y = f\pounds_X Y - X\pounds_Y f \neq f\pounds_X Y$. Hence $\pounds_X A$ does not depend only on the direction of X at a point p, but also on the rate of change of X; thus $X_p = Y_p$ does not imply $\pounds_X A = \pounds_Y A$ at p.

3. In local coordinates,

$$\begin{aligned}\pounds_X f &= X^i\partial_i f,\\ \pounds_X Y &= X^i\partial_i Y^j - Y^i\partial_i X^j,\\ \pounds_X\alpha &= X^i\partial_i\alpha_j + \alpha_i\partial_j X^i,\\ \pounds_X A &= X^i\partial_i A_j^k - A_j^i\partial_i X^k + A_i^k\partial_j X^i.\end{aligned}$$

   Proof: $Y_t = Y_0 + tX^i\partial_i Y + o(t)$, so $\phi_t^* Y_t = (I + t\partial_i X^j + o(t))^{-1}Y_t = Y_0 + t(X^i\partial_i Y^j - \partial_i X^j Y^i) + o(t)$.

4. $\pounds_X Y = [X,Y]$ is a Lie product,

$$\begin{gathered}\pounds_Y X = -\pounds_X Y, \quad \pounds_X X = 0,\\ \pounds_X\pounds_Y Z + \pounds_Z\pounds_X Y + \pounds_Y\pounds_Z X = 0\end{gathered}$$

   Then $[\pounds_X, \pounds_Y] = \pounds_{[X,Y]}$. So vector fields form a Lie algebra.

   The product generalizes to totally skew-symmetric contravariant tensors (Schouten product).

5. $[X,Y]=0$ iff the flows commute, i.e., $\Phi_t^X\Phi_s^Y=\Phi_s^Y\Phi_t^X$.

6. (Frobenius) A sub-bundle of the tangent space (i.e., a smooth choice of vector subspaces of T_pM) is the tangent space of some local sub-manifold (called 'integrable') $\Leftrightarrow$ it is a Lie subalgebra.

7. Example: Let $X:=\binom{-y}{x}$ on $\mathbb{R}^2$. Then its flow is $\Phi_t\binom{x_0}{y_0}=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\binom{x_0}{y_0}$.
Let $Y:=\binom{x^2}{xy}$. $\Phi_t^*Y(\Phi_t(\boldsymbol{x}_0))=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}(x_0\cos t-y_0\sin t)^2\\ (x_0\cos t-y_0\sin t)(y_0\cos t+x_0\sin t)\end{pmatrix}=$ $\begin{pmatrix}x_0(x_0\cos t-y_0\sin t)\\ y_0(x_0\cos t-y_0\sin t)\end{pmatrix}$. So the derivative of Y wrt X,
$\pounds_X Y=\lim_{t\to0}\frac{\Phi_t^*Y(\Phi_t(\boldsymbol{x}_0))-Y(\boldsymbol{x}_0)}{t}=\lim_{t\to0}\frac{1}{t}\begin{pmatrix}x_0(-x_0+x_0\cos t-y_0\sin t)\\ -y_0(x_0-x_0\cos t+y_0\sin t)\end{pmatrix}=$ $\begin{pmatrix}-x_0y_0\\ -y_0^2\end{pmatrix}$ (more easily obtained from $X^i\partial_iY^j-Y^i\partial_iX^j$).

Differential of an anti-symmetric tensor field ('form'), $\mathrm{d}:\Omega_k\to\Omega_{k+1}$ defined by

$$\mathrm{d}\alpha(X_1,\ldots,X_{k+1}):=\alpha(X_2,\ldots,X_{k+1})'X_1-\alpha(X_1,X_3,\ldots)'X_2+\ldots$$

1.
$$\mathrm{d}(\alpha+\beta)=\mathrm{d}\alpha+\mathrm{d}\beta,\quad \mathrm{d}(\lambda\alpha)=\lambda\,\mathrm{d}\alpha,$$
$$\mathrm{d}(\alpha\wedge\beta)=\begin{cases}\mathrm{d}\alpha\wedge\beta+\alpha\wedge\mathrm{d}\beta, & \alpha\in\Omega_k \text{ even}\\ \mathrm{d}\alpha\wedge\beta-\alpha\wedge\mathrm{d}\beta, & \text{odd}\end{cases}$$
$$\mathrm{d}^2\alpha=0,\quad \mathrm{d}f=f',\quad \mathrm{d}A|_V=\mathrm{d}(A|_V)$$

2. In local coordinates $\mathrm{d}\alpha=\partial_k\alpha_{ij\cdots}\,\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\cdots$,
$$\begin{aligned}\mathrm{d}f&=\partial_i f,\ \text{(grad)}\\ \mathrm{d}\alpha&=\partial_i\alpha_j-\partial_j\alpha_i,\ \text{(curl)}\\ \mathrm{d}A&=\partial_iA_{jk}+\partial_jA_{ki}+\partial_kA_{ij}\end{aligned}$$

3. For example, in $\mathbb{R}^n$, the 1-form $\mathrm{d}x_1$ is $(x_1,\ldots)\mapsto x_1$.
Change of coordinates: e.g. from Cartesian to polar, then $\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z=\mathrm{d}(r\cos\theta\cos\phi)\wedge\mathrm{d}(r\cos\theta\sin\phi)\wedge\mathrm{d}(r\sin\theta)=-r^2\cos\theta\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi$.

4. $\pounds_X=\mathrm{d}\iota_X+\iota_X\mathrm{d}$ (i.e., $\pounds_X\alpha=\mathrm{d}(X\cdot\alpha)+X\cdot\mathrm{d}\alpha$.)
Proof: by induction $\pounds_XA=\pounds_X(\mathrm{d}f\wedge B)=\pounds_X(\mathrm{d}f)\wedge B+\mathrm{d}f\wedge\pounds_XB=\mathrm{d}(\mathrm{d}f\wedge B)(X)+\mathrm{d}(\mathrm{d}f\wedge B(X))=\mathrm{d}A(X)+\mathrm{d}(A(X)))$.
Hence $\mathrm{d},\pounds_X$ commute, and for $A\in\Omega_k$
$$\begin{aligned}\mathrm{d}\alpha(X,Y)&=\pounds_X(\alpha(Y))-\pounds_Y(\alpha(X))-\alpha([X,Y]),\\ \mathrm{d}\alpha(X,Y,Z)&=\pounds_X(\alpha(Y,Z))+\pounds_Y(\alpha(Z,X))+\pounds_Z(\alpha(X,Y))\\ &\quad-\alpha([X,Y],Z)-\alpha([Z,X],Y)-\alpha([Y,Z],X),\\ \pounds_X(A\wedge B)&=(\pounds_XA)\wedge B+A\wedge(\pounds_XB),\\ \pounds_{fX}A&=f\pounds_XA+\mathrm{d}f\wedge\iota_XA,\end{aligned}$$

(since $\pounds_{fX}A = \mathrm{d}A(fX) + \mathrm{d}(A(fX)) = \cdots = f\pounds_X A + \mathrm{d}f \wedge A(X)$);

5. Morphisms preserve d, that is, $F^* \,\mathrm{d}\alpha = \mathrm{d}(F^*\alpha)$.

6. Forms that are 'closed', $\mathrm{d}\alpha = 0$, form a subspace $Z^k(M) := \ker(\mathrm{d}) \cap \Lambda^k(M)$.

   Forms that are 'exact', $\alpha = \mathrm{d}\beta$, form a subspace $B^k(M) := \mathrm{im}(\mathrm{d}) \cap \Lambda^k(M) \subseteq Z^k(M)$.

   Closed $\Leftrightarrow$ locally exact.

   Proof: Transfer the closed form to the simply connected Banach space; let $X_t(x) := x/t$, generates a flow $\phi_t(x) = tx$, so $\pounds_{X_t}\alpha = d\iota_{X_t}\alpha$, so $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\alpha = \phi_t^* d\iota_{X_t}\alpha = d\phi_t^*\iota_{X_t}\alpha$, so $\alpha - \phi_t^*\alpha = d\int_t^1 \phi_s^*\iota_{X_s}\alpha\,ds$, so $\alpha = d\int_0^1 \phi_s^*\iota_{X_s}\alpha\,ds$.

   Morphisms $f : M \to N$ map $f^* : Z^k(N) \to Z^k(M)$ and $B^k(N) \to B^k(M)$.

7. The quotient spaces $H^k := Z^k/B^k$ form a co-homology.

   Morphisms $f : M \to N$ map $f^* : H^k(N) \to H^k(M)$, invariant under homotopies (i.e., f, g homotopic implies $f^* = g^*$), i.e., $H^k(M)$ depends only on the homotopy class of M.

   The dimensions of H^k are called the manifold's Betti numbers (when finite): $\dim H^0$ is equal to the number of connected components. For $k > \dim M$, $\dim H^k = 0$. (Proof: Z^0 consists of locally constant functions; $B^0 = 0$.)

8. In oriented finite-dimensional manifolds, the volume form μ is closed, $\mathrm{d}\mu = 0$. There is a *co-differential*, or *divergence*, acting on n-vectors:

   $$\delta := \bar{*}\,\mathrm{d}* : \Lambda^n M \to \Lambda^{n-1} M.$$

   Then $\delta^2 = 0$.

   $\pounds_X \mu = \mathrm{d}(X \cdot \mu) = \mathrm{d}(*X) = (\delta X)\mu$,
   $\pounds_X f\mu = (\pounds_X f + f\delta X)\mu = \delta(fX)\mu$.

9. There is a canonical 1-form $\theta : M \to T^*T^*M$ defined by

   $$\theta_{(p,\alpha)}(v, w) = (\pi^*\alpha)(v, w) = \alpha v,$$

   where $\pi : T^*M \to M$ is the canonical projection; in local coordinates (x^i, ξ_j), $\theta = \xi_i \,\mathrm{d}x^i$. For any 1-form α, $\alpha^*\theta = \alpha$.

10. (Poincare-Hopf) For any vector field with isolated zeros, on a compact manifold without boundary, $\sum_i \mathrm{index}_{p_i}(X) = \chi(M)$. For example, if there exists a non-degenerate vector field, then $\chi(M) = 0$.

Integration

The line **integral** of a 1-form on a curve is defined by

$$\int_\gamma \alpha := \int \gamma^*\alpha = \int \alpha\cdot\gamma' = \int_{\mathbb{R}} \alpha_i(\gamma(t)){\gamma'}^i(t)\,\mathrm{d}t$$

Similarly for the integral of an n-form on an n-dimensional patch

$$\int_\Omega \omega = \int_{\phi U} \phi_*\omega\,\mathrm{d}\mu = \int_U \omega_{i\ldots j}\,\mathrm{d}x^i\ldots\mathrm{d}x^j.$$

The integral of an n-form on an orientable paracompact T_2 n-submanifold M with a given partition of unity λ_U subordinate to the charts ϕ_U is

$$\int_M \omega := \sum_U \int_{\phi U} \lambda_U \phi_*\omega\,\mathrm{d}\mu$$

(Well-defined, independent of partition of unity and charts: $\sum_U f_U = \sum_{U\cap V} f_U g_V$ and $\phi_* = (\psi\phi^{-1})^*(\psi^{-1})^* = \psi_*$.)

Cartan's theorem:

If ω is an $(n-1)$-form in an n-dim orientable (sub)manifold M, with ∂M having compatible orientation, then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega|_{\partial M}$$

($\omega|_{\partial M} = \iota^*\omega$ where $\iota : \partial M \to M$ is the embedding of the boundary.)

Proof: On a small interval, $\int_x^{x+h} \partial f\,\mathrm{d}x = \frac{1}{h}\int_x^{x+h}(f(x+h)-f(x))\,\mathrm{d}x + o(h) = [f]_x^{x+h}$. Hence, by subdividing a patch U into small cuboids, $\iint_x^{x+h} \partial_i\omega_{j\ldots}\,\mathrm{d}x^i\,\mathrm{d}x^j\ldots = [\int \omega_{j\ldots}\,\mathrm{d}x^j\ldots]_x^{x+h}$, so cancelling and extending to the boundary. Therefore $\int_M \mathrm{d}\omega = \sum_U \int_U \lambda_U\,\mathrm{d}\omega = \sum_U \int_U \mathrm{d}(\lambda_U\omega) = \sum_U \int_{\iota^{-1}U} \lambda_U\omega|_{\partial U} = \int_{\partial M} \omega_{\partial M}$ (since $\sum_U \mathrm{d}(\lambda_U\omega) = \mathrm{d}(\sum_U \lambda_U)\wedge\omega + \sum_U \lambda_U\,\mathrm{d}\omega$.)

1. *Change of variables*: If $f : M \to N$ is an isomorphism, then

$$\int_M \omega = \int_{fM} f_*\omega.$$

   (Proof: $\int_{fU} f_*\omega = \int_{\psi fU} \psi_* f_*\phi^*\phi_*\omega = \int_{\phi U} \phi_*\omega = \int_U \omega$.)

2. *Integration by parts*: $\int_M f\,\mathrm{d}\omega = \int_{\partial M} f\omega - \int_M \mathrm{d}f\wedge\omega$.

3. For a compact manifold without boundary, $\int_M \mathrm{d}\omega = 0$. In particular, a volume form μ cannot be exact.

4. Special cases:

   (a) *Fund. Th. Calculus*: For a scalar C^1 function f, $\int_\gamma f' = \int_\gamma \mathrm{d}f = [f]_{\gamma(0)}^{\gamma(1)}$. For exact 1-forms, can define $\int_p^x f'$; in particular, $\oint f' = 0$.

   (b) *Stokes*: For a surface in $\mathbb{R}^3$,

   $$\int_S (\partial_i F_j - \partial_j F_i)\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\,\mathrm{d}u\,\mathrm{d}v = \int_{\partial S} F_i x'^i\,\mathrm{d}t$$

   Green: For a closed curve in $\mathbb{R}^2$,

   $$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_{\partial A} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

   Cauchy: For a complex analytic function, $\oint f(z)\,\mathrm{d}z = 0$ (since $\oint(u+iv)(\,\mathrm{d}x + i\,\mathrm{d}y) = \int_A -(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + i(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})\,\mathrm{d}A = 0$). So the path in $\int_\gamma f(z)\,\mathrm{d}z$ can be deformed as long as f remains analytic.

   Also $\int_S \partial_j f(\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v} - \frac{\partial x^j}{\partial u}\frac{\partial x^i}{\partial v})\,\mathrm{d}u\,\mathrm{d}v = \int_{\partial S} f(x^i)'\,\mathrm{d}t$. (Take $F_i = fa_i$.)

   (c) *Gauss*: For a volume $V \subseteq \mathbb{R}^3$, $\int_V \partial_i F^i\,\mathrm{d}V = \int_{\partial V} \epsilon^i_{jk} F_i\,\mathrm{d}x^j\,\mathrm{d}x^k$.

   Also, $\int_V \partial_i f\,\mathrm{d}V = \int_{\partial V} f\epsilon_{ijk}\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\,\mathrm{d}u\,\mathrm{d}v$.

5. *Mean Value Theorem*: For a curve $\gamma : [a,b] \to M$, $\frac{1}{b-a}\int_a^b \gamma'$ belongs to the convex hull of $\gamma'[a,b]$.
   (Proof: Split the curve into small pieces so $(\gamma'(t_{n+1}) - \gamma'(t_n))\delta t_n < \epsilon$, hence integral becomes finite convex sum of $\gamma'(t_n)$.)

   For Banach spaces, $f \circ \gamma : [0,1] \to X \to \mathbb{R}$, $\gamma(t) = (1-t)a + tb$, $\exists c \in [a,b]$,

   $$f(b) - f(a) = \int_0^1 (f\circ\gamma)'\,\mathrm{d}t = f'(c)(b-a).$$

1.1 Poisson Manifolds

have an anti-symmetric bivector field π^{ij} such that

$$\pi^{i\bullet}\partial_\bullet\pi^{jk} + \pi^{j\bullet}\partial_\bullet\pi^{ki} + \pi^{k\bullet}\partial_\bullet\pi^{ij} = 0.$$

Equivalently, it has a *Poisson product* on scalar fields, $\{f,g\} := \frac{1}{2}(\,\mathrm{d}f \wedge \mathrm{d}g)\pi = \pi^{ij}\partial_i f\partial_j g$, which is a Lie product (bilinear, anti-commutative, Jacobi) that satisfies

$$\{f, gh\} = \{f,g\}h + \{f,h\}g.$$

The morphisms preserve π: $F_*\pi = \pi$, i.e., $\{f,g\}_{F(x)} = \{f\circ F, g\circ F\}_x$.
Products $M \times N$ are again Poisson.

Examples: T^*M with $\pi = \omega^{-1}$, $\omega := \mathrm{d}\theta$; in local coordinates, $\omega = \mathrm{d}\xi_i \wedge \mathrm{d}x^i$. Diffeomorphisms preserve this ω.

Every covector field is associated to a vector field: $\pi^{ij}\alpha_i$.
In particular, every scalar field is associated to its *Hamiltonian* vector field $H_f := \pi\,\mathrm{d}f = \pi^{ij}\partial_i f$.

1. The Lie product generalizes to covectors as $[\alpha,\beta] := \pounds_{\pi\alpha}\beta - \pounds_{\pi\beta}\alpha - \mathrm{d}(\pi\alpha\beta)$. Then

   (a) $\pi[\alpha,\beta] = [\pi\alpha,\pi\beta]$

   (b) $[\,\mathrm{d}f,\,\mathrm{d}g] = \mathrm{d}\{f,g\}$

   (c) $[\alpha, f\beta] = [\alpha,\beta]f + (\pounds_{\pi\alpha}f)\beta$

2. Hamiltonian vector fields satisfy

$$[H_f, H_g] = H_{\{f,g\}}, \quad \pounds_{H_f} g = \{f,g\}, \quad \pounds_{H_f}\pi = 0, \quad F_* H_f = H_{F_* f}$$

   Proof: $\pounds_{H_f} g = (\,\mathrm{d}g)(\pi\,\mathrm{d}f) = \{f,g\}$; $H_{F_*f} = \pi\,\mathrm{d}(F_*f) = F_*(\pi\,\mathrm{d}f) = F_*H_f$.

   Thus, the flow is a Poisson morphism; a function g is conserved along the flow of a Hamiltonian vector field H_f if $\{f,g\} = 0$ (in particular f is conserved).

3. At each point, the image of π^{ij}, generated by the Hamiltonian vectors, is a subspace that integrate into foliated immersed connected sub-manifolds, called *leaves*.

