Elementary Linear Algebra

1   Matrices

1.1   Definition

A **matrix** is a rectangular array of numbers, arranged in rows and columns.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & \ddots & \\ a_{m1} & \dots & & & a_{mn} \end{pmatrix}$$

We write the array in short as $[a_{ij}]$ with i and j denoting the indices for the rows and columns respectively.

We say that a matrix with m rows and n columns is of *size* $m \times n$. An $m \times n$ matrix has mn elements.

If $m = n$ the matrix is called a *square* matrix. A matrix with just one column is called a *vector* while one with just one row is called a *covector* or *row vector*. A 1×1 matrix is called a *scalar* and is simply a number.

Examples:

$$\text{a } 2 \times 3 \text{ matrix,} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}; \qquad \text{a } 2 \times 2 \text{ square matrix,} \quad \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix};$$

$$\text{a vector,} \quad \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}; \quad \text{a row vector,} \quad \begin{pmatrix} 5 & 2 \end{pmatrix}; \quad \text{a scalar, } 9.$$

1.2   Equality of Matrices

Two matrices are **equal** when all their respective elements are equal.

$$\mathbf{A} = \mathbf{B} \qquad \text{when } a_{ij} = b_{ij} \text{ for all } i, j.$$

Note that for this to make sense, the size of the matrices must be the same.

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}.$$

1.3 Addition

The **addition** of two matrices is defined by

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Example:

$$
\begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 1 & 5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1+3 & 2+4 \\ 3+1 & 5+5 \\ 4+2 & 3+2 \end{pmatrix}
$$
$$
= \begin{pmatrix} 4 & 6 \\ 4 & 10 \\ 6 & 5 \end{pmatrix}
$$

Note that it is not possible to add matrices together if they are of different sizes.

It does not matter which order the matrices are added i.e. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

1.4 Multiplication by Scalars

A matrix can be multiplied by a number (scalar),

$$\lambda[a_{ij}] = [\lambda a_{ij}].$$

Example:

$$
2 \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 2 \times 1 & 2 \times 2 \\ 2 \times -2 & 2 \times 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -4 & 8 \end{pmatrix}
$$

1.5 Multiplication of Matrices

Multiplication of matrices is defined by

$$\mathbf{AB} = [a_{ij}][b_{jk}] = [\sum_j a_{ij} b_{jk}].$$

Example:

$$
\begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times -1 + -1 \times 2 \\ 3 \times 1 + -2 \times -1 + 0 \times 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix},
$$

$$
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 17 & 19 \end{pmatrix}.
$$

Note that matrices must be compatible for multiplication, i.e. the number of columns of the first matrix must equal the number of rows of the second matrix. In general, an $l \times m$ matrix multiplied by an $m \times n$ matrix gives an $l \times n$ matrix.

The usual *dot product* of vectors can also be expressed as matrix multiplication:

$$\begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 1 \times -1 + 0 \times 2 + 2 \times 3 = 5.$$

The following multiplication of vectors gives a very different result:

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \end{pmatrix}.$$

From this example it follows that the order in which we multiply matrices is important. This may be true even for square matrices:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 7 \\ 22 & 15 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 13 & 20 \end{pmatrix}.$$

In general, $\mathbf{AB} \neq \mathbf{BA}$.

1.6 Zero Matrix

The **zero** matrix, $\mathbf{0}$, is defined as that matrix with zero elements, $[0]$.

Example: The 2×3 zero matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are zero matrices of any size.

When the matrices are compatible, $\mathbf{0} + \mathbf{A} = \mathbf{A}$ and $\mathbf{0A} = \mathbf{0}$; also $0\mathbf{A} = \mathbf{0}$. What's written symbolically here could be a reference to the following three operations:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 0 & 3 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

$$0 \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

1.7   Identity Matrix

The **identity** matrix, **I**, is the square matrix,

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\vdots & & & \ddots
\end{pmatrix}.
$$

It can be written in short as $\mathbf{I} = [\delta_{ij}]$ where

$$
\delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases},
$$

is called the Kronecker delta.

It is easy to check that for compatible matrices $\mathbf{IA} = \mathbf{A}; \quad \mathbf{AI} = \mathbf{A}$.

1.8   Transpose

The **transpose** of a matrix is that matrix, written $\mathbf{A}^\top$, obtained by interchanging the rows and columns,

$$
[a_{ij}]^\top = [a_{ji}].
$$

Example:

$$
\begin{pmatrix} 1 & 0 & 2 \\ 4 & 2 & -1 \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 2 & -1 \end{pmatrix}
$$

An $m \times n$ matrix becomes an $n \times m$ matrix. In particular the transpose of a vector is a covector and vice versa.