   Complementary to them, *Casimir functions* satisfy $H_f = 0$, so they are constant on the leaves.

4. Each leaf has a *symplectic* 2-form $\omega_{ij} = (\pi^{-1})_{ij}$; it is nowhere-degenerate, anti-symmetric, and closed $\mathrm{d}\omega = 0$ (from $[\pi,\pi] = 0$). If finite-dimensional, the dimension is even, and the leaf is orientable with $\omega^n := \omega\wedge\cdots\wedge\omega$ as a volume form (called Liouville volume form).

   (Darboux) Symplectic leaves (of same dimension) are locally isomorphic since $\omega_{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ in a local chart.

5. A Hamiltonian vector field H_f on a symplectic leaf satisfies $\omega X_f = \mathrm{d}f$ and $\{f,g\} := \omega(H_f, H_g)$. A vector field is called *locally Hamiltonian* when $\mathrm{d}(\omega X) = 0$.

6. For the cotangent manifold, a Hamiltonian vector field $H_f = \omega^{-1}\,\mathrm{d}f$, is associated with its *Lagrangian* scalar function $L := \theta(X) + f$, and its flow is a symplectic morphism,

$$\pounds_X\theta = \mathrm{d}L, \qquad \pounds_X\omega = 0$$

   Thus the volume form is preserved, $\pounds_X\omega^n = 0$. When a curve is moved along the flow of X, then $\Delta\int\theta = \int_2 L - \int_1 L$.

Proof: $\pounds_X\theta = \mathrm{d}\iota_X\theta + \iota_X\,\mathrm{d}\theta = \mathrm{d}(\theta(X)) + \omega X = \mathrm{d}(\theta(X)+E) = \mathrm{d}L$. $\pounds_X\omega = \pounds_X\,\mathrm{d}\theta = \mathrm{d}\pounds_X\theta = \mathrm{d}^2L = 0$.

For the function $f(x,\xi)$, the flow of H_f is $\dot{x} = -\frac{\partial f}{\partial\xi}$, $\dot{\xi} = \frac{\partial f}{\partial x}$.

Proof: $\omega((H_f,\xi),(X,\psi)) = \mathrm{d}f(X,\psi)$, i.e., $\psi H_f - \xi X = \frac{\partial f}{\partial H_f}X + \frac{\partial f}{\partial\xi}\psi$, so $H_f = \frac{\partial f}{\partial\xi}$ and $\xi = -\frac{\partial f}{\partial H_f}$.

2 Complex Manifolds

An **almost-complex** manifold is a differentiable manifold with a $(1,1)$-tensor field J such that $J^2 = -1$. If finite, the dimension is even (since $\det(-1) > 0$) and the manifold is oriented. The tangent space splits locally into $T_p^+M \oplus T_p^-M$ (also T_l^kM), where $Jv_+ = v_-$, $Jv_- = -v_+$.

A **complex** manifold is a differentiable manifold which is locally analytically diffeomorphic to a complex Banach space, i.e., for local charts $f = v \circ u^{-1}$, $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$,

$$f(\boldsymbol{z}+\boldsymbol{h}) = f(\boldsymbol{z})+(A+iB)(\boldsymbol{h}_x+i\boldsymbol{h}_y)+o(\boldsymbol{h}) = \begin{pmatrix}\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\\ \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})\end{pmatrix}+\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\begin{pmatrix}\boldsymbol{h}_x\\ \boldsymbol{h}_y\end{pmatrix}+o(\boldsymbol{h})$$

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}} = \frac{\partial\boldsymbol{v}}{\partial\boldsymbol{y}}, \quad \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{y}} = -\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}}.$$

i.e., $f'J = Jf'$.

Equivalently (Newlander-Nirenberg), an almost complex manifold in which $J \sim \left(\begin{smallmatrix}0 & -I\\ I & 0\end{smallmatrix}\right)$ throughout; equivalently, $[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY] = 0$. Morphisms are required to be analytic, i.e., $f'J = Jf'$, hence smooth.

Examples:

- $\mathbb{C}^n$ and its quotients, the complex tori $\mathbb{C}^n/\mathbb{Z}^{2n}$.
- Complex projective space $(\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^\times$.
- Any 2-D orientable metrizable manifold admits a complex manifold structure (by checking condition $N_J(X,Y) = 0$).
- Hopf manifolds, Stein manifolds.

1. Locally, an analytic function $f(z) = u(x,y)+iv(x,y)$ satisfies $\partial_{y_i}v = \partial_{x_i}u$, $\partial_{y_i}u = -\partial_{y_i}v$.

2. Not every almost-complex manifold is complex, but the only known examples are of (real) dimension 4.

3. In $\mathbb{C}$, an analytic (iff conformal) 1-1 map $f : A \to B$, $A,B \subseteq \mathbb{C}$, is automatically invertible with f^{-1} analytic; therefore there is an equivalence relation of regions A with f acting as morphisms.

4. Most finite-dimensional complex manifolds cannot be embedded in any $\mathbb{C}^m$ (e.g. the compact ones).

5. Any two complex surfaces are locally conformal.

   A compact complex surface is a *variety*, i.e., the zero set of some polynomial; it can be embedded in $\mathbb{CP}^3$.

6. For a complex manifold, the co-homology groups are of type $H^{p,q}$ (Chern classes), with $\dim H^k = \sum_{p+q=k} \dim H^{p,q}$ (note a k-form in complex Banach spaces is of type $dz_1 \wedge \ldots dz_p \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_q$);

7. For compact complex manifolds: (Riemann-Roch, special case of Atiyah-Singer) arithmetic genus $:= h_0 - h_1 + \ldots \pm h_N$ is an invariant (h_i :=dimension of abelian forms of degree i).

   Every compact complex surface corresponds to an irreducible polynomial $p(z, w)$ and can be obtained by gluing the sides of a polygon in $\mathbb{C}$, $\mathbb{S}^2$ or $\mathbb{H}^2$.

3 Geometry

Up to this point the manifold can only have global invariants (dimension, genus, orientability, etc.) as locally it is a Banach space; to allow for local invariants a connection of nearby tangent spaces is needed. Locally, a curve in TM takes the form of a roving vector, $u(t) = (p(t), x(t)) \in U \times X$ ($U \subseteq M$); it can be approximated in a chart by $u(t) = (p(t), x(t)) = (p, x) + (v, w)t + o(t)$, so $TTM \cong TU \times X^2$ locally. Those roving vectors which remain at one point p, i.e., $v = 0$, form a vector space called the *vertical space* at p, which is properly defined invariant of the charts as $V_u := \ker \pi'_u$. A *connection* between tangent spaces is a choice of a subspace of vectors $(p, x) \mapsto w$ along which roving vectors are considered to move in M but not in X, to form a *horizontal space* complementary to the vertical space.

A **geometry** is a differentiable manifold with a differentiable selection $u \mapsto H_u$, ($u \in TTM$), where H_u is a linear subspace of $T_u TM$ complementary to V_u, i.e., $H_u \oplus V_u = T_u TM$.

Parallel Transport

For any path $p : [-1, 1] \to M$ with $p(0) = p$, and any vector $x \in T_pM$, there is a unique differentiable path $u : [-1, 1] \to TM$, called its 'horizontal lift', whose tangent vectors are in H_u (by projection of the tangents to H_u):

$$u'(t) \in H_{u(t)}, \qquad \pi \circ u(t) = p(t), \qquad u(0) = (p, x).$$

It extends to tensors, e.g. $A_{p(t)}x(t) = (Ax)(t)$ for any $A \in B(X)$; so x or A move in a unique way along the path. This *parallel transport* of vectors and tensors is denoted $\tau_t : T_pM \to T_{p(t)}M$, a linear isomorphism 'connecting' tangent spaces at different points. Thus $B(X)$ acts on TM locally in a covariant manner, making the manifold a local *geometry* in the Klein sense. For a Banach space, $\tau_t = I$.

The **covariant derivative** ∇_X of a tensor field A in the direction of a vector X is defined by

$$\nabla_X A(p) := \lim_{t\to 0} \frac{\tau_t^{-1} A_{p(t)} - A_p}{t}$$

where τ_t is parallel transport along a path $p(t)$, $p = p(0)$, $p'(0) = X_p$.
For M a Banach space, $\nabla_X A(p) = \lim_{t\to 0} \frac{A_{p+tX_p} - A_p}{t}$.

1. The mapping $\nabla A : X \mapsto \nabla_X A$ is tensorial:

$$\nabla_{X+Y} A = \nabla_X A + \nabla_Y A, \quad \nabla_{fX} A = f\nabla_X A.$$

2. ∇ acts linearly on tensor fields as:

$$\nabla(A + B) = \nabla A + \nabla B, \quad \nabla(\lambda A) = \lambda \nabla A,$$
$$\nabla(A \otimes B) = (\nabla A) \otimes B + A \otimes (\nabla B),$$
$$\nabla A^i_i = (\nabla A)^i_i, \quad \nabla f = f'$$

   In particular $\nabla(fA) = (\nabla f)A + f\nabla A$.

3. For a local basis, $\nabla e_i = \Gamma^k_{ji} e_k : U \subseteq M \to B(X)$, called the *Christoffel* symbol. It contains the information about how the manifold curves; e.g. a Banach space is *flat*, $\Gamma = 0$ (in standard basis). In local coordinates,

$$\begin{aligned}
\nabla_X A &= X^i \nabla_i A \\
\nabla f &= \partial_i f \\
\nabla X &= \partial_i X^j + \Gamma^j_{ik} X^k, \\
\nabla \alpha &= \partial_i \alpha_j - \Gamma^k_{ij} \alpha_k, \\
\nabla A &= \partial_i A^k_j + \Gamma^k_{i\bullet} A^\bullet_j - \Gamma^\bullet_{ij} A^k_\bullet
\end{aligned}$$

   Proof: If $X = X^j e_j$, then $\nabla X = (\partial_i X^j) e_j + X^k \Gamma^j_{ik} e_j$. $\nabla(\alpha \cdot X) = (\nabla\alpha)_{ji} X^i + \alpha_i(\partial_j X^i + \Gamma^i_{jk} X^k) = (\partial_j \alpha_i) X^i + \alpha_i(\partial_j X^i)$.

4. A tensor A is parallel transported along the curve $x(t)$ when $\nabla_{x'} A = 0$. In general, the derivative of a tensor along a path is $\frac{\mathrm{d}A}{\mathrm{d}t} = \nabla_{x'} A = \frac{\mathrm{d}x^i}{\mathrm{d}t} \nabla_i A$.
   In coordinates, a parallel transported vector $X(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} X^i = -\Gamma^i_{jk} (x')^j X^k, \qquad X(0) = X_{x(0)}$$

5. A diffeomorphism $f : M \to N$ induces a connection on the manifold N, $\nabla_X A = \nabla_{f^* X} f^* A$.

6. A Poisson manifold has also a *contravariant* derivative, $\nabla_\alpha := \nabla_{\pi\alpha}$. It satisfies
$$\nabla_{f\alpha} = f\nabla_\alpha, \qquad \nabla_\alpha(fA) = f\nabla_\alpha A + (\pounds_{\pi\alpha} f)A$$

7. Higher derivatives, e.g. $\nabla^2 f = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f$ in local coordinates.

8. Parallel transport around a knot gives an element of the connection that depends only on the knot-type (i.e., a knot-invariant); the Loop transform is a mapping from connections to the knot-invariants, it generalizes the Fourier transform.

Torsion and Curvature

The **torsion** and the **curvature** tensor fields are defined by

$$\Theta(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

1. The torsion and curvature are both tensorial and anti-symmetric in X,Y.

   Proof:
$$\begin{aligned}\Theta(fX,Y) &= f\nabla_X Y - \nabla_Y(fX) - [fX,Y]\\ &= f(\nabla_X Y - \nabla_Y X - [X,Y]) = f\Theta(X,Y)\\ R(fX,Y) &= f\nabla_X(\nabla_Y) - \nabla_Y(f\nabla_X) - \nabla_{f[X,Y]+fY\nabla_X - X\nabla_Y f}\\ &= f(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}) = fR(X,Y)\end{aligned}$$

2. In coordinates,
$$\begin{aligned}\Theta^k_{ij} &= \Gamma^k_{ij} - \Gamma^k_{ji},\\ R_{ijk}{}^l &= \partial_i\Gamma^l_{jk} + \Gamma^l_{i\bullet}\Gamma^\bullet_{jk} - \partial_j\Gamma^l_{ik} - \Gamma^l_{j\bullet}\Gamma^\bullet_{ik}\end{aligned}$$

   Proof: $R_{ijk}{}^l X^i Y^j Z^k = (X^i\nabla_i)(Y^j\nabla_j)Z^k - (Y^j\nabla_j)(X^i\nabla_i)Z^k - [X,Y]^i\nabla_i Z^k$.

3. The skew-symmetric part of $\nabla^2 A$ is
$$(\nabla_i\nabla_j - \nabla_j\nabla_i)A^l_m = R^l_{ij\bullet}A^\bullet_m - \Theta^\bullet_{ij}\nabla_\bullet A^l_m - R^\bullet_{ijm}A^l_\bullet - \Theta^\bullet_{ij}\nabla_\bullet A^l_m$$

4. *Bianchi identities*

   (a) $\tilde{\mathrm{d}}\Theta = R'$
$$\nabla_{[i}\Theta^l_{jk]} = \nabla_{[i}\Theta^l_{jk]} + \Theta^\bullet_{[ij}\Theta^l_{\bullet k]} = R^l_{[ijk]},$$

   (b) $\tilde{\mathrm{d}}R = 0$
$$\nabla_{[i}R^m_{jk]l} + \Theta^\bullet_{[ij}\Theta^m_{\bullet k]l}$$

5. In finite dimensions, the *Ricci* curvature is $\mathrm{Ric}_{ij} = R_{i\bullet j}{}^{\bullet}$; in harmonic coordinates, $\mathrm{Ric}_{ij} = \frac{1}{2}\triangle g_{ij} + \cdots$.

   Scalar curvature $R := R^i_i$.

   $\frac{\mathrm{d}^2\mu}{\mathrm{d}\tau^2} = R_{ab}V^aV^b\mu$.

6. For Poisson manifolds, there is a torsion $T(\alpha,\beta) = \nabla_\alpha\beta - \nabla_\beta\alpha - [\alpha,\beta]$.

7. On a loop, a frame is parallel transported to $Ax + b$, where b is related to the torsion.

Geodesics

are curves $x(t)$ which keep the same direction, $\nabla_{\dot{x}}\dot{x} = 0$.

1. In local coordinates, $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$.

2. The parametrization of a geodesic is unique up to affine changes.

   Proof: For $x(s(t))$, $0 = \nabla_{\dot{x}}\dot{x} = \dot{s}\nabla_{x'}(\dot{s}x') = \ddot{s}x' + \dot{s}^2\nabla_{x'}x' \Leftrightarrow \ddot{s} = 0 \Leftrightarrow s(t) = at + b$.

   The geodesic curve is determined only by the symmetric part of Γ_{ij}, not the torsion.

3. Given a vector X at a point x_0, there is a unique geodesic $x(t)$ with $x(0) = x_0$, $\dot{x}(0) = X$. Two such geodesics starting from the same point meet only at isolated points.

4. The *exponential* of a vector field is the flow along the geodesic in the direction of the vectors, $\exp(tX)(p) = \gamma_X(t)$.

$$\exp((t+s)X) = \exp(tX)\exp(sX), \quad \exp(0) = I$$

   $\exp(X)\exp(Y) = \exp((X+Y) + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots)$,

   in particular, $\exp(-Y)\exp(X)\exp(Y) = \exp(X + [X,Y] + \cdots)$.

   $\exp(\frac{1}{n}X)\exp(\frac{1}{n}Y)^n x \to \exp(X+Y)x$.

4 Ψ-Riemannian Manifolds

have a smooth nowhere-degenerate symmetric bilinear form g_{ij}.

Morphisms are the locally isometric differentiable maps, i.e., $F_*g = g$.

Examples:

- Minkowski space-time, $\mathbb{R}^{n+1}$ with $g(X,Y) = -X^0Y^0 + \sum_{i=1}^n X^iY^i$.
- Any metrizable smooth manifold can be given a ψ-metric.

1. The bilinear form extends to tensors of the same type, $g(A,B)$; in coordinates, $A_{i\cdots j}B^{i\cdots j}$.

2. Any immersion $M \to N$ pulls back g from N to M via $g_M(X,Y) = g(f'(X), f'(Y))$.

3. When two curves meet, the vertically opposite angles are equal.

4. There is a duality between X and X^* via g, manifested in coordinates as raising/lowering of indices, i.e., $X_i = g_{ij}X^j$, denoted by $\sharp$, $\flat$; the dual of a transformation $A^*_{ji} = g_{j\bullet}A^\bullet_i$; the exterior derivative now applies to k-vector fields as well by $\sharp\,\mathrm{d}\flat$.

5. Connections which preserve g, i.e., $\nabla g = 0$, have a unique symmetric part, namely
$$\Gamma_{ijk} = \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$
Proof: expand $\nabla_Z g(X,Y) - \nabla_Y g(Z,X) + \nabla_X g(Y,Z)$.

   For this unique torsionless connection,

   (a) $R_{ijkl} = -R_{ijlk} = R_{klij} = K_\pi(i,j)(g_{ik}g_{jl} - g_{ij}g_{kl})$
   (K_π is called the sectional curvature);

   (b) $\pounds_X g = \nabla_i X_j + \nabla_j X_i$, $\nabla_i X_j - \nabla_j X_i = (\mathrm{d}X)_{ij} - \Theta_{ij\bullet}X^\bullet$.

   (c) $g_{ij} = \delta_{ij} - \frac{1}{N-1}R_{i\bullet j\circ}x^\bullet x^\circ + O(3)$, $\Gamma^k_{ij} = 0 + O(1)$.

   (d) $\nabla_i \det(g) = (\det(g))g^{\bullet\circ}\partial_i g_{\bullet\circ} = 2\det(g)\Gamma^\bullet_{i\bullet}$

   (e) $\nabla\cdot X = \frac{1}{\sqrt{\det g}}\partial_i(\sqrt{\det g}X^i)$.

   (f) $\nabla_i\nabla_j X^l = R^l_{ij\bullet}X^\bullet + \frac{1}{2}g^{l\bullet}(\nabla_i \pounds_X g_{j\bullet} + \nabla_j \pounds_X g_{i\bullet} - \nabla_\bullet \pounds_X g_{ij})$
   $(\nabla_i\nabla_j - \nabla_j\nabla_i)A^a_b = R_{ij\bullet}{}^a A^\bullet_b - R_{ijb}{}^\bullet A^a_\bullet$

   For a submanifold, the connection associated with the inherited g is the same as the restriction of ∇.

6. The "energy" density of a morphism $F : M \to N$ (with metrics g, h) is $\|f^*h\|^2 = h_{ab}g^{ij}\partial_i F^a \partial_j F^b$; in particular the energy density of a curve is $h(\dot{x},\dot{x}) = \dot{x}^i\dot{x}_i$.

   For any geodesic, $g(\dot{x},\dot{x})$ is constant. A vector is parallel transported along it when $X' = 0$ so $g(X,\dot{x})$ and $g(X,X)$ are constant.

   If X is a family of geodesic curves (so $X_\bullet X^\bullet = 1$, $X^\bullet\nabla_\bullet X^j = 0$) and $\pounds_X Y = 0$, then $\frac{\mathrm{d}}{\mathrm{d}s}(X_\bullet Y^\bullet) = 0$, $\frac{\mathrm{d}^2}{\mathrm{d}s^2}Y^l = R^l_{\bullet\circ\star}X^\bullet Y^\circ X^\star$.

   Proof: $\frac{\mathrm{d}}{\mathrm{d}s}(X_iY^i) = X_iX^j\nabla_jY^i = Y^j(X_i\nabla_jX^i) = \frac{1}{2}Y^j\nabla_j(X_iX^i) = 0$. Expand $X^i\nabla_i(X^j\nabla_jY^l)$.

7. $R = \sum_{i,j} K_\pi(i,j)$;
   $\mathrm{Ric}_{ij} = \frac{1}{n}Rg_{ij} + S_{ij}$ (S traceless);
   $R_{ijkl} = \frac{1}{(n-1)n}R(g_{ik}g_{jl} - g_{ij}g_{kl}) - \frac{1}{N-2}(S_{il}g_{jk} + S_{jk}g_{il} - S_{ik}g_{jl} - S_{jl}g_{ik}) + C_{ijkl}$ (C traceless, called *Weyl* tensor).

A manifold has constant curvature when $\nabla R = 0$.

An *isotropic* manifold is one with $C_{ijkl} = 0 = S_{ij}$. If the index is not 2, then the manifold is of constant curvature. When $S_{ij} = 0$ and the index is not 2, then R is constant. When the index is 0, M is locally conformal to a constant R manifold.

A manifold is *maximally symmetric* when $K_\pi(i,j) = \kappa$ constant, so $R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk})$, $\mathrm{Ric}_{ij} = (n-1)\kappa g_{ij}$, $R = n(n-1)\kappa$ $(S_{ij} = 0 = C_{ijkl})$.

A manifold is *flat* when $\kappa = 0$.

8. A vector field is called *Killing* when its flow preserves the bilinear form, $\pounds_X g = 0$.
   (a) $\pounds_X$ commutes with lowering/raising of indices,
   (b) For any geodesic, $X^i \dot{x}_i$ is constant;
   (c) $\nabla_i \nabla_j X^k = R^k_{ijl} X^l$;
   (d) Their flow are isometries: so group of isometries of manifold have the Lie algebra of Killing vector fields.

9. Two metrics are *conformally* equivalent when $\tilde{g} = e^\sigma g$, so $\tilde{g}(X,Y) = 0 \Leftrightarrow g(X,Y) = 0$.

   The angle between two vectors is invariant under local conformal mappings.

   A vector field which gives rise to a conformal flow satisfies $\pounds_X g = e^\sigma g$.

   Under a conformal mapping, $g \mapsto \tilde{g} = e^\sigma g$,

$$\begin{aligned}
\Gamma^k_{ij} &\mapsto \Gamma^k_{ij} + \tfrac{1}{2}(\delta^k_j \partial_i \sigma + \delta^k_i \partial_j \sigma - g_{ij} g^{kl} \partial_l \sigma), \\
\Theta^k_{ij} &\mapsto \Theta^k_{ij} \\
R^l_{ijk} &\mapsto R^l_{ijk} + \tfrac{1}{2}(\delta^l_j \nabla_i \nabla_k \sigma - \delta^l_i \nabla_j \nabla_k \sigma + g_{ik} \nabla_j \nabla^l \sigma - g_{jk} \nabla_i \nabla^l \sigma) \\
&\quad + \tfrac{1}{4}(\delta^l_i \nabla_j \sigma \nabla_k \sigma - \delta^l_j \nabla_i \sigma \nabla_k \sigma + g_{jk} \nabla_i \sigma \nabla^l \sigma - g_{ik} \nabla_j \sigma \nabla^l \sigma + (\delta^l_j g_{ik} - \delta^l_i g_{jk}) \nabla^\bullet \sigma \nabla_\bullet \sigma) \\
\mathrm{Ric}_{ij} &\mapsto \mathrm{Ric}_{ij} + \frac{n-2}{2} \nabla_i \nabla_j \sigma + \frac{1}{2} g_{ij} \triangle \sigma + \frac{n-2}{4}(g_{ij} \nabla^\bullet \sigma \nabla_\bullet \sigma - \nabla_i \sigma \nabla_j \sigma) \\
S_{ij} &\mapsto S_{ij} + \frac{n-2}{2} \nabla_i \nabla_j \sigma - \frac{n-2}{2n} g_{ij} \triangle \sigma + \frac{n-2}{4n} g_{ij} \nabla^\bullet \sigma \nabla_\bullet \sigma - \frac{n-2}{4} \nabla_i \sigma \nabla_j \sigma \\
R &\mapsto e^{-\sigma}(R + (n-1)\triangle \sigma + \frac{(n-1)(n-2)}{4} \nabla^\bullet \sigma \nabla_\bullet \sigma) \\
C &\mapsto C
\end{aligned}$$

10. An embedded sub-manifold $M \subsetneq \widetilde{M}$ inherits the bilinear form of $\widetilde{M}$. It gives rise to a decomposition of the tangent space: $T_p\widetilde{M} = T_pM \times T_pM^\perp$, with projections $X \mapsto (X^i, X^a)$. Then
    (a) The inherited bilinear form is $g_{ij} = \tilde{g}_{kl} \partial_i r^k \partial_j r^l$, where $r : M \to \widetilde{M}$ is the embedding; called the *first fundamental form*.

(b) The covariant derivative induces derivatives on tangent and normal vectors:

$$\widetilde{\nabla}_i X^j = (\nabla_i X^j, \mathrm{II}^a_{ij} X^j) \qquad \text{(Gauss)}$$
$$\widetilde{\nabla}_i X^a = (-\mathrm{II}^j_{ib} X^b, D_i X^a) \qquad \text{(Weingarten)}$$

where

i. $\mathrm{II}^a_{ij} = \Gamma^a_{ij}$ is called the *second fundamental form* of M; $\mathrm{II}^a_{ij} X^i X^j$ gives the normal curvature (for X unit); the eigenvalues of II^a are the *principal curvatures*. A local isometry preserves both g and II (Theorema Egregium). The *mean curvature* tensor is $H^a = \mathrm{II}^{ia}_i$;

ii. $D_i X^a = \partial_i X^a + \Gamma^a_{ib} X^b$ is called the *normal form* of M.

Example: For a surface in $\mathbb{R}^3$, $\boldsymbol{r}_i := \partial_i \boldsymbol{r}$,

$$g_{ij} = [\boldsymbol{r}_i \cdot \boldsymbol{r}_j] = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \sqrt{\det g} = \sqrt{|\boldsymbol{r}_1|^2 |\boldsymbol{r}_2|^2 - |\boldsymbol{r}_1 \cdot \boldsymbol{r}_2)|^2} = |\boldsymbol{r}_1 \times \boldsymbol{r}_2|$$

$$\boldsymbol{n} = \frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{|\boldsymbol{r}_1 \times \boldsymbol{r}_2|}, \quad \mathrm{II}_{ij} = [\partial_i \partial_j \boldsymbol{r} \cdot \boldsymbol{n}] = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

$$\partial_i \boldsymbol{r}_j = \Gamma^k_{ij} \boldsymbol{r}_k + \mathrm{II}_{ij} \boldsymbol{n}, \quad \partial_i \boldsymbol{n} = -\mathrm{II}^j_i \boldsymbol{r}_j,$$

$$[\partial_i \boldsymbol{n}] = -\mathrm{II}^j_i = -\mathrm{II}_{ik} g^{jk} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

so $\kappa = \det \mathrm{II} / \det g = \frac{LN - M^2}{EG - F^2}$. $\kappa_n := \boldsymbol{r}'' \cdot \boldsymbol{n} = L(u')^2 + 2Mu'v' + N(v')^2$.

$$\boldsymbol{r}(u,v) = \boldsymbol{a} + u\boldsymbol{r}_u + v\boldsymbol{r}_v + \tfrac{1}{2}(\boldsymbol{r}_{uu} u^2 + 2\boldsymbol{r}_{uv} uv + \boldsymbol{r}_{vv} v^2) + o(2),$$

so $y(u,v) := \langle \boldsymbol{r} - \boldsymbol{a}, \boldsymbol{n} \rangle = \frac{1}{2}\mathrm{II}(u,v) + o(2)$.

(c)

$$\begin{aligned}
\tilde{R}_{ijk}{}^{l} &= R^l_{ijk} + \mathrm{II}^a_{ik}\mathrm{II}^l_{ja} - \mathrm{II}^a_{jk}\mathrm{II}^l_{ia}, && \text{(Gauss)} \\
\tilde{R}_{ijk}{}^{a} &= \nabla_i \mathrm{II}^a_{jk} - \nabla_j \mathrm{II}^a_{ik} = \nabla_i \mathrm{II}^a_{jk} - \nabla_j \mathrm{II}^a_{ik} + 2\Gamma^{\bullet}_{jk}\mathrm{II}^a_{i\bullet} - 2\Gamma^{\bullet}_{ik}\mathrm{II}^a_{j\bullet}, && \text{(Codazzi-Mainardi)} \\
\tilde{R}_{ijb}{}^{a} &= R^{\perp}{}_{ijb}{}^{a} + \mathrm{II}^a_{i\bullet}\mathrm{II}^{\bullet}_{jb} - \mathrm{II}^a_{j\bullet}\mathrm{II}^{\bullet}_{ib} \\
&= \partial_i \Gamma^a_{jb} - \partial_j \Gamma^a_{ib} + \Gamma^a_{i\bullet}\Gamma^{\bullet}_{jb} - \Gamma^a_{j\bullet}\Gamma^{\bullet}_{ib} && \text{(Ricci)}
\end{aligned}$$

(d)

$$\begin{aligned}
\pounds_N g &= -2\mathrm{II}^a N_a, \\
\pounds_N^2 g &= 2R_{iajb} N^a N^b - 2\mathrm{II}^b_{ia}\mathrm{II}^{ac}_j N_a N_c \\
\pounds_N \mu_g &= -\mathrm{II}^i_i \mu_g
\end{aligned}$$

Proof. $\pounds_N g(X,Y) = \nabla_N g(X,Y) = g(\nabla_N X, Y) + g(X, \nabla_N Y) = g(\nabla_X N, Y) + g(X, \nabla_Y N) = -2\mathrm{II}(X,Y))$, $\pounds_N \mathrm{II}(X,Y) = \nabla_N \mathrm{II}(X,Y) = -\nabla_N g(\nabla_X N, Y) = -g(\nabla_N \nabla_X N, Y) - g(\nabla_X N, \nabla_N Y) = R(X,N,Y,N) - \mathrm{II}^2(X,Y)$. $\pounds_N \sqrt{\det g} = \frac{1}{2}\frac{\det g}{\sqrt{\det g}} g^{ij}(-2\mathrm{II}_{ij})$

Given g, II, and Γ^a_{ib}, that satisfy the Gauss, Codazzi-Mainardi and Ricci equations, then there is a (simply-connected) manifold locally embedded in $\mathbb{R}^n$ with those metrics and forms.

11. A point on a hypersurface M is *elliptical* when II is positive self-adjoint; *parabolic* when $0 \in \sigma(\mathrm{II})$ (in particular *planar* $\mathrm{II} = 0$); *hyperbolic* otherwise; *umbilical* when $\mathrm{II} = \kappa$.

    A curve on a submanifold is called *asymptotic* when $\mathrm{II}(\dot{x}, \dot{x}) = 0$; it is a *line of curvature* when $\mathrm{II}^i_j \dot{x}^j(t) = \lambda(t)\dot{x}^i(t)$.

    A *line of curvature* is one whose tangent is along eigenvectors of curvature. For surfaces, $\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ L & M & N \end{pmatrix} = 0$.

    Manifolds with zero mean curvature are called *minimal*.

12. For finite-dimensional manifolds, the signature of g is well-defined (constant on components).

    For a complete finite-dimensional manifold, the set of isometries (preserving g) form a Lie group; the stabiliser subgroup that fixes a point p is a compact subgroup.

13. For an oriented finite-dimensional manifold, there is a natural volume form

    $$\mu_{i\cdots j} = \sqrt{\det(g)}\epsilon_{i\cdots j}$$

    (because in $\mathbb{R}^n$, the unit volume of a subspace $\operatorname{im} A$ is $\sqrt{\det A^T g A}$.)
    $\sqrt{\det[g]} = 1 - \frac{1}{6}R_{ij}x^i x^j + O(3)$.

    The *Hodge dual* then becomes a map $\Lambda^k M \to \Lambda^{N-k} M$ defined by $B \wedge *A = g(A, B)\mu$; in coordinates $(*A)_{i\cdots j} = \frac{1}{k!(N-k)!}\mu_{i\cdots j}{}^{\bullet\cdots\bullet}A_{\bullet\cdots\bullet}$. It is its own inverse up to a sign $** = \operatorname{sgn}(g)$ (but $-\operatorname{sgn}(g)$ when n is even, k odd); in particular, $*\mu = 1$, $*1 = \mu$.

    It gives rise to the following operations:
    Cross product $A \times B := *(A \wedge B)$, in coordinates $\mu^{k\cdots k'}{}_{\bullet\cdots\bullet\circ\cdots\circ}A^{\bullet\cdots\bullet}B^{\circ\cdots\circ}$.
    Grad $\boldsymbol{\nabla} f = \sharp\, \mathrm{d}f$.
    Curl $\sharp * \mathrm{d}\flat : \Lambda^n M \to \Lambda^{N-n-1}M$.
    Divergence $\delta = \bar{*}\,\mathrm{d}* : \Lambda^n M \to \Lambda^{n-1}M$;
    Laplacian $\triangle := (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\,\mathrm{d} = \mathrm{d}*\mathrm{d}* + *\mathrm{d}*\mathrm{d}$. It commutes with d, δ, $*$.

    Example: In $\mathbb{R}^3$,

    (a) $\boldsymbol{X} \times \boldsymbol{Y} = \epsilon^i{}_{\bullet\circ}X^\bullet Y^\circ$; $\boldsymbol{\nabla} f = \delta^{i\bullet}\nabla_\bullet f$; $\nabla \times \boldsymbol{X} = \epsilon^{ij}{}_\bullet \nabla_j X^\bullet$; $\nabla \cdot \boldsymbol{X} = \nabla_\bullet X^\bullet$.

    (b) The identities of $\mathrm{d}(\alpha \wedge \beta)$ become
    $\nabla \cdot (f\boldsymbol{F}) = \nabla f \cdot \boldsymbol{F} + f\nabla \cdot \boldsymbol{F}$, $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{G} - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})$,
    $\nabla \times (f\boldsymbol{F}) = \nabla f \times \boldsymbol{F} + f\nabla \times \boldsymbol{F}$,
    $\nabla \times (\boldsymbol{F} \times \boldsymbol{G}) = (\nabla \cdot \boldsymbol{G} + \boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla)\boldsymbol{G}$.

(c) $\nabla\times\nabla f = 0$, $\nabla\cdot\nabla\times\boldsymbol{F} = 0$ (since $\mathrm{d}^2 = 0$).

(d) $\nabla\times\boldsymbol{F} = \boldsymbol{0} \Leftrightarrow \boldsymbol{F} = \nabla f$ locally; $\nabla\cdot\boldsymbol{F} = 0 \Leftrightarrow \boldsymbol{F} = \nabla\times\boldsymbol{A}$ locally.

14. On a submanifold of dimension k, the volume form induces a local form $\mu_M(v_1,\ldots,v_k) = \mu_{\widetilde{M}}(v_1,\ldots,v_k,N_{k+1},\ldots,N_n)$, where N_i are orthonormal vectors, normal to M.

    In particular, for $k=1$, $\flat X = (X\cdot T)\mu(\cdot,N_2,\ldots,N_n) = (X\cdot T)\mu_\gamma =: X\cdot\mathrm{d}\boldsymbol{s}$. For $k=n-1$, $*\flat X = (X\cdot N)\mu(\ldots,N) =: X\cdot\mathrm{d}\boldsymbol{S}(=\boldsymbol{n}\,\mathrm{d}S.)$.

    Cartan' theorem becomes:

    (a) *Stokes' theorem*: For a surface $S\subseteq\mathbb{R}^3$,

    $$\int_S \nabla\times\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = \int_{\partial S}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s}$$

    $\int_S \nabla f\times\mathrm{d}\boldsymbol{A} = -\int_{\partial S} f\,\mathrm{d}\boldsymbol{s}$

    (b) *Gauss*: For a compact submanifold $V\subseteq\mathbb{R}^n$,

    $$\int_V \nabla\cdot\boldsymbol{F}\,\mathrm{d}V = \int_{\partial V}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$$

    ($\boldsymbol{F}$ can be any tensor $\nabla_i A^{ijk}\mapsto A^{ijk}N_i$.)

    Also, $\int_V \nabla f\,\mathrm{d}V = \int_{\partial V} f\,\mathrm{d}\boldsymbol{S}$, $\int_V \nabla\times\boldsymbol{F}\,\mathrm{d}V = -\int_{\partial V}\boldsymbol{F}\times\mathrm{d}\boldsymbol{S}$.

    Proof: $\int_S \sharp{*}\mathrm{d}\flat\boldsymbol{F}\,\mathrm{d}A = \int_S \mathrm{d}\flat\boldsymbol{F} = \int_\gamma \flat\boldsymbol{F} = \int_\gamma \boldsymbol{F}\cdot\mathrm{d}S$.
    $\int_V *\mathrm{d}*F\mu = \int_V \mathrm{d}(F\cdot\mu) = \int_{\partial V} F\cdot N\mu_S$. For corollaries, dot with $\boldsymbol{a}$.