Some properties of the transpose:

$$
(\lambda A)^\top = \lambda A^\top, \qquad (A + B)^\top = A^\top + B^\top,
$$

$$
(A^\top)^\top = A, \qquad (AB)^\top = B^\top A^\top.
$$

1.9   Simultaneous Linear Equations

If we multiply out the following two matrices, we get

$$
\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x + 3y.
$$

This means that we can write an equation such as $5x - 2y = 3$ as

$$\begin{pmatrix} 5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3.$$

Moreover if we have two simultaneous equations of the same type we can combine them in matrix notation:

$$\begin{aligned} 5x - 2y &= 3 \\ 2x + y &= -1 \end{aligned}$$

becomes

$$\begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

This idea of transforming simultaneous equations into matrix equations can be generalized to any number of variables, not just two.

Example: The equations

$$\begin{aligned} 2x - 5y + 3z &= -1 \\ 3y + x - z &= 3 \\ 3x - 4y + 3z &= 1 \end{aligned}$$

become the matrix equation

$$\begin{pmatrix} 2 & -5 & 3 \\ 1 & 3 & -1 \\ 3 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}.$$

As long as the equations are *linear*, that is, only terms like x and y occur and not terms like x^3, $\sin x$, etc., then it will always be possible to write the equations as a single matrix equation. The number of rows of the matrix would equal the number of given equations, while the number of columns equals the number of variables.

We can therefore write a general system of linear equations as

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{x}$ refers to the vector of variables x, y, etc.

If at all possible, we would like to find the values for x, y, ..., in other words the vector $\mathbf{x}$, to make the equation true.

In order to proceed, we must look at the simplest matrix equation, one for which there is only one variable x and the matrix A is of size 1×1, that is it is just a scalar.

Example:
$$5x = 3$$

To solve this equation, we need to divide both sides of the equation by 5, or rather multiply both sides by 1/5. Why not multiply with 1/3 or any other such number? Obviously because doing so only gives us another equation without telling us directly what x is. Only by multiplying with 1/5 are we able to eliminate the 5 in front of the unknown x, because $\frac{1}{5} \times 5 = 1$ and $1x = x$, so that we get $x = 3/5$.

Repeating the same argument to the matrix $\mathbf{A}$ (compare with the 5), we realize that what we need is another matrix $\mathbf{B}$ (compare with the 1/5) such that $\mathbf{BA} = \mathbf{I}$. We call such a matrix the inverse of $\mathbf{A}$ and denote it by $\mathbf{A}^{-1}$. (The notation $\frac{1}{A}$ should be avoided, because it suggests $\frac{1}{AB} = \frac{1}{A}\frac{1}{B}$ which is false.)

1.10 Inverses

The **inverse** of a matrix $\mathbf{A}$ is another matrix, denoted by $\mathbf{A}^{-1}$ with the property that
$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}, \qquad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$
(It is not apparent, but true for matrices, that if $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, then automatically $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$)

If we are able to find a method which will give us the inverse of a matrix we would then be able to solve simultaneous equations, even if they had a thousand variables. (In fact, there are many applications ranging from predicting the weather to modeling the economy to simulating the shape of an airplane wing, that require this many variables.)

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \mathbf{b} \\
\mathbf{A}^{-1}\mathbf{A}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{I}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}
\end{aligned}
$$

Example: By a method that will be done later on, the inverse of the matrix $\mathbf{A}$ shown previously is
$$
\begin{pmatrix} 2 & -5 & 3 \\ 1 & 3 & -1 \\ 3 & -4 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 3 & -4 \\ -6 & -3 & 5 \\ -13 & -7 & 11 \end{pmatrix}.
$$

Let us check this.
$$
\begin{pmatrix} 5 & 3 & -4 \\ -6 & -3 & 5 \\ -13 & -7 & 11 \end{pmatrix} \begin{pmatrix} 2 & -5 & 3 \\ 1 & 3 & -1 \\ 3 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Therefore the solution of the simultaneous equations shown above is

$$
\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} 5 & 3 & -4 \\ -6 & -3 & 5 \\ -13 & -7 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.
$$

For which matrices is it possible to find an inverse? **Non-square matrices do not have an inverse.** How do we know this? Suppose we have an $m \times n$ matrix $\mathbf{A}$. From the requirement that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ we see that the number of columns of $\mathbf{A}$, which is n, is the same as that of $\mathbf{I}$. We also have the requirement that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. This means that the number of rows of $\mathbf{A}$, which is m, must equal that of the identity matrix. But the identity matrix is square, therefore $m = n$.

Even if a matrix is square, it does not necessarily mean that it has an inverse. Some matrices are analogous to the number 0, for which the reciprocal $1/0$ does not make sense. In the next section we will find ways of determining which matrices are invertible and if they are, how to find the inverse.