    For a measure concentrated on a hypersurface, $\nabla\cdot\boldsymbol{F} = \Delta\boldsymbol{F}\cdot\boldsymbol{n}$.

15. The integral of f on a piecewise smooth curve is $\int f\,\mathrm{d}\boldsymbol{s}$, where $\mathrm{d}\boldsymbol{s} = \dot{\boldsymbol{r}}\,\mathrm{d}t = \boldsymbol{t}\|\dot{\boldsymbol{r}}\|\,\mathrm{d}t$.
    The 'interval' of a curve is $\int\sqrt{g(\dot\gamma,\dot\gamma)}\,\mathrm{d}t$; invariant under reparametrizations (with $\|\dot\gamma\|\neq 0$). A geodesic is a stationary curve for energy and intervals locally.

4.1 Riemannian Manifolds

have a positive-definite g_{ij}, i.e., a real inner product.

Examples:

- $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{T}^n$, ℓ^2.
- Immersed submanifolds, inheriting g.

1. The length of a curve, $L[\gamma] := \int_\gamma \|r'(t)\|\,\mathrm{d}t$ is positive.
   The curve can be reparametrised using arclength, so $r'(s)$ is of unit length, called the *unit tangent* at $r(s)$.

2. For points on a curve-connected subset, there is a distance function

$$d(x,y) = \inf\{\, L[r(t)] : r(t) \text{ connects } r(0) = x \text{ to } r(1) = y \,\}$$

   If there is a minimum distance between points, then it is achieved by a geodesic. For a Hilbert space, $\|[r]_a^b\| = \|\int_a^b r'\| \leqslant \int_\gamma \|r'\| = L(\gamma)$, so straight lines are shortest.

   (Hopf-Rinow) For manifolds without boundary, the metric is complete $\Leftrightarrow$ it is geodesically complete; then the distance is achieved (by a geodesic). (Nomizu-Ozeki) Every Riemannian manifold is conformal to a geodesically complete manifold, and conformal to a bounded manifold (if both, then the manifold is compact).

3. (Serret-Frenet) For any curve $r(t)$, can orthogonalise $\dot{r}$, $\ddot{r}$, …, to get vectors T, N_1, N_2, …, where

$$T' = \kappa_1 N_1, \quad N_i' = -\kappa_i N_{i-1} + \kappa_{i+1} N_{i+1},$$

   i.e., $\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{N} = K\boldsymbol{N}$ with K skew-symmetric. κ_i are called the *curvatures* of the curve.

   Proof: By construction, $N_i' - \sum_{j<i} g(N_j, N_i') N_j = N_i' + \kappa_i N_{i-1} =: \kappa_{i+1} N_{i+1}$ since $\frac{\mathrm{d}}{\mathrm{d}s} g(N_j, N_i) = 0$.

   For a manifold embedded in another, the curvatures can be decomposed into tangential and normal.

Finite-Dimensional Riemannian manifolds

1. (Crofton) The length of a rectifiable curve is $\frac{1}{4}\int n(\gamma)\,\mathrm{d}\gamma$ where $n(\gamma)$ is the number of times that a geodesic γ intersects the curve and $\mathrm{d}\gamma$ is the natural measure of geodesics.

2. Any paramcompact T_2 finite-dimensional manifold can be given a Riemannian metric (since it can be immersed in some $\mathbb{R}^N$ inheriting its g). Any Riemannian manifold can be embedded in $\mathbb{R}^{2N+1}$, almost always uniquely, up to translations/isometries for $N \geqslant 3$ (Nash); even with negative Ricci curvature.

3. The Lie group of isometries at a point is $O(n)$; for an oriented manifold, it is $SO(n)$; and $H^0 = \mathbb{R}$.

4. The shortest curve between two submanifolds is a geodesic perpendicular to both.

5. Manifolds of constant curvature are locally conformally flat.

   The only simply connected complete manifolds of constant curvature are $\mathbb{S}^N$, $\mathbb{R}^N$, H^N. Isotropic hyper-surfaces must be cylindrical (only one non-zero principal curvature) or umbilic; have constant mean curvature $\|H^a\|$.

Every orientable complete 2-manifold can be given a metric with constant curvature.

Each prime compact 3-manifold without boundary can be decomposed along tori into components that are either $\mathbb{S}^3$, H^3, $\mathbb{R}^3$, $\mathbb{S}^2\times\mathbb{R}$, $\mathsf{H}^2\times\mathbb{R}$, $SL_2(\mathbb{R})$, `Nil` ($=\begin{pmatrix}1&*&*\\0&1&*\\0&0&1\end{pmatrix}$), or `Sol` ($=\begin{pmatrix}*&*\\0&*\end{pmatrix}$) (each has a unique differentiable atlas).

6. Complete manifolds with $\mathrm{Ric}>0$ have volume of balls less than those of same radius in $\mathbb{R}^n$ (Bishop).

   Complete non-compact manifolds with $K_\pi>0$ are diffeomorphic to $\mathbb{R}^N$. (Cheeger-Gromoll)

   Complete simply-connected manifolds with $K_\pi\leqslant 0$ are diffeomorphic to $\mathbb{R}^N$. (Cartan-Hadamard)

7. *Liouville surfaces* have a first fundamental form of the type $E=G=f(u)+g(v)$, $F=0$. Geodesics satisfy $f(u)\sin^2\theta-g(v)\cos^2\theta=c$, where θ is the angle it makes with a parallel.

Compact Riemannian manifolds

1. All distances are equivalent (since $a\leqslant\frac{g_1(X,X)}{g_2(X,X)}\leqslant b$). Manifold is complete, hence geodesically complete.

2. Orientable manifolds are conformal to a constant-curvature manifold.

   Proof: let $\tilde g_{ij}=\phi^m g_{ij}$, with $m=4/(N-2)$, then $\tilde R=\phi^{-m-2}(R\phi^2+m(N-1)\phi\triangle\phi+m(N-1)(mN/4-m/2-1)\nabla\phi\cdot\nabla\psi)$; let $B(\phi,\psi):=\int R\phi\psi+m(N-1)\nabla\phi\cdot\nabla\psi$, then $\|\phi\|_{L^{(\frac12-\frac1N)^{-1}}}\leqslant cB(\phi,\psi)$ where c is called the Yamabe constant; hence can solve $m(N-1)\triangle\phi+R\phi=+1,0,-1\phi^{m+1}$.

3. (Chern) For even dimensions,

$$\frac{1}{(2\pi)^{n/2}}\int_M \mathrm{Pf}(\Omega)\mu=\chi(M)+\int_{\partial M}\Phi\cdot N\mu_{n-1}$$

   where $\mathrm{Pf}(\Omega)=\sqrt{\det\Omega}$, $\Omega=\mathrm{d}\Phi$ is the curvature form. For odd dimensions, both sides of the equation reduce to 0. (Special case of Atiyah-Singer theorem with $D=\mathrm{d}+\mathrm{d}*$.)

   In particular, for no boundary, $\frac{1}{(2\pi)^{n/2}}\int\mathrm{Pf}(\Omega)=\chi(M)$.

   For $4k$-dimensions, (special case of Atiyah-Singer) $\forall\alpha,\beta\in H^2,\int\alpha\wedge\beta=\mathrm{index}(M)$, in particular $\frac{1}{24\pi^2}\int\Omega_i^j\wedge\Omega_j^i=\mathrm{index}(M)$.

   (Gauss-Bonnet) For polygons in a surface,

$$\int_M\kappa=2\pi-\sum_i(\pi-\theta_i)-\int_{\partial M}\kappa_g$$

In particular for the whole oriented compact surface $\frac{1}{2\pi}\int_M \kappa = \chi(M)$. For example, geodesic polygons, $\int_M \kappa = \sum_i \theta_i - (n-2)\pi$.

Proof: first show for small local triangles $\int_T \kappa = 2\pi - \sum_i \theta'_i - \int_{\partial M} \kappa_g$; then add up for a triangulation; thus $\sum_i \theta'_i = 2\pi E_{int} + \pi E_{ext} - \sum_i(\pi - \theta_i)$; $E_{ext} = V_{ext}$; $\sum_i 2\pi = 2\pi F$.

Corollaries:

(a) Compact surfaces of positive curvature are homeomorphic to sphere (since $\chi > 0$).

(b) If $\kappa \leqslant 0$ then any two geodesics that meet at two points must contain a 'hole'.

(c) If $\kappa > 0$, then any two closed simple geodesics intersect (else they enclose a sub-surface of $\chi = 0$)

4. For negatively curved compact manifolds, geodesics are ergodic.

5. For compact Riemannian manifolds without boundary,

   There are a countable number of diffeomorphic classes of compact Riemannian manifolds, increasing in $|K_\pi|$, diameter, decreasing in volume. (Cheeger)

   Ric $< 0 \ \Rightarrow$ isometry group is discrete (so no Killing vector fields);
   Ric $\leqslant 0 \ \Rightarrow$ every Killing vector field is parallel;
   Ric $\geqslant 0 \ \Rightarrow$ first Betti number $\leqslant n$ (Bochner);
   Ric $> 0 \ \Rightarrow$ every Killing vector field must have a 0.
   Ric $\geqslant c > 0$, complete, connected $\Rightarrow$ compact. (Bonnet-Myers)
   Ric $\geqslant \frac{n-1}{r^2} g \ \Rightarrow$ diam$(M) \leqslant \pi r$ (e.g. $K_\pi \geqslant 1/r^2$)

   Complete simply connected with $1/4 + \epsilon \leqslant K_\pi/K \leqslant 1$ is homeomorphic to $\mathbb{S}^n$.

   If $K_\pi > 0$ then each Betti number of its components is less than some C_n (Gromov).

4.1.1 Hermitian Manifolds

An **almost-Hermitian** manifold is a Riemannian manifold with a compatible almost complex structure J, $g(Ju, Jv) = g(u, v)$. (At least the symmetric part $\tilde{g}(u, v) := \frac{1}{2}(g(u, v) + g(Ju, Jv))$ preserves J.) A paracompact T_2 almost-complex manifold can be given an almost-Hermitian structure.

Equivalently, a Riemannian manifold with a skew-symmetric 2-form J_{ij}; then $J_i^j = J_{i\bullet} g^{\bullet j}$, $g_{ij} = J_{i\bullet} J_j^\bullet$, $J_{ij} = J_i^\bullet g_{\bullet j}$.

A **Hermitian** manifold is a differentiable manifold with a complex inner product h; equivalently, a complex manifold with a Riemannian metric, $g = \mathrm{Re}(h)$, $J = \mathrm{Im}(h)$.

- Complex Hilbert spaces.

- Grassmannians of $\mathbb{C}^n$ (acted upon by $GL(n)$).

There is a unique connection, called the *Chern* connection, satisfying $\nabla h = 0$, so $\nabla_X h(A,B) = h(\nabla_X A, B) + h(A, \nabla_X B)$.

Every 2-D Hermitian manifold is conformal to one with constant curvature.

4.1.2 Kähler Manifolds

An **almost-Kähler** manifold is a Riemannian manifold with a symplectic form ω, i.e., $\mathrm{d}\omega = 0$ (hence skew-symmetric); equivalently, an almost-Hermitian manifold with $\nabla J = 0$.

A **Kähler** manifold is a Hermitian manifold with a symplectic ω.

Example:

- $\mathbb{S}^6$
- Algebraic varieties embedded in a projective space.
- $\mathbb{CP}^n$ with Fubini-Study metric.

1. The connection as a Riemannian manifold coincides with the Hermitian one.
2. The Lie group of isometries is $U(n/2)$.
3. *Calabi-Yau manifolds* are compact Kähler manifolds which are Ricci flat; their Lie group is a subgroup of $SU(n/2)$.

   They are for real dimension $n = 2$ the tori, for $n = 4$ the torus $\mathbb{T}^4$ or the K_3 surfaces; (for $n = 6$ unknown).

4.1.3 Hyper-Kahler manifolds

are differentiable manifolds with a quaternionic inner-product.

Their Lie group is $USp(n/4)$; they are Ricci-flat.

The only compact ones of dimension 4 are $\mathbb{T}^4$ and the K_3 surfaces.

4.2 Lorentzian Manifolds

are pseudo-Riemannian manifolds whose g has signature $(-1, 1, \ldots, 1)$.

Vectors classify as (i) *time-like* if $g(X,X) < 0$, (ii) *null* if $g(X,X) = 0$, (iii) *space-like* if $g(X,X) > 0$.

Examples: Every non-compact metrizable manifold can be given a Lorentz metric.

1. Lorentzian manifolds with a non-degenerate time-like vector field have $\chi(M) = 0$; e.g. compact Lorentzian manifolds.
   A manifold may be complete for time-like geodesics but not for space-like geodesics, and vice-versa. A compact Lorentzian manifold with a timelike Killing vector field is geodesically complete (Romero-Sanchez).

2. *Causal* manifolds have no closed time-like/null curves; they have a nowhere-degenerate time-like vector field; they are non-compact.
   Strongly causal manifolds do not have time-like/null curves do return arbitrarily close to itself; they have distinct past/futures for distinct points (so are causal). They are conformal to a time-like/null geodesically complete manifold.
   Globally hyperbolic manifolds have Cauchy surfaces with a time-like vector field, $\Leftrightarrow$ strongly causal with the intersection of past and future of distinct points being compact. In the future and past closed sets, any two points are joined by a maximal geodesic of finite length, and is homeomorphic to $N \times \mathbb{R}$.

5 Riemannian Manifolds of Constant Curvature

The simply connected smooth Riemannian manifolds of constant sectional curvature are unique and of three types, according to curvature. Hence smooth manifolds of the same constant curvature are locally isometric.

5.1 Flat, Euclidean Space $\mathbb{R}^n$

1. The metric is δ_{ij} in Cartesian coordinates; in spherical-polar coordinates,

$$\begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & r^2 & & & \\ & & r^2\cos^2\theta_1 & & \\ \vdots & & & \ddots & \\ & & & & r^2\cos^2\theta_1\cdots\cos^2\theta_{n-2} \end{pmatrix}$$

2. The group of isometries (dim $= \frac{n(n+1)}{2}$) is generated by translations $\mathbb{R}^n$, rotations $SO(n)$, and a reflection; and similarity (scalar multiplication) is a conformal mapping.

   Translations are generated by ∇_i, rotations by $x_i\nabla_j - x_j\nabla_i$, scaling by $x^i\nabla_i$.

3. $\Gamma = 0$, parallel transport preserves the coordinates of vectors.

4. Similar shapes of dimension k have 'volume' proportional to sides^k.

   The volume of a ball of radius r is $V_n(r) = \frac{\pi^{n/2}}{(n/2)!}r^n$ ($= \frac{2^{k+1}\pi^k}{n!!}$ for $n = 2k+1$); its surface area is $A_n(r) = \frac{\mathrm{d}}{\mathrm{d}r}V_n(r) = n\frac{\pi^{n/2}}{(n/2)!}r^{n-1}$

5. Compact convex sets: (Helley) if every $N+1$ members of a family of compact convex sets have non-empty intersection, then the whole family has non-empty intersection; implies

(a) if A and B are two compact sets, and if every $N+2$ points in $A \cup B$ can be separated by hyperplane (into A and B points), then the two sets can be separated by a hyperplane;

(b) every compact convex set has a point such that chords through it are divided in a ratio $\leqslant 2N$;

(c) every open set V of dimension N has a point x such that every hyperplane through x contains at least $\frac{1}{N+1}\mathrm{vol}(V)$;

(d) if $I_1, \ldots, I_n$ are intervals in $\mathbb{R}$ such that any $N+2$ have a polynomial of degree N passing through them, then there is a polynomial of degree N passing through all the I_n;

6. (Cauchy) The average projected 'area' of a convex body (in all directions) is a constant k_{n-1} of the surface area, where k_n is the ratio of the volume to surface area of the unit ball in $\mathbb{R}^n$; e.g. in $\mathbb{R}^3$, $k_2 = \frac{1}{4}$.

   For line segments, $k_0 = \frac{2}{\pi}$, and extends to any curve (by summing); hence for a closed curve, $\pi \mathrm{diam}(\gamma) \leqslant L[\gamma]$.

Curves

1. Geodesics are the straight lines $\boldsymbol{a} + t\boldsymbol{e}$.

   There is a unique line passing through a point and parallel to a given line; any two points can be joined by a straight line; two straight lines meet in at most one point; parallel lines never meet.

   Angles: when a line meets two parallel lines (or subspaces), the alternate angles are equal and the interior angles sum to π.

2. A point on a hyper-plane has one perpendicular (normal); three parallel hyper-planes cut any (non-parallel) line in the same ratio.

3. $x_1, \ldots, x_n$ are collinear iff $\det A = 0$ where $A_{ij} = d(x_i, x_j)^2$ for $i, j \leqslant n$, $A_{i,n+1} = A_{n+1,i} = 1$ for $i \leqslant n$, $A_{n+1,n+1} = 0$.

4. The external angles of a planar polygon sum to 2π; hence the internal angles sum to $(n-2)\pi$; in particular, the angles of a triangle is π.

5. The points equidistant from two points form a perpendicular hyper-plane; the points equidistant from two hyper-planes form an angle bisector hyper-plane; points equidistant from an r-plane and an s-plane form a second order "quadric", in general.

6. Triangles with SAS, ASA or SSS (or RHS) equal are congruent; triangles with AAA (or SAS, ASA, SSS with sides proportional) are similar; AA equal iff SS equal (isosceles).

   The area of a triangle is half base times height $\frac{1}{2}\|a \times b\|$ $(= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)})$;

$AB^2 = BC^2 + AC^2 - 2AC.BC\cos C$ (in particular Pythagoras' theorem).

Medians meet at one point (of trisection).Let X be on BC, cutting it in ratio $\alpha : \beta$ ($\alpha + \beta = 1$); then $AX^2 = \alpha AB^2 + \beta AC^2 - \alpha\beta BC^2$; AX is the bisector of $A \Leftrightarrow \frac{BX}{CX} = \frac{BA}{CA} \Leftrightarrow AB.AC = \alpha AB^2 + \beta AC^2 = AX^2 + BX.XC$.

The perpendiculars from the vertices meet at a point (the orthocenter) and form the pedal triangle, bisecting its angles; removing the pedal triangle gives triangles that are similar to the original (in particular for right-angled triangles).

Isosceles triangles: median iff perpendicular.

7. Quadrilaterals: Area equals half diagonal times altitude.

   Trapezium area is mean of parallel sides times altitude.

   Parallelogram (pairs of parallel lines) $\Leftrightarrow$ opposite angles are equal $\Leftrightarrow$ opposite sides equal $\Leftrightarrow$ diagonals bisect; diagonal bisects parallelogram into congruent triangles; sum of squares on sides equals sum of squares on diagonals.

8. Envelope curves: The curve which is tangent to a family of curves $r_s(t)$ satisfies $\frac{\partial F}{\partial t}(x) = 0$, $F_t(x) = 0$, i.e., $\det[\partial_i r^j] = 0$ (since it is the limit of the intersection of two curves as $t \to t_0$).

   Evolute/Involute: the *evolute* is the envelope of the normal lines to a curve, $r + n/\kappa$ (its tangent is n); the *involute* is the unwinding of a curve along its tangent, $r(s) - st(s)$; they are inverses of each other in 2-D.

   The *pedal* curve of a curve is the projection of a fixed point to its tangent line $\boldsymbol{r} + \boldsymbol{t} \cdot (\boldsymbol{a} - \boldsymbol{r})\boldsymbol{t}$.

9. Tangent of a planar curve is of form $(\cos\theta, \sin\theta)$, so $\kappa n = t' = \theta' n$, hence $\kappa = \theta'$. A planar curve with constant curvature is a circle.

   For a closed planar curve: $\int_\gamma \kappa \, \mathrm{d}s = [\theta] = 2\pi m$ (m called the 'winding number'). If $\kappa \leqslant \kappa_0$ then diam$\geqslant 2/\kappa_0$ (else enclosed in circle touching at a point), so length$\geqslant 2\pi/\max\kappa$.

   A closed planar curve is convex iff it is simple and has $\kappa \geqslant 0$; has winding number 1; it has at least four 'vertices' (i.e., max/min of κ); area$\leqslant \frac{1}{4\pi}(\text{length})^2$.

   (Fenchel) For a simple closed curve, $\int\kappa \geqslant 2\pi$; equality holds iff convex planar. (Fary-Milnor) For a knot, $\int\kappa \geqslant 4\pi$.

10. A *spiral* is a planar curve with positive (or negative) curvature. The evolute of a curve with increasing/decreasing curvature has non-vanishing curvature; so (Kneser) $|e(t) - e(t_0)| \leqslant \int_{t_0}^t |r'n| = r(t_0) - r(t)$; so the curve cannot self-intersect and is a spiral.

Hypersurfaces

1. (Hartman-Nirenberg) A complete hyper-surface with $K_\pi = 0$ is a cylinder over some curve.

2. (Alexandrov-Hadamard) A connected compact hyper-surface with constant mean curvature is diffeomorphic to a sphere.

3. A connected umbilical hypersurface in $\mathbb{R}^3$ (all points umbilical) is part of a sphere or plane.

   Proof: $\mathrm{II} = \kappa$, so for any two indices, $N_u = \kappa \boldsymbol{g}_u$, $N_v = \kappa \boldsymbol{g}_v$, so $0 = N_{uv} - N_{vu} = \kappa_v \boldsymbol{g}_u - \kappa_u \boldsymbol{g}_v$, so $\kappa_u = 0 = \kappa_v$ by independence of tangent vectors. So κ is locally constant, hence globally. If $\kappa = 0$ then $N_u = N_v = 0$ so N is constant, so $\boldsymbol{r} \cdot N =$const. If $\kappa \neq 0$ then $\boldsymbol{r} - \frac{1}{\kappa} N$ is constant $\boldsymbol{c}$, so $|\boldsymbol{r} - \boldsymbol{c}| = \frac{1}{\kappa}$.

4. The only complete, embedded, simply-connected minimal surfaces in $\mathbb{R}^3$ are the plane and helicoid.

   For an embedded minimal surface in $\mathbb{R}^3$, each end is asymptotic to the end of a plane or catenoid.

Projective Spaces

$\mathbb{PR}^n$ is the space of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. The morphisms are the maps in $\mathsf{PGL}(n+1)$.

1. Any two subspaces generate a higher plane $u \wedge v$ (the join) and intersect in a lower plane $(u \vee v) := (u^* \wedge v^*)^*$ (meet) where $*$ is the Hodge dual, (so $u \vee v = u^* \cdot v$.

   $Tu \wedge Tv = (\det T) u \wedge v$; $T(u \vee v) = \det T^{-1} T(u) \vee T(v)$.

2. A straight line is given by the equation $u \wedge v \wedge x = 0$; 3 lines are coincident when $(u \vee v) \wedge w = 0$; Desargues' theorem.

3. $\mathbb{PR}$ is homeomorphic to $\mathbb{S}^1$, $\mathbb{PC}$ to $\mathbb{S}^2$, $\mathbb{PH}$ to $\mathbb{S}^4$, and $\mathbb{PO}$ to $\mathbb{S}^8$. $\mathbb{PO}^2$ has a metric, and is compact.

4. Projective space $\mathbb{RP}^n$; $\chi = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$.

Varieties

A variety is the complex manifold with singularities that arises from simultaneous polynomial equations.

The morphisms are polynomial functions $\boldsymbol{p} : \mathbb{C}^n \to \mathbb{C}^m$, $\boldsymbol{x} \mapsto (p_1(\boldsymbol{x}), \cdots, p_m(\boldsymbol{x}))$; or rational functions. Finite unions, intersections, and products of varieties are again varieties.

1. Polynomials of degree > 2 can be reduced to quadratic by introducing variables, e.g. $x^2yz^4 = uv$ where $u = x^2$, $v = yt$, $t = r^2$, $r = z^2$.

2. In projective space, the polynomials are homogeneous. The tangent plane is $0 = (x-a)\cdot\nabla p(a) = x\cdot\nabla p(a)$ since $a\cdot\nabla p(a) = \frac{\mathrm{d}}{\mathrm{d}t}p(ta) = k1^{k-1}p(a) = 0$.

   Points with $[\nabla_i p_j(a)]$ of less than full rank are singular, e.g. $\nabla p(a) = 0$ for one polynomial.

   A point at "infinity" is a solution of the homogeneous equation $p(\boldsymbol{x}, z) = 0$ with $z = 0$, i.e., the highest-degree part of p is zero; its "asymptote" is the direction $\boldsymbol{x}$.

3. Varieties in $\mathbb{C}^n$ can be represented by ideals of $\mathbb{C}[\boldsymbol{x}]$, since every such ideal is finitely generated (see Rings); so the intersection and union of varieties is another variety. They are not compact.

   $W \subseteq V(J) \Leftrightarrow J \trianglelefteq I(W)$, so $U \subseteq W \Rightarrow I(W) \trianglelefteq I(U)$. $I(V(J))$ is the radical of J; so radical ideals are the 'closed' ones, in 1-1 correspondence with varieties. Maximal ideals correspond to points.

   The ideal $I(V)$ decomposes into prime ideals (so that $p \in I(V) \Leftrightarrow p = \sum_{i=1}^k a_i(x)p_i(x)$ for some $p_i(x) \in I(V)$); a prime ideal corresponds to an irreducible variety, i.e., , not the union of two varieties; irreducible varieties are preserved by regular maps.

   Each irreducible variety V gives rise to a unique reduced integral domain $R(V) = \mathbb{C}[\boldsymbol{x}]/I(V)$ (called the coordinate ring of V); morphisms $V \to W$ correspond to algebra-morphisms $R(W) \to R(V)$; J/I is a prime ideal in $R(V) \Leftrightarrow J$ is prime in $\mathbb{C}[x]$; the subvarieties of V correspond to the ideals in $R(V)$.

4. Every irreducible subvariety W in V gives rise to an integral domain called the local ring $R_W(V)$, i.e., , the ring of fractions generated from $R(V)$, α/β where $\alpha, \beta \in R(V)$, but $\beta \notin I(W)/I(V)$; if $W_1 \subseteq W_2$ then $R_{W_1}(V) \subseteq R_{W_2}(V)$; $R_V(V)$ is in fact a field (of fractions); $\bigcap_{\dim W=0} R_W(V) = R(V)$.

   The free group generated by the irreducible subvarieties of V, each of dimension m, is called $G_m(V)$; each element is of the type $\sum_i n_i W_i$ (called an m-cycle); any two irreducible varieties P, Q gives $P \cap Q = \sum_i n_i W_i$ with $\dim W_i = \dim P + \dim Q - \dim V$; so $G_m(V) \cap G_n(V) = G_{m+n-\dim V}(V)$; two m-cycles are "homotopic" when there is an $m+1$-cycle whose boundaries are the two m-cycles; the m-cycles that are homotopic to 0 is called $g(V) \trianglelefteq G(V)$, and so $A_m(V) := G_m(V)/g_m(V)$, and so $A(V) := \sum_{i=0}^N A_i(V)$, which is a ring (Chow ring) but not necessarily preserved by regular maps.

5. Algebraic curves are varieties in $\mathbb{R}^2$, satisfying some polynomial $p(x, y) = 0$. They cross any line in a finite number of points (at most $\deg p$).

   Over $\mathbb{C}$, the 'curve' is a surface. They can be represented by rational functions in one variable. The genus of a non-singular algebraic curve is

$\binom{m-1}{2} - s$ where m is the degree of the polynomial and s the number of ordinary singularities (immersion overlaps).

(a) genus 0 - have 0 or ∞ number of rational points (a 'conic');

(b) genus 1 - are either 'elliptic' curves (with finitely-generated abelian group of rational points) or have no rational points;

(c) genus $\geqslant 2$ - have a finite number of rational points.

6. (Chow) Every compact projective complex manifold is a projective variety (there is algebraic structure), i.e., every compact submanifold of $\mathbb{CP}^n$ is of the type $Z(p_1, \ldots, p_n)$.

7. Generalized Riemann-Roch theorem - $\chi(V) = \dim H^0 - \dim H^1 + \ldots$.

Lattices

1. Lattices in $\mathbb{R}^n$ (discrete subgroups) are isomorphic to $\mathbb{Z}^n$ via a basis $\sum_i \mathbb{Z}v_i$. Products of lattices are themselves lattices.

   Every finite spanning set with $\langle u_i, u_j \rangle \in \mathbb{Q}$ generates a lattice. (Proof: Reduce to a basis; the rest are rational linear combinations by inverting Gram matrix; there is a largest denominator.)

   In particular, the root lattices

   (a) A_n generated by $e_i - e_{i+1}$ in $\mathbb{R}^{n+1}$; $\{\, (a_i) \in \mathbb{Z}^{n+1} : \sum_i a_i = 0 \,\}$, or $\{\, (\frac{a_1+\cdots+a_i-ia_{i+1}}{\sqrt{i(i+1)}}) \in \mathbb{R}^n : (a_i) \in \mathbb{Z}^{n+1},\ \sum_{i=1}^{n+1} a_i = 0 \,\}$.

   (b) B_n generated by e_i; $\mathbb{Z}^n$.

   (c) D_n generated by $2e_i$ and $e_i - e_{i+1}$; $\{\, (a_i) \in \mathbb{Z}^n : \sum_i a_i \in 2\mathbb{Z} \,\}$.

   (d) D_n^+, $\{\, (a_i) \in \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n : \sum_i a_i \in 2\mathbb{Z} \,\}$, two copies of D_n; only for n even; e.g. diamond lattice.

2. The densest lattices in n-dimensions for $n = 1, 2, \ldots$, are $\mathbb{Z} = A_1$, triangular A_2, D_3, D_4, D_5, E_6, E_7, E_8 $(= D_8^+)$, ...

3. The only lattice in $\mathbb{R}$ is $\mathbb{Z}$; there are two types of 'frieze patterns', with/out reflections.

4. $\mathbb{R}^2$ has 5 types of lattices, by symmetry, generated by:

parallelograms	(p1, p2),
rectangles	(pg, pm, pmg, pmm, pgg),
rhombi	(cm, cmm),
squares	(p4, p4m, p4g),
equilateral triangles	(p3, p3m1, p31m, p6, p6m)

   (17 wall-paper patterns in brackets). The one-dimensional lattice can have 7 frieze patterns. The regular tessellations are square and triangular/hexagonal.

If the lattice points are treated as complex numbers ω, then the lattice is determined by the numbers $g_2 := \sum_{\omega\neq 0}\frac{1}{\omega^4}$ and $g_3 = \sum_{\omega\neq 0}\frac{1}{\omega^6}$. By removing the degenerate lattices ($\Delta := (60g_2)^3 - 27(140g_3)^2 = 0$) and fixing the fundamental area (to factor out scaling), the space of lattices is isomorphic to $S^3 \smallsetminus \Delta$ (a sphere minus a trefoil knot). The curve $t \mapsto (e^t g_2, e^{-t} g_3)$ is called modular flow: it contains periodic orbits which are geodesic knots; horocyclic flow is $(e^t, se^t + ie^{-t})$, with complicated knots.

5. $\mathbb{R}^3$ has 14 lattices in 7 'systems':

name	types	faces
triclinic	P	3parallelograms
monoclinic	P C	2 rectangles, 1 parallelogram
orthorhombic	PIFC	3 rectangles
tetragonal	PI	1 square, 2 rectangles
rhombohedral	P	3 rhombi60°
hexagonal	P	2 rectangles, 1 rhombus60°
cubic	PIF	3squares, $D_3 = A_3 =$cF

P=primitive, I=body-centered, F=face-centered, C=base-centered; P has points at vertices of repeating cuboid, IFC have extra points at center of body, all faces, pair of opposite faces; e.g. FCC is cF, BCC is cI. There are 219 space groups (plus 11 chiral copies). The one-dimensional lattice can have 13 patterns; the 2-D lattices can have 80 patterns. The only regular tessellation is cubic.

6. $\mathbb{R}^4$ has 64 lattices of three tessellation types: cubic, 16-cell, and its dual 24-cell; has 4783 space groups.

   $\mathbb{R}^5$ has 189 lattices (all of hypercubes) and 222018 space groups. Every tessellation of $\mathbb{R}^n$, $n \geqslant 5$, is of hypercubes.

5.2 Positive, Spheres $\mathbb{S}^n$

1. Spherical-polar coordinates $\boldsymbol{r}(\theta_1, \ldots, \theta_n) = R\begin{pmatrix}\cos\theta_1\cdots\cos\theta_n\\ \cos\theta_1\cdots\sin\theta_n\\ \vdots\\ \sin\theta_1\end{pmatrix}$;

   First fundamental form
$$R^2\begin{pmatrix}1 & 0 & \cdots & \\ 0 & \cos^2\theta_1 & & \\ \vdots & & \ddots & \\ & & & \cos^2\theta_1\cdots\cos^2\theta_{n-1}\end{pmatrix}$$

   Stereographic projection $\boldsymbol{r} \mapsto \boldsymbol{x} = \frac{\boldsymbol{r}-(\boldsymbol{r}\cdot\boldsymbol{a})\boldsymbol{a}}{1\pm\boldsymbol{x}\cdot\boldsymbol{a}}, \mathbb{S}^n \smallsetminus \{\pm\boldsymbol{a}\} \to \mathbb{R}^n$; inversely, $\boldsymbol{r}(\boldsymbol{x}) = \frac{2\boldsymbol{x}+(|\boldsymbol{x}|^2-1)\boldsymbol{a}}{1+|\boldsymbol{x}|^2}$.

2. Geodesics are great circles.

3. The group of isometries is $O(n+1)$; the conformal group is $O(n+1,1)$.

4. Orientable (since its stereographic atlas of two charts has a connected overlap). $\mathbb{S}^2$ is a complex manifold, but in addition only $\mathbb{S}^6$ has an almost hermitian structure.

5. In $\mathbb{S}^2$, geodesic triangles satisfy the sine rule
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

6. The Betti numbers of $\mathbb{S}^n$ are $1, 0, \ldots, 0, 1$,
so $\chi(\mathbb{S}^n) = 1 + (-1)^n = \begin{cases} 2 & n \text{even} \\ 0 & n \text{odd} \end{cases}$.
Hence the area of a polygon in $\mathbb{S}^2$ is $\frac{1}{\kappa}(\sum_i \theta_i - (n-2)\pi)$.

7. Lattices:
$\mathbb{S}^1$ has lattices with any number of points (polygons), with symmetry I_m.
$\mathbb{S}^2$ has cylindrical lattices of any n (with cyclical symmetry nn, $*nn$, $n\times$, $n*$, or dihedral $2*n$, $22n$, $*22n$), and three proper lattices: tetrahedron, cube/octahedron, dodecahedron/icosahedron ($H_3 \supset A_5$), (each with symmetries that are chiral or full (with reflection); the tetrahedron can also have symmetry $3*2$ with inversion); space groups: 332, *332, 432, *432, 3*2, 532, *532.
$\mathbb{S}^3$ has 6 proper lattices: simplex, hypercube/orthoplex (tesseract), 24-cell (F_4), 120-cell/600-cell (H_4).
$\mathbb{S}^n$ ($n \geqslant 4$) has 3 proper lattices: simplex ($A_n = S_{n+1}$), hypercube/orthoplex ($BC_n \supset S_n$).

5.3 Negative, Hyperbolic spaces H^n

1. Poincaré model: Unit ball in $\mathbb{R}^n$, $p = \frac{\boldsymbol{x}}{1+\sqrt{1+|\boldsymbol{x}|^2}}$, with metric $\frac{\delta_{ij}}{(1-|p|^2)^2}$.
Upper-half region $\mathbb{R}^{n+}$ with $\frac{\delta_{ij}}{p_n^2}$ (nth component of p).
Hyperboloid model $\boldsymbol{r} = (\sqrt{1+|\boldsymbol{x}|^2}, \boldsymbol{x})$.
First fundamental form for surfaces: $\begin{pmatrix} 1 & 0 \\ 0 & \frac{\sinh^2 \sqrt{|\kappa|}\rho}{|\kappa|} \end{pmatrix}$.

2. H^n is diffeomorphic to $\mathbb{R}^n$. The isometries are $O^+(1,n)$ (no similarity). For $n=2$, $SO^+(1,2) \cong \mathsf{PSL}^{\pm}(\mathbb{R}^2)$.

3. Geodesics are arcs of circles in the Poincaré model, intersecting the boundary at right angles.

   There are many geodesics through a point that do not intersect another geodesic.

   The area of a geodesic polygon is $\text{Area} + \sum_i \theta_i = (n-2)\pi$. So rectangles do not exist. Triangles are congruent iff they have the same angles.