The inverses of matrices obey certain laws:

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ How do we know this? Let $\mathbf{B} = \mathbf{A}^{-1}$. We want to show that $\mathbf{B}^{-1} = \mathbf{A}$, that is, that the inverse of the matrix $\mathbf{B}$ is nothing else but the original matrix $\mathbf{A}$. To show this, all we need is to show that $\mathbf{B}\mathbf{A} = \mathbf{I}$ which is true.

- $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. We can show this by multiplying out

$$
\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{A}\mathbf{B} = \mathbf{B}^{-1}\mathbf{I}\mathbf{B} = \mathbf{I}
$$

Hence the inverse of $\mathbf{A}\mathbf{B}$ is $\mathbf{B}^{-1}\mathbf{A}^{-1}$.

1.11   Exercises

1. Let
$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -2 & 5 \end{pmatrix}; \qquad \mathbf{B} = \begin{pmatrix} -2 & 0 & -1 \\ 4 & -6 & -3 \end{pmatrix}.$$

   If possible find the following: $3\mathbf{A}$, $\mathbf{A} + \mathbf{B}$, $3\mathbf{A} - 2\mathbf{B}$, $\mathbf{AB}$.

2. If $\mathbf{A}$ is a matrix of size 3×5, which of the following may be true?

   (a) $\mathbf{AB}$ has 3 rows and $\mathbf{B}$ has 5 columns;

   (b) $\mathbf{B}$ has 5 rows and $\mathbf{AB}$ has 3 rows;

   (c) $\mathbf{BA}$ has 5 columns and $\mathbf{B}$ has 3 rows;

   (d) $\mathbf{B}$ has 3 columns and $\mathbf{BA}$ has 5 columns.

3. Find a matrix $\mathbf{C}$ such that the following sum is the zero matrix:

$$3 \begin{pmatrix} 1 & 2 & 0 \\ -2 & 3 & 4 \\ 6 & -2 & 1 \end{pmatrix} - 4 \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix} + 2\mathbf{C}$$

4. Perform the following matrix multiplications if possible,

$$\begin{pmatrix} -5 & 6 \\ 1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 7 & 1 & 4 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -3 & 5 \\ 5 & 1 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

5. Without doing any calculations, find the inverse of the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

6. If we want to define $\mathbf{A}^2$ by $\mathbf{AA}$, on what type of matrices would such a definition make sense?

   Let
$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}.$$

   Find $\mathbf{A}^2$ and $\mathbf{A}^3$.

7. Suppose we take some functions of x and apply their McLaurin series to matrices:
$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \dots,$$
$$(\mathbf{I} + \mathbf{A})^{-1} = \mathbf{I} - \mathbf{A} + \mathbf{A}^2 - \mathbf{A}^3 + \dots.$$

By considering the matrix $\mathbf{A} = \begin{pmatrix} 0.1 & 0.2 \\ -0.1 & 0.3 \end{pmatrix}$, find approximations for $e^{\mathbf{A}}$, $e^{-\mathbf{A}}$ and $(\mathbf{I} + \mathbf{A})^{-1}$ by calculating the first few terms in the series. Check whether

   (a) $(\mathbf{I} + \mathbf{A})^{-1}$ is indeed the inverse of $\mathbf{I} + \mathbf{A}$;

   (b) $e^{-\mathbf{A}}$ is the inverse of $e^{\mathbf{A}}$.

2   Gaussian Elimination

Our aim in this chapter is to describe a method for solving simultaneous equations of the type $\mathbf{Ax} = \mathbf{b}$ for any number of variables, and also to find a method for calculating the inverse of a matrix $\mathbf{A}$ when possible.

Before we learn how to run, let us consider how we normally solve simultaneous equations with two or three variables.

2.0.1   Example

Consider the simultaneous equations

$$\begin{array}{rcrcr} 2x & + & 3y & = & 4 \\ 5x & - & y & = & -7. \end{array}$$

Our usual method is to multiply both equations by different multiples in such a way that one of the variables can be eliminated by subtraction. In other words, multiplying the first equation by 5 and the second by 2, we get

$$\begin{array}{rcrcr} 10x & + & 15y & = & 20 \\ 10x & - & 2y & = & -14. \end{array}$$

Subtracting the first equation from the second we get

$$\begin{array}{rcrcr} 10x & + & 15y & = & 20 \\ & - & 17y & = & -34. \end{array}$$

Once the variable x has been eliminated from the second equation we can solve it directly to get $y = 2$. Substituting into the first equation we then get $10x + 30 = 20$ which implies that $x = -1$.

Let us rewrite this example using matrix notation so that the procedure becomes clearer.

$$\begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 15 \\ 10 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ -14 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 15 \\ 0 & -17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ -34 \end{pmatrix}$$

The steps that we have taken in solving the equations become simple operations on the rows of the matrix. In particular, we have multiplied whole rows by scalars, and we have subtracted away a whole row from a whole row, *including* the vector **b**.

Let us take another example, this time involving three variables. We will write the equations in matrix form on the right hand side for comparison.

$$\begin{array}{rcr} 2x + 3y - 4z & = & -4 \\ x - y + 3z & = & 8 \\ 3x + y - 2z & = & -1 \end{array} , \qquad \begin{pmatrix} 2 & 3 & -4 \\ 1 & -1 & 3 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -1 \end{pmatrix} .$$

To simplify things slightly, we will switch round the first and second rows. Obviously this does not make the slightest difference to the solution.

$$\begin{array}{rcr} x - y + 3z & = & 8 \\ 2x + 3y - 4z & = & -4, \\ 3x + y - 2z & = & -1 \end{array} \qquad \begin{pmatrix} 1 & -1 & 3 \\ 2 & 3 & -4 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ -1 \end{pmatrix} .$$

We can eliminate the x variable from the second and third equation by subtracting 2 times the first row and 3 times the first row respectively.

$$\begin{array}{rcr} x - y + 3z & = & 8 \\ 5y - 10z & = & -20, \\ 4y - 11z & = & -25 \end{array} \qquad \begin{pmatrix} 1 & -1 & 3 \\ 0 & 5 & -10 \\ 0 & 4 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -20 \\ -25 \end{pmatrix} .$$

The second equation simplifies to

$$\begin{array}{rcr} x - y + 3z & = & 8 \\ y - 2z & = & -4 \\ 4y - 11z & = & -25 \end{array} , \qquad \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 4 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ -25 \end{pmatrix} ,$$

and now we can eliminate the y variable from the third equation by subtracting away 4 times the second equation,

$$\begin{array}{rcr} x - y + 3z & = & 8 \\ y - 2z & = & -4, \\ -3z & = & -9 \end{array} \qquad \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ -9 \end{pmatrix} .$$

This gives us that $z = 3$ and we can therefore substitute back into the second equation to get $y - 2 \times 3 = -4$ which gives $y = 2$. Substituting into the first equation gives $x - 2 + 3 \times 3 = 8$ i.e. $x = 1$.

We will try to use these two examples to find a general method of solving any number of simultaneous linear equations. We notice that we have used three rules throughout:

1. We can multiply an equation by any number (except 0 of course!);
2. We can add/subtract an equation to another equation;
3. We can switch any two equations.

We have used these three rules to eliminate x from all the subsequent equations, then to eliminate y from the rest of the equations, and so on until we reach the last equation in which only one variable remains i.e. a value for it is found. Substituting back into the previous equation solves for the other variables.

It is clearer to work with matrices when a large number of variables are involved, and we will adapt our rules and method for this case.

2.1 Preliminary Method

Rules Given a matrix equation, we can apply the following **row operations** to the rows without changing the solution:

1. Multiply a row with a non-zero scalar;

2. Add/subtract a row to another row;

3. Switch two rows.

Method Using the three operations listed above, we will try to bring the lower left corner of the matrix into zeros.

In what follows, we will simplify the notation slightly by not writing the vector **x** and replacing it with a vertical line.

2.1.1 Example

$$\left(\begin{array}{ccc|c} 2 & -2 & 6 & 2 \\ 4 & -3 & 14 & -22 \\ -9 & 6 & -32 & 50 \end{array} \right)$$

First simplify the first row by dividing by 2,

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 4 & -3 & 14 & -22 \\ -9 & 6 & -32 & 50 \end{array} \right),$$

then eliminate the 4 in the second row, by subtracting 4 times the first row from the second, and eliminate the -9 in the third row by adding 9 times the first row to the third,

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & 2 & -26 \\ 0 & -3 & -5 & 59 \end{array}\right).$$

Next we use the second row to eliminate the -3 of the third row by adding 3 times the second row to the third,

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & 2 & -26 \\ 0 & 0 & 1 & -19 \end{array}\right).$$

At this stage we have achieved our aim of reducing the left-bottom corner of the matrix to zeros (i.e. we have done the necessary eliminations), and we can therefore write out the equations in their normal form and solve for x, y and z. However we can keep on applying the same rules to eliminate the -1 and 3 in the first row and the 2 in the second.

Adding the second row to the first eliminates the -1 of the first row:

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 5 & -25 \\ 0 & 1 & 2 & -26 \\ 0 & 0 & 1 & -19 \end{array}\right).$$

Similarly subtracting 5 times the third row from the first and subtracting 2 times the third row from the second gives,

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 70 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & -19 \end{array}\right),$$

which can be expanded to $x = 70$, $y = 12$ and $z = -19$.