4. Every complete manifold of constant curvature is the quotient of H^n by a lattice, e.g. $\mathsf{H}^2/\mathbb{Z}^2$.

   (Hilbert) No H^2/Γ can be isometrically immersed in $\mathbb{R}^3$.

5. Lattices:

   H^2 has tessellations of any type (p,q) with p-faces and q-vertices such that $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$, symmetry $*pq2$; including *apeirogons* with an infinite number of sides.

   H^3 has 4 regular tessellations ('honeycombs'): cubic, icosahedral, dodecahedral of orders 4 or 5; and 11 regular ones touching the boundary.
   H^4 has 5 regular tessellations and 2 touching the boundary.
   H^n, $n \geqslant 5$, has no regular tessellations, except H^5 has 5 touching the boundary.

6 Lie Groups

A **Lie group** is a manifold that is also a topological group, such that the group operations are differentiable.

Lie subgroup iff subgroup submanifold. Products are again Lie groups.
Morphisms are the differentiable group morphisms.

Examples:

- Banach spaces with translation.
- The unit circle $\mathbb{S}^1$, the unit quaternions $\mathbb{S}^3$, with multiplication.
- The Heisenberg group, $\mathbb{R}^3$ with $(x_1,y_1,z_1)*(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2)$.
- $GL(\mathbb{R}^n)$, the invertible $n\times n$ matrices; since it is the open submanifold $\det^{-1}\mathbb{R}^\times$ of $\mathbb{R}^{n^2}$. It is disconnected into $GL(\mathbb{R}^n)^+$ and $GL(\mathbb{R}^n)^-$. More generally, the group of invertibles of $B(X)$ for X a Banach space.
- More generally, $GL(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$.
- The isometries of a ψ-Riemannian manifold (with compact-open topology).

A *complex Lie group* is a complex manifold with group operations that are analytic.

1. A vector field is *left-invariant* when $L_{g*}X_h = X_{gh}$ for all $g \in G$. The tangent vectors at 1 extend to all of G by $X_g = L_{g*}X_1$; it is left-invariant.

2. So tangent spaces are all isomorphic to a Lie algebra $\mathrm{Lie}(G)$ (with dimension $\dim G$), via left-invariance;
$$L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y] = [X,Y].$$
Hence the tangent manifold is trivial $TG \cong X \times G$; hence orientable, with left-invariant measure obtained by translating a volume form at one point.

3. The *exponential* map $\exp : \mathrm{Lie}(G) \to G$ is
$$\exp(X) := x(1), \quad x'(t) = X_{x(t)}, \quad x(0) = 1$$
The morphism $\exp_X : \mathbb{R} \to G$ maps to a one-parameter subgroup, e.g., $\exp(X)^{-1} = \exp(-X)$. The solution of $x'(t) = X_{x(t)}$, $x(0) = g$, for a left-invariant vector field is $x(t) = g\exp(tX)$; for right-invariant vector field, it is $\exp(tX)g$.

4. As locally connected topological groups, any connected neighborhood of 1 generates G_1, a normal clopen subgroup, via the exponentials; so G/G_1 is discrete. (If abelian, $G/G_1 \cong \mathbb{Z}^k \times H$ with H finite.) Any discrete normal subgroup of G_1 is in the center.

   First countable Lie groups have a norm and translation invariant metric, with $d(g,h) = \|g^{-1}h\|$.

   Hilbert's 5th problem: every locally Euclidean group is a Lie group.

5. If $f : G \to H$ is a morphism, then f'_p is a Lie algebra morphism; $\exp(X)$ is mapped to $\exp(f'X)$.

6. The Killing form is the symmetric quadratic form $B(X,Y) = \mathrm{tr}[X,[Y,\cdot]]$. Its kernel $\{\, X : B(X,Y) = 0, \forall Y \,\}$ is an ideal.

   For an automorphism, $B(AX,AY) = B(X,Y)$,
   for an inner automorphism, $B(AX,Y) + B(X,AY) = 0$.

   Proof: Differentiate $[e^{tA}X, e^{tA}Y] = [X,Y]$.

   Example: The Killing form of $\mathfrak{gl}(n)$ is $2(n\,\mathrm{tr}(X^2) - (\mathrm{tr}\,X)^2)$.

7. For any c, $\nabla_X Y = c[X,Y]$ is a connection.
   $\Theta(X,Y) = (2c-1)[X,Y]$, $R(X,Y) = c(c-1)\nabla_{[X,Y]}$.
   (a) $c = 1$ gives positive torsion, no curvature.
   (b) $c = \frac{1}{2}$ gives zero torsion, negative curvature.
   (c) $c = 0$ gives negative torsion, no curvature.

8. Lie groups act on homogeneous topological spaces (e.g. G/H where H is a subgroup of G); they are parallelizable, so orientable, Riemannian manifolds, with a (left)-translation-invariant volume form ω.

Structure of Lie Groups

1. There is a correspondence between the connected immersed Lie subgroups and the Lie subalgebras; $H \trianglelefteq G_1 \Leftrightarrow \mathrm{Lie}(H) \trianglelefteq \mathrm{Lie}(G)$ (as ideals).

   Proof: $\mathfrak{h} = T_I H \subseteq T_I G = \mathfrak{g}$. Conversely, $(L_x)'\mathfrak{h}$ is an integrable vector field.

   For example, $\mathrm{Lie}(Z(G)) = Z(\mathrm{Lie}(G))$, an abelian ideal of $\mathrm{Lie}(G)$. Simple connected Lie groups correspond to simple Lie algebras.

2. A Lie subgroup is embedded iff it is closed.

3. H covers $G_1 \Leftrightarrow G_1 \cong H/N$ a discrete subgroup of the center $Z(H)$. (They all have the same Lie algebra.)

   If H is simply connected and $\phi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra morphism, then there is an associated Lie group morphism $f : H \to G$ with $f' = \phi$.

   Hence Lie algebras are in 1-1 correspondence with simply connected Lie groups, up to isomorphism; all connected Lie groups with this Lie algebra are discrete subgroups of this group's center.

4. The map $G \to GL(\mathrm{Lie}(G))$, $g \mapsto A_g$, where $A_g(X) = (L_g R_{g^{-1}})'X$, is a morphism; for G_1, the kernel is $Z(G_1)$.

   For $GL(X)$ with X a Banach algebra, the Lie algebra product is $xy - yx$. (Let $L_g h := gh$, then $L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y] = [X,Y]$ if X,Y are left-invariant; so the left-invariant vector fields form a lie subalgebra of the algebra of vector fields - it is the Lie algebra of G. Right-multiplication gives $R_g h = hg$, $[R_g X, R_g Y] = -[L_g X, L_g Y]$)

   The component of 1 is an open normal subgroup generated by e^X. If X is commutative, then the component of 1 is e^X (since $e^x e^y = e^{x+y}$).

   If $X = B(H)$ (complex), then $GL(B(H))$ is connected and generated by e^X. $\det : GL(X) \to \mathbb{R}^\times$ is a morphism with $\det' = \mathrm{tr}$. $\exp(X) = e^X = \sum_n \frac{1}{n!} X^n$. $\mathrm{ad}(g)X = gXg^{-1}$.

   In $U(H)$, $\|[S,T] - 1\| \leqslant 2\|S-1\|\|T-1\|$ (since $[S,T]-1 = S^{-1}T^{-1}ST - 1 = S^{-1}T^{-1}(TS - ST)$; so $\|[S,T]-1\| = \|ST - TS\| = \|(S-1)(T-1) - (T-1)(S-1)\|$).

   The affine transformations $(Tx + a, T \in GL(X))$ form a lie group. Similarly, the isometries $Ux + a, U \in U(X)$.

5. The Lie algebra of $GL(\mathfrak{g})$ is $\mathfrak{g}$. So every Lie algebra is the tangent space of some Lie group.

   The subgroup of automorphisms has Lie subalgebra of derivations. It contains the subgroup of inner automorphisms, generated by the inner derivations, isomorphic to $G/Z(G)$.

   Proof: $A[X,Y] = [AX, AY]$, so $\mathrm{Aut}(\mathfrak{g})$ is a subgroup. If B is a derivation, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}[e^{tB}X, e^{tB}Y] = [BX,Y] + [X,BY] = B[X,Y] = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} e^{tB}[X,Y]$, so $[e^{tB}X, e^{tB}Y] = e^{tB}[X,Y]$, so $e^B \in \mathrm{Aut}(\mathfrak{g})$.

6. A Lie group is called Lie-simple (semi-simple) when its Lie algebra is simple (semi-simple); its only connected normal subroups are 1 and G_1, but it may have other discrete normal subgroups.

   The center of a semi-simple Lie group is discrete (since $\mathrm{Lie}(Z(G)) = Z(\mathfrak{g}) = 0$).

   The Lie-simple abelian connected Lie groups are $\mathbb{R}$ and its quotient $\mathbb{S}$; thus the abelian connected Lie groups are $\mathbb{R}^n \times \mathbb{T}^m$, with trivial Lie algebra. Then $\exp(X+Y) = \exp(X)\exp(Y)$.

   The Betti numbers of $\mathbb{T}^n$ are $\binom{n}{i}$.

7. G_1 contains maximal compact subgroups, all conjugates (Cartan-Iwasawa-Malcev).

   The maximal connected solvable normal subgroup of G_1 is called its *solvable radical* G_{sol}. G_1 is a product of G_{sol}, simple groups, and a discrete group. The derived subgroup $[G_{sol}, G_{sol}]$ is a nilpotent group.

8. (Iwasawa) If G is semi-simple, then G_1 is diffeomorphic to $K \times A \times N \to G_1$, with K compact, A abelian, N nilpotent, i.e., every $g \in G_1$ can be written as $g = kan$ (non-uniquely, not a group morphism; generalized QR decomposition).

   If G has nilpotent Lie algebra, then G_1 is diffeomorphic to $\mathbb{R}^n$ and is generated by $\exp\mathfrak{g}$; but the nilpotent Lie groups are not classified.

9. Connected semi-simple or nilpotent Lie groups are unimodular, i.e., $\mu(Ex) = \mu(E)$.

   Proof: $\mathrm{d}\Delta$ is a Lie algebra morphism $\mathrm{Lie}(G) \to \mathbb{R}$; if G is semi-simple then $\mathrm{d}\Delta(\mathrm{Lie}(G)) = \mathrm{d}\Delta[\mathrm{Lie}(G), \mathrm{Lie}(G)] = [\,\mathrm{d}\Delta\mathrm{Lie}(G), \mathrm{d}\Delta\mathrm{Lie}(G)] = 0$, so $\Delta = 1$; if G is nilpotent, then exp is onto G, so $\Delta(x) = |\det \mathrm{Adj} G| = |\det e^{\mathrm{ad} X}| = e^{\mathrm{tr}\,\mathrm{ad} X} = 1$.

Compact Lie Groups

1. $G_1 = \exp\mathfrak{g}$ is clopen, so G has a finite number of components (cosets of G_1); for G connected, $\chi(G) = 0$.

2. G has a metric, invariant under both left/right translations, and inversion $g \mapsto g^{-1}$.

   On $\mathfrak{g}$, $\langle X, Y\rangle := \int_G \langle g\cdot X, g\cdot Y\rangle\omega$, where ω is an invariant volume form and the dot product is any on $\mathfrak{g}$. On G, $\langle X, Y\rangle_g := \langle (L'_{g^{-1}})_g X, (L'_{g^{-1}})_g Y\rangle$.

   Then $\nabla_X Y = \frac{1}{2}[X, Y]$, $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$.

3. The Lie algebra is the sum of the center and a semi-simple algebra $[\mathfrak{g}, \mathfrak{g}]$.

   The center of a compact semi-simple Lie group is finite (since compact discrete).

A Lie algebra with trivial center cannot be the Lie algebra of a compact and a non-compact Lie group.

4. A *maximal torus* is the Lie subgroup generated by a maximal abelian subalgebra. Any two maximal tori are conjugate; together they cover all of the group. The *integral lattice* of G is $\{ X : \exp(2\pi i X) = 1 \}$ for X in this subalgebra.

   Every point of a Dynkin diagram of the Lie algebra corresponds to a reflection. The subgroup they generate is called the *Weyl* group, equal to $N_G(T)/T$, where $N_G(T)$ is the normalizer of the maximal torus.

5. The compact abelian Lie groups are $\mathbb{T}^m \times H$ where H is a finite (discrete) group.

6. Every connected compact group is the discrete quotient of a product of Lie-simple, simply connected, compact Lie groups and the maximal torus $\mathbb{T}^n$. The Lie-simple connected compact Lie groups are

simple Lie algebra	compact Lie group
$\mathbb{R}$	$\mathbb{S}$
A_n	$\mathsf{SU}(n+1)$
B_n	$\mathsf{SO}(2n+1)$
C_n	$\mathsf{Sp}(n)$
D_n	$\mathsf{SO}(2n)$
$G_2, \ldots$	G_2, F_4, E_6, E_7, E_8

   (but there may be other real compact, non-compact, or complex groups with the same Lie algebra, e.g. $\mathsf{Spin}(n)$ is the simply connected (and compact) cover of $\mathsf{SO}(n)$).

7. The finite-dimensional irreducible representations of a solvable Lie group are one-dimensional. When a semi-simple Lie group acts on a finite-dimensional vector space, the latter splits as a direct product into invariant-subspaces.

7 Examples

7.1 Curves

1. **Circle**: A diameter bisects the circle into congruent pieces. The tangent at a point is perpendicular to the radius.

   For an arc, length is $r\theta$, length of chord is $l = 2r \sin \frac{\theta}{2}$, distance of chord from center is $h = r \cos \frac{\theta}{2}$; the angle an arc makes with any point on the circle outside the arc is equal to half the angle at the center; the points with the same angle from two points form an arc of a circle; the angle between the tangent vector at an endpoint and the chord is $\theta/2$.

There is a unique circle passing through three non-collinear points (the circumscribed circle), with center the intersection of the perpendicular bisectors and diameter $AB.AC/h$; the feet of the perpendiculars from the circum-circle to the sides are collinear.
There is a unique circle passing through two points and tangent to a given line, or through a point and tangent to two lines, or tangent to three lines (e.g. the inscribed circle of a triangle, with center the intersection of the angle bisectors ie the centroid; also the escribed circles).
A triangle has a 9-point circle passing through the side midpoints, the feet of the perpendiculars from the vertices, and the midpoints of the vertices-orthocenter (its center is the midpoint between the orthocenter and the circum-center, its radius is half that of the circum-circle; the centroid, the circum-center, the orthocenter and the center of the 9-point circle are collinear).
Four points lie on a circle $\Leftrightarrow$ the opposite angles sum to π.

If two chords AB, CD meet at M (inside or outside circle), then $AM.MB = CM.MD$; in the limit, if CM is tangent, $AM.MB = CM^2$; the tangents from an external point to a circle are two and equal, and have the same angles.

Two circles are bisected by the line joining their centers; two circles meet at two points at most; the common chord of two circles and the line joining their centers bisect each other orthogonally; if two circles touch, then the centers and the point of contact are collinear.

2. **Conics**: quadratics in $\mathbb{R}^2$; $r(1 + e\cos\theta) = l$, the points whose distance from a point (the focus) is a constant multiple (eccentricity) to the distance from a line (the directrix).

   Ellipse, $e < 1$ (circle $e = 0$); $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $\binom{a\cos t}{b\sin t}$, $\frac{1}{1+t^2}\binom{a(1-t^2)}{2bt}$; has area πab; focus is eccentric by ea, $a^2 = b^2 + (ae)^2$; sum of distances from foci is constant $2a$ and reflect;
   parabola $e = 1$; $y = ax^2$; focus is $(0, a/4)$; equidistant between focus and directrix line; lines from focus reflect to parallel lines; the pedal line of the focus is a straight line; the envelope of lines $s(1-t)\boldsymbol{a} + t(1-s)\boldsymbol{b}$.
   hyperbola $e > 1$ (right hyperbola $e = \sqrt{2}$); $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $\binom{a\cosh t}{b\sinh t}$, $\frac{1}{2t}\binom{a(t^2+1)}{b(t^2-1)}$; $a^2 + b^2 = (ae)^2$; foci are at $\pm ae$; difference of distances to foci is constant $2a$; asymptotes are $y = \pm\frac{b}{a}x$; lines from one focus reflect out of other focus.

3. *Cubics*: cissoid $y^2 = xr^2$ $\frac{2t^2}{1+t^2}\binom{1}{t}$ the pedal curve and inverse of parabola in its vertex;
   strophoid $y^2(1-x) = x^2(1+x)$, folium of Descartes $x^3 + y^3 = 3xy$,

Tschirnhausen $x^3 = x^2 - 3y^2$ $(1-t^2)\binom{1}{t}$,
witch of Agnesi $y(x^2+1) = 1$, serpentine $x^2y = x - y$.

Quartics: cardioid $r = \cos\theta + 1$, conchoid $r = \sec\theta + d$, limacon $r = \cos\theta + d$ (the inverse of a conic in a focus),
lemniscate $r^2 = \cos 2\theta$, the inverse of a hyperbola in the center, product of distances from foci is constant,
Devil's curve $y^2(y^2-1) = ax^2(x^2-1)$, kappa curve $xr = y$.

Fermat parabolas $y^n = x^m$; Lamé curves $x^n + y^n = 1$; pearls of Sluze $y^n = (1-x)^k x^m$.

4. **Helix**: The curve with constant curvature $\frac{a}{a^2+b^2}$ and torsion $\frac{b}{a^2+b^2}$ in $\mathbb{R}^3$ is the circular helix $\begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix}$.

5. **Fourier Knots**: $\sum_n \boldsymbol{a}_n \cos(nt) + \boldsymbol{b}_n \sin(nt)$, e.g. Lissajous knots: one term each coordinate; figure-8 knot $\begin{pmatrix} \cos t + \cos(3t) \\ (\sin t)/2 + \sin(3t) \\ \sin(3t)/2 - \sin(6t) \end{pmatrix}$

   Roses: $r = \cos n\theta$, $\begin{pmatrix} \cos nt \cos t \\ \cos nt \sin t \\ \sin mt \end{pmatrix}$
   Astroids: $(\cos^n t, \sin^n t)$
   Hypo/Epitrochoids: $\begin{pmatrix} m\cos nt - a\cos mt \\ m\sin nt \mp a\sin mt \end{pmatrix}$, hypo/epicycloids $a = n$;
   Cycloids $\begin{pmatrix} t \\ 0 \end{pmatrix} - a\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$;
   Torus knots: $\begin{pmatrix} \cos nt(2+\cos mt) \\ \sin nt(2+\cos mt) \\ \sin mt \end{pmatrix}$, e.g. trefoil $(n,m) = (2,3)$.

6. Polynomial curves: $(p_1(t), \ldots, p_n(t))$.
   Polynomial knots: trefoil $\begin{pmatrix} t^3/3 - t \\ t^4/4 - t^2 \\ t^5/10 - t \end{pmatrix}$, figure-8 $\begin{pmatrix} t^3/5 - t \\ t^5/7 - 4t \\ t^7/32 - t^3 \end{pmatrix}$

7. **Spirals**: logarithmic $r = e^\theta$, $r = \theta^c$ (Archimedes' $c = 1$, Fermat's $c = \frac{1}{2}$, Cotes' $c = -\frac{1}{2}$, hyperbolic $c = -1$)

7.2 Surfaces

1. **Sphere** $\mathbb{S}$: simply connected, compact, constant positive curvature. Latitude-longitude map $\begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}$, $g = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$

2. **Quadrics**:

   Hyperbolic paraboloid: $\begin{pmatrix} u+v \\ u-v \\ uv \end{pmatrix}$, Ellipsoid: $\begin{pmatrix} a\cos u\cos v \\ b\cos u\sin v \\ c\sin u \end{pmatrix}$

   Hyperboloid of two sheets: $\begin{pmatrix} \tan u\cos v \\ \tan u\sin v \\ \sec u \end{pmatrix}$, of one sheet: $\begin{pmatrix} \sec u\cos v \\ \sec u\sin v \\ \tan u \end{pmatrix}$ or $\begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix} + v\begin{pmatrix} -\sin u \\ \cos u \\ 1 \end{pmatrix}$ (a ruled surface).

   Monkey saddle: $(u, v, u^3 - 3v^2u)$.

   Klein quartic $x^3y + y^3z + z^3x = 0$.

3. **Graphs** of functions: orientable (one chart).

   Implicit graphs: $\{\, x \in \mathbb{R}^n : f(x) = 0 \,\}$, the boundary of the manifold $\{\, x \in \mathbb{R}^n : f(x) > 0 \,\}$. Its tangent hyper-plane is $\nabla f(a) \cdot (x - a) = 0$.

   Example: *Scherk surface*: $\log \frac{\cos x}{\cos y}$, the only graph that is a minimal surface.
   $\sin z = \sinh x \sinh y$, a minimal surface.
   Fermat surface $x^n + y^n = z^n$.

4. **Torus** $\mathbb{T}$: compact, flat; $\begin{pmatrix} \cos u(2+\cos v) \\ \sin u(2+\cos v) \\ \sin v \end{pmatrix}$, thus $\mathbb{S} \times \mathbb{S} \cong \mathbb{T}^2$. More generally, $n\mathbb{T}$ of genus n, $\chi = 2 - 2n$.

5. **Projective space** $\mathbb{RP}^2$: $\begin{pmatrix} \cos(2u)\cos(v) \\ \cos(2u)\sin(v) \\ \frac{1}{4}(1 + 3\sin(2u) - \sin(2v) + \sin(2u)\sin(2v)) \end{pmatrix}$; compact, non-orientable, $\chi = 1$, metric $d(x, y) = \arccos \frac{x \cdot y}{\|x\|\|y\|}$, constant positive curvature, geodesics are great circles; any two lines intersect in one point; isotropic, homogeneous; acted upon by $O(n+1)/O(1)$.

   Möbius Strip: $\begin{pmatrix} \cos(2u)(2 + v\sin u) \\ \sin(2u)(2 + v\sin u) \\ v\cos u \end{pmatrix}$, $-1 < v < 1$, $0 \leqslant u < \pi$.

   More generally, $n\mathbb{P}$, $\chi = 2 - n$; $\mathbb{RP}^n$, $\chi = n + 1$.

   Klein bottle $2\mathbb{P}$: $\begin{pmatrix} \cos(2u)(\cos u\sin v - \sin u\sin(2v) + 2) \\ \sin(2u)(\cos u\sin v - \sin u\sin(2v) + 2) \\ \sin u\sin v + \cos u\sin(2v) \end{pmatrix}$

6. Addition of curves: $\boldsymbol{r}_1(u) + \boldsymbol{r}_2(v)$.

7. **Ruled surfaces**: $\boldsymbol{r}_1(u) + v\boldsymbol{r}_2(u)$; have negative curvature $\kappa \leqslant 0$, flat iff 'developable', i.e., can be reparametrized so $\boldsymbol{r}_2' \in [\![\boldsymbol{r}_1', \boldsymbol{r}_2]\!]$.

   (a) Cylinders $\boldsymbol{r}(u) + v\boldsymbol{a}$ (flat);

(b) Cones $v\boldsymbol{r}(u)$ (flat);

(c) Tangent surface: $\boldsymbol{r}(u) + v\boldsymbol{t}(u)$ (flat);

(d) Normal surface: $\boldsymbol{r}(u) + v\boldsymbol{n}(u)$; if lengths of curve and perpendicular line are a and b then area is ab.

Example: *Helicoid* $\begin{pmatrix} v\cos u \\ v\sin u \\ u \end{pmatrix}$, only ruled surface that is minimal (except plane); genus 0.

8. Tubular surface: $\boldsymbol{r}(u)+\epsilon(\boldsymbol{n}(u)\cos v+\boldsymbol{b}(u)\sin v)$. Area equals $2\pi\epsilon\times length$.

9. **Surfaces of Revolution**: $\begin{pmatrix} r(u)\cos v \\ r(u)\sin v \\ u \end{pmatrix}$; $g = \begin{pmatrix} 1+r'(u)^2 & 0 \\ 0 & r(u)^2 \end{pmatrix}$. Pappus: Area equals $2\pi\int_0^L r(s)\,\mathrm{d}s$. So are Liouville surfaces. Geodesics satisfy $r\cos\theta = c$.

   Example: *Tractroid* $r(u) = e^{-u}$, has constant negative curvature.
   Catenoid: $(\cosh u\cos v, \cosh u\sin v, u)$, the only surface of revolution that is a minimal surface; has total curvature -4π, genus 0; the only embedded minimal surface in $\mathbb{R}^3$ with finite topology and two ends.

   More generally $\begin{pmatrix} f(u)\cos v \\ g(u)\sin v \\ u \end{pmatrix}$.

10. Enneper's surface: $\begin{pmatrix} u(1-u^2/3+v^2) \\ v(1-v^2/3+u^2) \\ u^2-v^2 \end{pmatrix}$; a minimal surface; total curvature -4π; genus 0, one end.
    There are many more examples of minimal surfaces.

7.3 Lie Groups

1. $\mathbb{R}^n$, group of translations: abelian; $\exp(v) = v$; the dual of $\mathbb{R}^n$ is $\mathbb{R}^n$.

2. $\mathbb{R}^{\times n}$, group of scalings: $\exp(v) = (e^{v_i})$; 2^n connected components; 1-component is $\mathbb{R}^{+n}$. The dual of $\mathbb{R}^{\times}$ has measure $\mathrm{d}x/x$.

3. $\mathbb{S}$: $\exp(it) = \cos t + i\sin t$.

4. $\mathbb{S}^3$: $\exp(\boldsymbol{w}) = \cos|\boldsymbol{w}| + \sin|\boldsymbol{w}|\hat{\boldsymbol{w}}$; center is $\{\pm 1\}$.

5. Tori, $\mathbb{T}^n$, with pointwise multiplication; abelian; $\chi = 0$.

6. Unit Hyperbola: $\pm(\cosh t, \sinh t)$ with product $\binom{x_1}{y_1} * \binom{x_2}{y_2} := \binom{x_1x_2+y_1y_2}{x_1y_2+x_2y_1}$. $\exp(jt) = \cosh t + j\sinh t$, where $j^2 = 1$.

Matrix Groups

7. (a) $\mathsf{GL}(\mathbb{R}^n)$ **General Linear group**, the linear automorphisms of $\mathbb{R}^n$: preserve linearity;
– dimension $= n^2$;
– $\exp(A) = e^A = \sum_n \frac{1}{n!}A^n$; Lie algebra is $M_n(\mathbb{R})$;
– conjugacy classes are represented by real Jordan forms; – 2 connected components; 1-component is $\mathsf{GL}^+(n)$ ($\det > 0$);
$\mathsf{GL}(n) = \begin{cases} \mathsf{GL}^+(n) \times \{\pm 1\}, & n \text{ odd} \\ \mathsf{GL}^+(n) \rtimes \{I, P\}, & n \text{ even} \end{cases}$; the universal cover of $\mathsf{GL}^+(n)$ is not a matrix group.
– center is $Z = \mathbb{R}^\times$, and $\mathsf{GL}(n)/Z =: \mathsf{PGL}(n)$;
– not compact, maximal compact subgroups are $\mathsf{O}_Q(n)$, with Q positive definite (since it has a Haar measure and hence an invariant inner product $\langle x, y\rangle = \int_G \langle gx, gy\rangle \,\mathrm{d}g$); polar decomposition.

(b) $\mathsf{GL}(\mathbb{C}^n)$: preserve $AJ = JA$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$;
– dimension $= 2n^2$; connected;
– conjugacy classes are represented by Jordan forms;
– embedded in $\mathsf{GL}(\mathbb{R}^{2n})$ via $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$; contains $\mathsf{GL}(\mathbb{R}^n)$ as $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$; center is $\mathbb{C}^\times$, $\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$;
– not compact, maximal compact subgroups are $U_Q(n)$.

(c) $\mathsf{GL}(\mathbb{H}^n)$: preserve $\bar{J}$ (where $j(u+jv) = -v+ju$, $(u+jv)j = -\bar{v}+j\bar{u} = \bar{J}(u+jv)$); and $AJ_i = J_iA$ (with $J_2J_1 = -J_1J_2$)
(Note: $\det(AB) \neq \det A \det B$, $\operatorname{tr}(AB) \neq \operatorname{tr}(BA)$, but $(AB)^* = B^*A^*$ is true);
– dimension $4n^2$;
– embedded in $\mathsf{GL}(\mathbb{C}^{2n})$, by $A + jB \mapsto \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$ (since $Ajv = j\bar{A}v$).
– $A = Re^S$ with $R \in \mathsf{USp}(n)$, $S = S^*$; so $\mathsf{GL}(\mathbb{H}^n)$ is diffeomorphic to $\mathsf{USp}(n) \times \mathbb{R}^{n(2n-1)}$.

8. (a) $\mathsf{SL}^{\pm}(\mathbb{R}^n)$ **Special Linear group**, with $\det A = \pm 1$; shear matrices, preserve volume;
– dimension $n^2 - 1$;
– Lie algebra is of traceless matrices (semi-simple);
– 2 connected components; 1-component is $\mathsf{SL}(\mathbb{R}^n)$ ($= \ker\det$, a level surface)
– not compact; maximal compact subgroups are $SO_Q(n)$;
– there is a semi-norm $|||T||| := \ln(\|T\|\|T^{-1}\|)$ with zero set being $\mathsf{O}(\mathbb{R}^n)$;
– it is the commutator subgroup $[\mathsf{GL}(\mathbb{R}^n), \mathsf{GL}(\mathbb{R}^n)]$; $\mathsf{GL}(\mathbb{R}^n)/\mathsf{SL}(\mathbb{R}^n) = \mathbb{R}^\times$;
– center is $\mathsf{SZ}(\mathbb{R}^n) = \{I\}$ or $\{\pm I\}$; $\mathsf{SL}(\mathbb{R}^n)/\mathsf{SZ}(\mathbb{R}^n) =: \mathsf{PSL}(\mathbb{R}^n)$;

universal cover of $\mathsf{SL}(\mathbb{R}^n)$ is not a matrix group; $\mathsf{SL}(\mathbb{R}^{2n})$ contains $\mathsf{Sp}(\mathbb{R}^{2n})$;.
– For $\mathsf{SL}(\mathbb{R}^n)$, by polar decomposition $A = e^R e^B$ with B, R anti/symmetric traceless (dimensions $\binom{n+1}{2} - 1$, $\binom{n}{2}$)).
– $\mathsf{SL}(\mathbb{R}^2)$ of Möbius transformations consists of classes of rotation-like, shear-like, and inversion-like matrices (depending on their trace); the modular group $\mathsf{PSL}(2,\mathbb{Z})$ is generated by $z \mapsto -1/z$ and $z \mapsto z+1$; contains discrete subgroups (Fuchsian groups): the non-uniform lattice $\mathsf{SL}(2,\mathbb{Z})$, other subgroups $\Gamma(N)$ with $A = \pm I \mod N$, e.g. $\Gamma(2) \cong S_3$. More generally many groups (the semi-simple algebraic groups over a local field) $G(\mathbb{R})$ contain an *arithmetic group* $G(\mathbb{Z})$; all lattices in $SL(n)$ are arithmetic (Margulis). $\mathsf{SL}(\mathbb{R}^2)$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ (Iwasawa).

(b) $\mathsf{SL}(\mathbb{C}^n)$:
– dimension $2n^2 - 2$;
– simply connected; not compact, maximal compact subgroups are $SU_Q(n)$;
– polar decomposition, $A = e^R e^{iS}$ with R, S self-adjoint, traceless (each dimension $n^2 - 1$);
– center is $C_n = \{ e^{2\pi ki/n} \}$; $\mathsf{SL}(\mathbb{C}^4)$ covers $\mathsf{SO}(\mathbb{C}^6)$.
– $\mathsf{PGL}(\mathbb{C}^2) = \mathsf{PSL}(\mathbb{C}^2)$ is the group of Mobius transformations (automorphisms of Riemann sphere); elements are either (i) parabolic, i.e., translations with $a + d = 2$, have one fixed point, or (ii) elliptic (rotations) with $-2 < a + d < 2$, two fixed points, or (iii) hyperbolic (scaling) $|a + d| > 2$, or (iv) loxodromic (scaled rotation) with $a + d$ complex, two fixed points.

(c) $\mathsf{SL}(\mathbb{H}^n) = \mathsf{SL}(\mathbb{C}^{2n}) \cap \mathsf{GL}(\mathbb{H}^n)$.

9. $\mathsf{CO}(n)$ Conformal group: $\mathsf{O}(n)$ with scalings, preserve angles;
– $\mathsf{CO}(n) = \begin{cases} \mathsf{O}(n) \times \mathbb{R}^\times, & n \text{ odd}, \\ \mathsf{O}(n) \times \mathbb{R}^+, & n \text{ even} \end{cases}$; $\mathsf{CSO}(n) = \mathsf{SO}(n) \times \mathbb{R}^+$.
– $\mathsf{CO}(2) \cong \mathbb{S}^2$, $\mathsf{CO}(3)$ includes inversion as well.

10. $\mathsf{O}_Q(n) = \{ A : A^*QA = Q \}$ for a quadratic form Q;
– Lie algebra is $\mathfrak{so}_Q(n) = \{ A : A^*Q + QA = 0 \}$ (Proof: $\frac{\mathrm{d}}{\mathrm{d}t}(A(t)^*QA(t)) = A^*Q + QA$ for $A(t) = e^{tA}$);
– has subgroup $\mathsf{SO}_Q(n)$, and its 1-component $\mathsf{SO}^+_Q(n)$.
– the subgroups $O_Q(n)$ with Q positive definite, form one conjugacy class in $\mathsf{GL}(V)$;
– The Clifford group $\{ a \in \mathsf{Cl}(V,Q) : aVa^{*-1} \subseteq V \}$; it contains the subgroup of rotors $\mathsf{Pin}_Q(V) = \{ r : r^*r = 1 \}$, which covers $O_Q(V)$, and its subgroup $\mathsf{Spin}_Q(V) = \{ a \in \mathsf{Pin}_Q(V) : \det = 1 \}$, which covers $\mathsf{SO}^+_Q(V)$.

11. $\mathsf{O}(n)$ **Orthogonal group**: preserves norm/inner-product, have orthonormal columns/rows;

– dimension $\binom{n}{2}$;
– Lie algebra is set of skew-adjoint matrices (semi-simple);
– 2 connected components, compact;
– universal cover is $\mathsf{Pin}(n)$ (for $n > 2$); the center is 1 for n odd, ± 1 for n even;
– conjugacy classes consist of matrices with a number of 1s, -1s, and R_θs ($0 \leqslant \theta \leqslant \pi$), hence a product of n reflections at most; every 2×2 rotation is the product of two reflections $R_\theta = (R_{\theta/2}P)(PR_{\theta/2})$;
– has $\mathsf{O}(n-1)$ as subgroup, e.g. $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, $\mathsf{O}(n)/\mathsf{O}(n-1) \cong \mathbb{S}^n$ as manifolds.

$\mathsf{SO}(n)$, the 1-component of $\mathsf{O}(n)$, path-connected via e^{tA}; consists of 2D-rotations and possibly I; hence the maximal torus is $\mathbb{T}^{\lfloor n/2 \rfloor}$;
– for n odd, $\mathsf{SO}(n)$ is simple; for n even, $\mathsf{SO}(n)$ has largest normal subgroup $\{\pm 1\}$ (except $\mathsf{SO}(4)$ which also has $\mathbb{S}^3_L$ and $\mathbb{S}^3_R$);
– acts transitively on $\mathbb{S}^{n-1}$ with stabilizer group at a point being $\mathsf{SO}(n-1)$, so $\mathsf{SO}(n)/\mathsf{SO}(n-1) = \mathbb{S}^{n-1}$;
– its universal cover is $\mathsf{Spin}(n)$ for $n > 2$, with center $\mathbb{Z}_2$ if n is odd, $\mathbb{Z}_4$ if $n = 2 \pmod 4$, $\mathbb{Z}_2^2$ if $n = 0 \pmod 4$.

The lattices (discrete/finite subgroups) of $\mathsf{O}(n)$ generated by reflections are the Coxeter groups. A rotation/reflection in such a subgroup can be represented by an integer matrix such that $A^r = I$; if $r = \prod_i p_i^k$, the first dimension which allows such a rotation/reflection is $\sum_i \psi(p_i^k)$ where $\psi(p_i^k) = (p_i^k - p_i^{k-1})$ (and $\psi(2) = 0$) (so the dimension is even). – $\mathsf{SO}(2) \cong \mathbb{S}^1$ (via $R_\theta \mapsto e^{i\theta}$); contains all cyclic groups C_n; dual of $\mathbb{S}^1$ is $\mathbb{Z}$.