2.1.2 Example

Let us take another example, writing out the working with a minimum of words,

$$\begin{array}{rcr} x + y - 3z & = & 10 \\ x - 3y & = & 7 \\ -9x + 28z & = & -65 \end{array}$$

which, when written in matrix form, become

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 10 \\ 1 & 0 & -3 & 7 \\ -9 & 0 & 28 & -65 \end{array} \right),$$

$$\Rightarrow \begin{array}{c} \text{row}_2 - \text{row}_1 \\ \text{row}_3 + 9\text{row}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 10 \\ 0 & -1 & 0 & -3 \\ 0 & 9 & 1 & 25 \end{array} \right),$$

$$\Rightarrow \begin{array}{c} \\ \text{row}_3 + 9\text{row}_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 10 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array} \right),$$

$$\Rightarrow -\text{row}_2 \left(\begin{array}{ccc|c} 1 & 1 & -3 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right),$$

$$\begin{array}{c} \text{row}_1 - \text{row}_2 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right),$$

$$\begin{array}{c} \text{row}_1 + 3\text{row}_3 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right).$$

That is, $x = 1$, $y = 3$ and $z = -2$.

2.2   What can go wrong

Let us do a few examples where the outcome is not exactly as above:

2.2.1   Example

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & 10 \\ 3 & 6 & 2 & -23 \\ -7 & 2 & -6 & 11 \end{array} \right).$$

For this example, we cannot even start the elimination process because there is a zero in the key position. However we can apply the rule of switching the

first and the second row to bring a non-zero number at the beginning of the first row.

$$\begin{pmatrix} 3 & 6 & 2 & -23 \\ 0 & -1 & 1 & 10 \\ -7 & 2 & -6 & 11 \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} \\ -\text{row}_2 \\ 3\text{row}_3 \end{matrix} \begin{pmatrix} 3 & 6 & 2 & -23 \\ 0 & 1 & -1 & -10 \\ -21 & 6 & -18 & 33 \end{pmatrix},$$

$$\Rightarrow \quad \begin{matrix} \\ \\ \text{row}_3 + 7\text{row}_1 \end{matrix} \begin{pmatrix} 3 & 6 & 2 & -23 \\ 0 & 1 & -1 & -10 \\ 0 & 48 & -4 & -128 \end{pmatrix},$$

$$\Rightarrow \quad \begin{matrix} \\ \\ \frac{1}{4}\text{row}_3 \end{matrix} \begin{pmatrix} 3 & 6 & 2 & -23 \\ 0 & 1 & -1 & -10 \\ 0 & 12 & -1 & -32 \end{pmatrix},$$

$$\Rightarrow \quad \begin{matrix} \\ \\ \text{row}_3 - 12\text{row}_2 \end{matrix} \begin{pmatrix} 3 & 6 & 2 & -23 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 11 & 88 \end{pmatrix},$$

$$\Rightarrow \quad \begin{matrix} \text{row}_1 - 6\text{row}_2 \\ \\ \frac{1}{11}\text{row}_3 \end{matrix} \begin{pmatrix} 3 & 0 & 8 & 37 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 1 & 8 \end{pmatrix},$$

$$\Rightarrow \quad \begin{matrix} \text{row}_1 - 8\text{row}_3 \\ \text{row}_2 + \text{row}_3 \end{matrix} \begin{pmatrix} 3 & 0 & 0 & -27 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 8 \end{pmatrix},$$

which yields the solution $x = -3$, $y = -2$ and $z = 8$.

2.2.2   Example

In this example we end up with a final row of zeros.

$$\begin{pmatrix} 3 & -12 & -18 & 21 \\ 7 & -27 & -41 & 48 \\ 9 & -45 & -63 & 72 \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} \frac{1}{3}\text{row}_1 \\ \\ \frac{1}{9}\text{row}_3 \end{matrix} \begin{pmatrix} 1 & -4 & -6 & 7 \\ 7 & -27 & -41 & 48 \\ 1 & -5 & -7 & 8 \end{pmatrix},$$

$$\Rightarrow \quad \begin{matrix} \text{row}_2 - 7\text{row}_1 \\ \text{row}_3 - \text{row}_1 \end{matrix} \begin{pmatrix} 1 & -4 & -6 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix},$$

$$\Rightarrow \quad \begin{matrix} \\ \\ \text{row}_3 + \text{row}_2 \end{matrix} \begin{pmatrix} 1 & -4 & -6 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last equation has simply evaporated away and we have ended up with just two equations. This does not matter, and we can at least eliminate the -4 in the first row by adding 4 times the second row to it:

$$\text{row}_1 + 4\text{row}_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

We cannot do any other row operation to these equations without destroying some of the zeros. If we translate these equations back into long form we get,

$$\begin{array}{rcl} x \quad\quad -2z & = & 3 \\ y \quad +z & = & -1. \\ 0 & = & 0 \end{array}$$

which gives the "solution" $x = 3 + 2z$, $y = -1 - z$ and there is no restriction on z. That is, putting any number instead of z gives a valid solution. We can write these solutions as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 + 2\alpha \\ -1 - \alpha \\ \alpha \end{pmatrix},$$

where we have assigned the arbitrary value α to z.

Apply row operations to the following equations

$$\left(\begin{array}{ccc|c} 3 & -12 & -18 & 21 \\ 7 & -27 & -41 & 48 \\ 9 & -45 & -63 & 81 \end{array} \right),$$

and show that they reduce to the equations

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

In this case the last equation is impossible to achieve — there are no values for x, y and z which can possibly make $0 = 1$, and so there are no solutions.

2.2.3   Example

$$
\begin{pmatrix}
1 & 5 & -3 & -1 \\
2 & 10 & -4 & 2 \\
-1 & -5 & 6 & 7
\end{pmatrix}
$$

$$
\Rightarrow \quad
\begin{aligned}
\text{row}_2 - 2\text{row}_1 \\
\text{row}_3 + \text{row}_1
\end{aligned}
\begin{pmatrix}
1 & 5 & -3 & -1 \\
0 & 0 & 2 & 4 \\
0 & 0 & 3 & 6
\end{pmatrix},
$$

$$
\Rightarrow \quad
\begin{aligned}
\tfrac{1}{2}\text{row}_2 \\
\tfrac{1}{3}\text{row}_3
\end{aligned}
\begin{pmatrix}
1 & 5 & -3 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix},
$$

$$
\Rightarrow \quad
\begin{aligned}
\text{row}_1 + 3\text{row}_2 \\
\text{row}_3 - \text{row}_2
\end{aligned}
\begin{pmatrix}
1 & 5 & 0 & 5 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

In this example there are two zeros on the diagonal, and we cannot reduce the matrix any further. The equations written in full are

$$
\begin{aligned}
x + 5y &= 5 \\
z &= 2. \\
0 &= 0
\end{aligned}
$$

Therefore we have to take $z = 2$ but we can assign any value to y say $y = \alpha$ and then $x = 5 - 5\alpha$. The solution is therefore,

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 - 5\alpha \\ \alpha \\ 2 \end{pmatrix}.
$$

Notice that had the last element of **b** turned out to be non-zero, then it would have been impossible to satisfy the third equation, and therefore there would have been no solutions.

2.2.4   Example

$$
\begin{pmatrix}
3 & -5 & 5 & 7 \\
18 & -30 & 30 & 42 \\
-9 & 15 & -15 & -39
\end{pmatrix}
$$

$$
\Rightarrow \quad
\begin{aligned}
\tfrac{1}{6}\text{row}_2 \\
\tfrac{1}{3}\text{row3}
\end{aligned}
\begin{pmatrix}
3 & -5 & 5 & 7 \\
3 & -5 & 5 & 7 \\
-3 & 5 & -5 & -13
\end{pmatrix},
$$

$$
\Rightarrow \quad
\begin{aligned}
\text{row}_2 - \text{row}_1 \\
\text{row}_3 + \text{row}_1
\end{aligned}
\begin{pmatrix}
3 & -5 & 5 & 7 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -6
\end{pmatrix}.
$$

The last equation is impossible to solve, and hence there are no solutions.

2.2.5   Example

$$
\begin{pmatrix}
3 & -5 & 5 & | & 7 \\
18 & -30 & 30 & | & 42 \\
-9 & 15 & -15 & | & -21
\end{pmatrix}
\Rightarrow
\begin{matrix}
\\
\text{row}_2 - 6\text{row}_1 \\
\text{row}_3 + 3\text{row}_1
\end{matrix}
\begin{pmatrix}
3 & -5 & 5 & | & 7 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & | & 0
\end{pmatrix}.
$$

In this case the last two equations are trivial, and we end up with just one equation $3x - 5y + 5z = 7$. This means that we can choose any value for y, say $y = \alpha$, and any value for z, say $z = \beta$, but then the value for x has to be $x = (7 + 5\alpha - 5\beta)/3$. The solutions are therefore of the type

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
=
\begin{pmatrix} \frac{7}{3} + \frac{5}{3}(\alpha - \beta) \\ \alpha \\ \beta \end{pmatrix}.
$$

2.3   Gauss-Jordan Elimination Method

Summarizing the steps taken in the examples above, we deduce the following method:

   For each column, starting from the first,

1. if necessary and if possible, bring a non-zero number to pivot position by switching rows; this number will be called the pivot of the column; although not necessary, it is convenient to reduce the pivot to 1 by dividing its row by the value of the pivot;

2. eliminate each non-zero element in the column above and below the pivot, by adding or subtracting multiples of the pivot row;

3. repeat for the next column.