$\mathsf{SO}(3)$; consists of rotations $\cos\theta I + (1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^* + \sin\theta\boldsymbol{a}\times$, where $\boldsymbol{a}$ is the axis vector; $\cong \mathbb{RP}^3$ (solid ball with antipodes identified); the irreducible representations of $\mathsf{SO}(3)$ are the $n+1$ symmetric spinors $\psi_{(A} \cdots \phi_{B)}$ $(A, \ldots, B = 0, 1)$. $\mathsf{Spin}(3) \cong \mathbb{S}^3 \cong \mathsf{USp}(1)$ acts on $\mathsf{SO}(3)$ (by unit non-real quaternion x acting on vectors $x\boldsymbol{v}x^{-1}$).

Banach-Tarski: $\mathsf{SO}(3)$ contains the free subgroup generated by two rotations P (through $\arccos(1/3)$ in xy plane), Q (through $\arccos(1/3)$ in yz plane), acting on $\mathbb{S}^2$; the fixed points are countable, divide the rest into orbits M, and let $X_w := \{ wbx : \exists b, x \in M \}$; then $M = X_a \cup aX_{a^{-1}} = X_b \cup bX_{b^{-1}}$, even though all these sets are as large as M.

$\mathsf{Spin}(4) \cong \mathbb{S}^3 \times \mathbb{S}^3$ (via $v \mapsto qv\bar{r}$, with q, r non-real unit quaternions), $\mathsf{Spin}(5) \cong \mathsf{USp}(2)$, $\mathsf{Spin}(6) \cong \mathsf{SU}(4)$.

12. $\mathsf{O}(p,q) := \mathsf{O}(\mathbb{R}^{p+q})$; $\mathsf{O}(p,q) \cong \mathsf{O}(q,p)$; $\mathsf{O}(3,1)$ is called the Lorentz group;
– dimension $\binom{n}{2}$;
– Lie algebra is semi-simple
– 4 connected components; 1-component is $\mathsf{SO}^+(p,q)$;
– not compact; maximal compact subgroup is $\mathsf{O}(p) \times \mathsf{O}(q)$.
– universal cover is $\mathsf{Pin}(p,q)$; for n odd, $\mathsf{Spin}(n,1) \cong \mathsf{SL}(\mathbb{C}^{(n+1)/2})$, via $A_{\mu\nu} \mapsto A_{\mu\nu}\sigma^\mu_{AB'}\sigma^\nu_{CD'}$, $g_{\mu\nu} = \epsilon_{AC}\epsilon_{B'D'}$; hence $X_A = \epsilon_{AB}X^B$, $\nabla_\mu X^A =$

$\partial_\mu X^A + \Gamma^A_{B\mu} X^B$, $\nabla_\mu \epsilon_{AB} = 0$, $\nabla_\mu \sigma^\nu_{AB'} = 0$.

13. (a) $\mathsf{Sp}(\mathbb{R}^{2n})$, **Symplectic group**: preserve skew-symmetric form $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $A^\top \Omega A = \Omega$ (so a subgroup of $\mathsf{SL}(2n)$); generated by the subgroup $\begin{pmatrix} A & 0 \\ 0 & A^{-\top} \end{pmatrix}$ ($A \in \mathsf{GL}(n)$), the subgroup $\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$ ($A > 0$), and J; $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}^\top$;
– dimension $\binom{2n+1}{2}$; connected;
– Lie algebra: $A\Omega + \Omega A^\top = 0$, $\begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix}$, $B^\top = B$, $C^\top = C$;
– not compact; the unique compact real group is $\mathsf{USp}(\mathbb{R}^{2n})$; maximal compact subgroup is $\mathsf{U}(n)$
– center is $\pm I$; – polar decomposition $A = Re^S$ with $R \in \mathsf{U}(n)$, $S \in \mathfrak{sp}(\mathbb{R}^n)$, $S^* = S$; diagonalization $A = P_1 D P_2$ where D is diagonal (with λ_i, λ_i^{-1}) and P_i orthogonal.

(b) $\mathsf{Sp}(\mathbb{C}^{2n})$:
– simply connected;
– $A = Re^S$ with $R \in \mathsf{USp}(n, \mathbb{C})$, $S \in \mathfrak{sp}(n, \mathbb{C})$, $S^* = S$.
– maximal torus consists of diagonal matrices $\begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$;
– not compact; maximal compact subgroup is $\mathsf{USp}(n)$.

(c) $\mathsf{Sp}(p, q) := \mathsf{Sp}(\mathbb{C}^{2n}) \cap \mathsf{U}(p, q)$.

14. $\mathsf{U}(\mathbb{C}^n)$ **Unitary group**: preserve the inner-product on $\mathbb{C}^n$, $U^*U = 1$, i.e., $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $|\det A| = 1$; $\mathsf{U}(n) = \mathsf{O}(2n) \cap \mathsf{Sp}(\mathbb{R}^{2n}) = \mathsf{O}(2n) \cap \mathsf{GL}(\mathbb{C}^n)$ (by embedding of $\mathsf{GL}(\mathbb{C}^n)$);
– Lie algebra $\mathfrak{u}(n)$ is not a complex Lie algebra;
– dimension n^2; connected (via diagonal matrices $(e^{it\lambda})$), compact;
– diffeomorphic to $\mathsf{SU}(n) \times \mathbb{S}^1$;
– center is $e^{i\mathbb{R}}$; maximal torus is $\mathbb{T}^n$, consisting of the diagonal unitaries; conjugacy classes are classified by the spectra, i.e., , n-subsets of $e^{i\mathbb{R}}$;
– there is an outer automorphism $A \mapsto \bar{A}$ for $n \geqslant 3$;
– $\mathsf{U}(n)$ contains $\mathsf{O}(n)$; $\mathsf{U}(1) = \mathsf{O}(2)$;
– $\mathsf{U}(2)$ consists of matrices $\begin{pmatrix} a & -e^{i\theta}\bar{b} \\ b & \bar{a} \end{pmatrix} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = R_{(a,b)} F_\theta$;.

$\mathsf{SU}(\mathbb{C}^n)$: have $\det = 1$;
– simply connected; Lie algebra is semi-simple;
– dimension $n^2 - 1$,
– maximal torus consists of diagonal elements;
– contains $\mathsf{O}(\mathbb{R}^n)$, $\mathsf{USp}(n)$, and $\mathsf{SU}(m) \times \mathsf{SU}(n-m) \times \mathsf{U}(1)$;
– $\mathsf{SU}(2) \cong \mathbb{S}^3 \cong \mathsf{USp}(1)$ (for $(z, w) \in \mathbb{S}^3$ (i.e., $|z|^2 + |w|^2 = 1$), $(z, w) \mapsto R_{(z,w)}$) (the latitudes $t = c$ correspond to $\operatorname{tr} A = 2c$; longitudes to diagonal matrices in $\mathsf{SU}(2)$).

15. $\mathsf{USp}(n) = \mathsf{U}(\mathbb{H}^n)$: preserve inner-product of $\mathbb{H}^n$, $\langle q_1u, q_2v\rangle = \bar{q}_1\langle u, v\rangle q_2$ (extends the ip of $\mathbb{C}^{2n}$, so preserves complex inner-product and acts on unit sphere in $\mathbb{H}^n$);
– $\mathsf{USp}(n) = \mathsf{GL}(\mathbb{H}^n) \cap \mathsf{U}(\mathbb{C}^{2n}) = \mathsf{Sp}(\mathbb{C}^{2n}) \cap \mathsf{U}(\mathbb{C}^{2n}) = \mathsf{GL}(\mathbb{H}^n) \cap \mathsf{SU}(\mathbb{C}^{2n})$;
– dimension $\binom{2n+1}{2}$; Lie algebra is semi-simple;
– simply connected, compact;
– maximal torus of diagonal elements;
– contains $\mathsf{O}(n)$, $\mathsf{USp}(n-1)$, and $\mathsf{U}(n)$; $\mathsf{USp}(1) \cong \mathbb{S}^3$;
– $\mathsf{USp}(n) \subset \mathsf{SU}(2n) \subset \mathsf{U}(2n) \cap \mathsf{SO}(4n) \subset \mathsf{O}(4n)$

16. $\mathsf{Affine}(n)$: $\mathsf{GL}(n) \rtimes \mathbb{R}^n$ with $(A, a) * (B, b) = (AB, a + Ab)$;
– dimension $n(n + 1)$;
– embedded in $\mathsf{GL}(n+1)$ as $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$; every subgroup of $\mathsf{GL}(n)$ has an affine version, e.g. $\mathsf{Affine}^+(n)$, $\mathsf{CO}(n) \rtimes \mathbb{R}^n$, with scaling.
– $\mathsf{Affine}(1)$ is solvable but not nilpotent, not unimodular.

– **Euclidean group** $\mathsf{E}(n) := \mathsf{O}(n) \rtimes \mathbb{R}^n$, isometries of $\mathbb{R}^n$: translations, rotations, reflections/inversions, glides, screws;
– dimension $\binom{n+1}{2}$;
– 1-component of the orientation-preserving isometries $\mathsf{E}^+(n)$ ($= \mathsf{SO}(n) \rtimes \mathbb{R}^n$, screws).

17. $\mathsf{IUT}(\mathbb{R}^n)$ of invertible upper triangular matrices.
– dimension $\binom{n+1}{2}$
– Lie algebra of upper triangular matrices;
– 2^n connected components, according to sign of diagonal elements; 1-component is $\mathsf{IUT}^+(n)$ with positive diagonal;
– solvable; (Lie) any simply connected solvable Lie group is embedded in $\mathsf{IUT}(n)$.

– $\mathsf{IUT}_1(\mathbb{R}^n)$ of *unipotent* matrices, with 1s on the main diagonal;
– dimension $\binom{n}{2}$
– nilpotent; any simply connected nilpotent Lie group is embedded in it.
– contains the *Heisenberg group* H_n of matrices $\begin{pmatrix} 1 & \boldsymbol{u}^\top & a \\ 0 & I & \boldsymbol{v} \\ 0 & \boldsymbol{0}^\top & 1 \end{pmatrix}$, of dimension $2n - 3$; simply connected.

A non-exhaustive list of the first few connected Lie groups (bold = simple):

Dimension	Group	
0	$\{I\}$	$= \mathsf{SO}(1) = \mathsf{SU}(1) = \mathsf{SL}(1) = \mathsf{SL}(\mathbb{C}) = \mathsf{SO}(\mathbb{C})$
1	$\mathbf{R}$	$\cong \mathbb{R}^+ =\cong \mathsf{GL}^+(1) \cong \mathsf{CSO}(1) \cong \mathsf{SO}^+(1,1) \cong \mathsf{IUT}_1(2)$
		$\mathbf{S^1} = \mathbb{R}/\mathbb{Z} \cong \mathsf{U}(1) = \mathsf{SO}(2) \cong \mathsf{Spin}(2)$
2	$\mathbb{R}^2$	$\mathbb{T}^2$, $\mathbb{T}\times\mathbb{R} \cong \mathbb{C}^\times = \mathsf{GL}(\mathbb{C}) \cong \mathbb{R}^\times \times \mathbb{S}^1 \cong \mathsf{SO}(\mathbb{C}^2)$
	$\mathsf{Aff}^+(1)$	$= \mathbb{R}^+ \rtimes \mathbb{R}$
3	$\mathbb{R}^3$	$\mathbb{T}^3$, $\mathbb{T}^2\times\mathbb{R}$, $\mathbb{T}\times\mathbb{R}^2$
	$\mathsf{E}^+(2)$	
	$\mathbf{S}^3$	$\cong \mathsf{USp}(1) = \mathsf{SU}(2) = \mathsf{Spin}(3)$
		$\mathbf{SO(3)} = \mathsf{PSU}(2)$, covered by $\mathbb{S}^3$
	$\mathbf{SL(2)}$	$= \mathsf{Sp}(\mathbb{R}^2) \cong \mathsf{SU}(1,1) \cong \mathsf{Spin}^+(1,2)$
		$\mathbf{SO^+(1,2)}$ in $\mathsf{Spin}^+(1,2)$ covered by $\mathsf{SL}(2)$
	H_3	$= \mathsf{IUT}_1(3)$
	$\mathsf{IUT}(2)$	
4	$\mathbb{R}^4$	$\mathbb{T}^4$, $\mathbb{T}^3\times\mathbb{R}$, ...
	$\mathsf{GL}^+(2)$	
	$\mathbb{H}^\times = \mathsf{GL}(\mathbb{H})$	
	$\mathsf{U}(2)$	
5	$\mathbb{R}^5$	$\mathbb{T}^5$, $\mathbb{T}^4\times\mathbb{R}$,...
	H_4	
6	$\mathbb{R}^6$	$\mathbb{T}^6$, $\mathbb{T}^5\times\mathbb{R}$, ...
	$\mathsf{Aff}^+(2)$	
	$\mathsf{E}^+(3)$	
	$\mathsf{IUT}(3)$	
	$\mathsf{IUT}_1(4)$	
	$\mathbf{SL}(\mathbb{C}^2)$	$\cong \mathsf{Sp}(\mathbb{C}^2)$
		$\mathbf{SO}(\mathbb{C}^3) \cong \mathsf{PSL}(\mathbb{C}^2) \cong \mathsf{SO}^+(1,3)$
	$\mathsf{SO}(4)$	covered by $\mathsf{SU}(2)\times\mathsf{SU}(2) \cong \mathsf{Spin}(4)$
	$\mathsf{SO}(2,2)$	
7	$\mathbb{R}^7$	$\mathbb{T}^7$, $\mathbb{T}^6\times\mathbb{R}$, ...
	H_5	
8	$\mathbb{R}^8$	$\mathbb{T}^8$, $\mathbb{T}^7\times\mathbb{R}$, ...
	$\mathsf{GL}(\mathbb{C}^2)$	
	$\mathbf{SL(3)}$	
	$\mathbf{SU(3)}$	
	$\mathbf{SU(1,2)}$	
9	$\mathbb{R}^9$	$\mathbb{T}^9$, $\mathbb{T}^8\times\mathbb{R}$, ...
	$\mathsf{GL}^+(3)$	
	$\mathsf{U}(3)$	
	H_6	

Dimension	Group	
10	$\mathbb{R}^{10}$	$\mathbb{T}^{10}$, $\mathbb{T}^9 \times \mathbb{R}$, ...
	$\mathsf{E}^+(4)$	
	$\mathbf{USp}(\mathbf{2})$	$= \mathsf{Spin}(5)$
	$\mathbf{Sp}(\mathbf{4})$	
	$\mathbf{Sp}(2,2)$	
	$\mathbf{SO}(\mathbf{5})$	
	$\mathbf{SO}^+(\mathbf{1},\mathbf{4})$	
	$\mathbf{SO}^+(\mathbf{2},\mathbf{3})$	
	$\mathsf{IUT}(4)$	
	$\mathsf{IUT}_1(5)$	
11	$\mathbb{R}^{11}$	$\mathbb{T}^{11}$, $\mathbb{T}^{10} \times \mathbb{R}$, ...
	H_7	
12	$\mathbb{R}^{12}$	$\mathbb{T}^{12}$, $\mathbb{T}^{11} \times \mathbb{R}$, ...
	$\mathsf{Aff}^+(3)$	
	$\mathsf{SO}(\mathbb{C}^4)$	covered by $\mathsf{SL}(\mathbb{C}^2)^2$
13	$\mathbb{R}^{13}$	$\mathbb{T}^{13}$, $\mathbb{T}^{12} \times \mathbb{R}$, ...
	H_8	
14	$\mathbb{R}^{14}$	$\mathbb{T}^{14}$, $\mathbb{T}^{13} \times \mathbb{R}$, ...
	$\mathbf{G_2}$	subgroup of $\mathsf{O}(7)$, contains $\mathsf{O}(3)$
15	$\mathbb{R}^{15}$	$\mathbb{T}^{15}$, $\mathbb{T}^{14} \times \mathbb{R}$, ...
	$\mathsf{E}^+(5)$	
	$\mathbf{SL}(\mathbf{4})$	
	$\mathbf{SU}(\mathbf{4})$	$\cong \mathsf{Spin}(6)$
	$\mathbf{SU}(\mathbf{1},\mathbf{3})$	
	$\mathbf{SU}(\mathbf{2},\mathbf{2})$	
	$\mathsf{SO}(6)$	
	$\mathsf{SO}(3,3)$	
	$\mathsf{SO}(2,4)$	
	$\mathsf{SO}(1,5)$	
	$\mathbf{SL}(\mathbb{H}^{\mathbf{2}})$	
	$\mathsf{IUT}(5)$	
	$\mathsf{IUT}_1(6)$	
	H_9	
16	$\mathbb{R}^{16}$	$\mathbb{T}^{16}$, $\mathbb{T}^{15} \times \mathbb{R}$, ...
	$\mathbf{SL}(\mathbb{C}^{\mathbf{3}})$	
	$\mathsf{GL}^+(4)$	
	$\mathsf{GL}(\mathbb{H}^2)$	
	$\mathsf{U}(4)$	
17	$\mathbb{R}^{17}$	$\mathbb{T}^{17}$, $\mathbb{T}^{16} \times \mathbb{R}$, ...
	H_{10}	
18	$\mathbb{R}^{18}$	$\mathbb{T}^{18}$, $\mathbb{T}^{17} \times \mathbb{R}$, ...
	$\mathsf{GL}(\mathbb{C}^3)$	
19	$\mathbb{R}^{19}$	$\mathbb{T}^{19}$, $\mathbb{T}^{18} \times \mathbb{R}$, ...
	H_{11}	
20	$\mathbb{R}^{20}$	$\mathbb{T}^{20}$, $\mathbb{T}^{19} \times \mathbb{R}$, ...
	$\mathsf{Aff}^+(4)$	
	$\mathbf{Sp}(\mathbb{C}^4)$	covers $\mathbf{SO}(\mathbb{C}^{\mathbf{5}})$

Exceptional Groups:
G_2 contains $\mathsf{O}(3)$, $\mathsf{SU}(3)$, and $\mathsf{USp}(1)$
F_4 contains $\mathsf{O}(9)$ and $\mathsf{USp}(4)$
E_6 contains $\mathsf{O}(10)$ and $\mathsf{SU}(6)$
E_7 contains $\mathsf{O}(12)$ and $\mathsf{SU}(8)$
E_8 contains $\mathsf{O}(16)$