   It is easy to realize that the end product of such operations will be a matrix of the following type, called the **Gauss echelon form**.:

$$
\begin{pmatrix}
0 & \cdots & 0 & 1 & * & \cdots & 0 & * & | & * \\
0 & \cdots & 0 & 0 & \cdots & & 0 & 1 & * & | & * \\
& & & 0 & \cdots & & 0 & 0 & * & | & * \\
\vdots & & & & \vdots & & & & \vdots & | & *
\end{pmatrix}.
$$

   The resulting number of non-zero equations is called the **rank** of the matrix equation. For most matrices the rank will equal the number of variables

and the Gauss echelon form will be exactly the same as the identity matrix. In this case there will be a **unique solution** to $\mathbf{x}$.

If the rank is less than the number of variables, the difference between the number of variables and the rank must be made up by a number of rows of zeros at the bottom. In such a case, there can be

- either many solutions, when the bottom equations are **consistent** i.e. of the type $0 = 0$;

- or no solutions, when one of the bottom equations is of the **inconsistent** type $0 = * \neq 0$.

2.4   Exercises

Apply the Gauss-Jordan elimination method to solve the following systems of equations,

1.
$$\begin{aligned} 2x + 4y &= 10 \\ 3x + 5y &= 13\text{'} \end{aligned}$$

2.
$$\begin{aligned} 2x + 4y + 6z &= 4 \\ 4x + 5y + 6z &= 5, \\ 3x + y - 2z &= 0 \end{aligned}$$

3.
$$\begin{aligned} x - 3y + 4z &= -5 \\ 2x - 5y + 7z &= -8, \\ z - y &= -2 \end{aligned}$$

4.
$$\begin{aligned} 3x + y &= 5 \\ x - y + 2z &= 5, \\ x + y + z &= 6 \end{aligned}$$

5.
$$\begin{aligned} x + 2y + z - w &= 4 \\ 2x + 3y + 2z - 4w &= -2 \\ x + y - 2w &= -3 \\ 2x + 3y + 5z - 7w &= -11 \end{aligned}$$

6. Find the value of k for which the following equations do not have a solution

$$\begin{aligned}
2x - y + 4z &= 3 \\
5z - x &= -1 \\
19x - 7y + kz &= 26
\end{aligned}$$

2.4.1   Simultaneous matrix equations

It is sometimes required to solve several matrix equations simultaneously, $A\mathbf{x}_1 = \mathbf{b}_1, \ldots A\mathbf{x}_n = \mathbf{b}_n$. This system of equations is equivalent to $AX = B$ where $X = [\mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_n]$, and $B = [\mathbf{b}_1 \ldots \mathbf{b}_n]$. So one can solve them all at one go by applying Gaussian elimination to the matrices $[A|B]$.

For example, to solve the three sets of equations

$$\begin{array}{ccc}
x + 2y = 3 & x + 2y = -1 & x + 2y = 1 \\
3x + 4y = 1 & 3x + 4y = 5 & 3x + 4y = -2
\end{array}$$

we can put them in compact form

$$\left(\begin{array}{cc|ccc} 1 & 2 & 3 & -1 & 1 \\ 3 & 4 & 1 & 5 & -2 \end{array} \right)$$

and apply row operations to reduce the left-hand matrix to the identity if possible:

$$\Rightarrow \left(\begin{array}{cc|ccc} 1 & 2 & 3 & -1 & 1 \\ 0 & -2 & -8 & 8 & -5 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{cc|ccc} 1 & 0 & -5 & 7 & -4 \\ 0 & 1 & 4 & -4 & 5/2 \end{array} \right)$$

The three solutions can then be read off the columns of the right-hand matrix.

2.5   Finding the Inverse

Consider the following matrix multiplication:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We notice that multiplying a matrix $\mathbf{A}$ by the first matrix is the same as multiplying the second row of $\mathbf{A}$ by λ i.e. applying a row operation of the first type. In fact this is true also of the other row operations, they are all equivalent to multiplication by particular matrices called **elementary row matrices**.

We have just found that

1. Multiplying a row by a scalar λ is equivalent to multiplying on the left by the matrix

$$
E_I = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \qquad e.g. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \, 4\mathrm{row}_3
$$

2. Adding two rows is equivalent to multiplying on the left by the matrix

$$
E_{II} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \qquad e.g. \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{row}_1 + \mathrm{row}_2
$$

3. Switching two rows is equivalent to multiplying on the left by the matrix

$$
E_{III} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & 1 & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & 1 & & 0 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{pmatrix}, \qquad e.g. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mathrm{row}_2 \leftrightarrow \mathrm{row}_3
$$

Notice that the three matrices can be obtained by applying the respective row operations to the identity matrix.

Now let us review what the Gauss-Jordan elimination method does using these matrices for notation. The method applies various row operations to the equation

$$
\mathbf{Ax} = \mathbf{b}
$$

to get

$$E_r \ldots E_1 \mathbf{A}\mathbf{x} = E_r \ldots E_1 \mathbf{b}$$

until the matrix is reduced to the identity matrix (if possible)

$$\mathbf{I}\mathbf{x} = E_n \ldots E_1 \mathbf{A}\mathbf{x} = E_n \ldots E_1 \mathbf{b},$$

which is the same as $\mathbf{x} = E_n \ldots E_1 \mathbf{b}$. But we have already seen in chapter 1 that the solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ when the inverse exists. Combining the two results we get that

$$\mathbf{A}^{-1} = E_n \ldots E_1.$$

This means that if we keep track of what row operations we perform on $\mathbf{A}$, convert them to row matrices and multiply these out in order, we get precisely the inverse $\mathbf{A}^{-1}$. To avoid having to convert the row operations into matrices and multiplying, we can simplify things by applying the row operations directly to the identity matrix, since this would be equivalent to $E_n \ldots E_1 \mathbf{I}$. We can compactify the method even further by placing the matrices $\mathbf{A}$ and $\mathbf{I}$ side by side and performing row operations to both until $\mathbf{A}$ is reduced to the identity matrix.

We start with

$$[\mathbf{A}|\mathbf{I}],$$

apply row operations to both

$$[E_r \ldots E_1 \mathbf{A}|E_r \ldots E_1 \mathbf{I}],$$

until we get

$$[\mathbf{I}|\mathbf{A}^{-1}].$$

2.6   Examples

1. Let us find the inverse of

$$\begin{pmatrix} -3 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 4 & 2 \end{pmatrix}.$$

We first put the identity matrix adjacent to it, and continue performing row operations to both until the given matrix is reduced to the identity matrix,

as follows:

$$
\begin{pmatrix}
-3 & -1 & 2 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 \\
1 & 4 & 2 & 0 & 0 & 1
\end{pmatrix}
$$

$\Rightarrow$ $3\mathrm{row}_2 + \mathrm{row}_1$
$\quad\;\; 3\mathrm{row}_3 + \mathrm{row}_1$
$$
\begin{pmatrix}
-3 & -1 & 2 & 1 & 0 & 0 \\
0 & -1 & -1 & 1 & 3 & 0 \\
0 & 11 & 8 & 1 & 0 & 3
\end{pmatrix}
$$

$(\mathrm{row}_1 - \mathrm{row}_2)/3$
$\Rightarrow$
$(\mathrm{row}_3 + 11\mathrm{row}_2)/3$
$$
\begin{pmatrix}
-1 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & -1 & 1 & 3 & 0 \\
0 & 0 & -1 & 4 & 11 & 1
\end{pmatrix}
$$

$\mathrm{row}_1 + \mathrm{row}_3$
$\Rightarrow$
$\mathrm{row}_2 - \mathrm{row}_3$
$$
\begin{pmatrix}
-1 & 0 & 0 & 4 & 10 & 1 \\
0 & -1 & 0 & -3 & -8 & -1 \\
0 & 0 & -1 & 4 & 11 & 1
\end{pmatrix}
$$

$-\mathrm{row}_1$
$\Rightarrow$ $-\mathrm{row}_2$
$-\mathrm{row}_3$
$$
\begin{pmatrix}
1 & 0 & 0 & -1 & -10 & -1 \\
0 & 1 & 0 & 3 & 8 & 1 \\
0 & 0 & 1 & -4 & -11 & -1
\end{pmatrix}
$$

The right-hand matrix is therefore the required inverse.

2. Find the inverse of $\begin{pmatrix} 2 & 2 & 1 \\ 5 & -4 & 5 \\ 4 & 0 & 3 \end{pmatrix}$.

$$
\begin{pmatrix}
2 & 2 & 1 & 1 & 0 & 0 \\
5 & -4 & 5 & 0 & 1 & 0 \\
4 & 0 & 3 & 0 & 0 & 1
\end{pmatrix}
$$

$\Rightarrow$ $2\mathrm{row}_2 - 5\mathrm{row}_1$
$\quad\;\; \mathrm{row}_3 - 2\mathrm{row}_1$
$$
\begin{pmatrix}
2 & 2 & 1 & 1 & 0 & 0 \\
0 & -18 & 5 & -5 & 2 & 0 \\
0 & -4 & 1 & -2 & 0 & 1
\end{pmatrix}
$$

$9\mathrm{row}_1 + \mathrm{row}_2$
$\Rightarrow$
$9\mathrm{row}_3 - 2\mathrm{row}_2$
$$
\begin{pmatrix}
18 & 0 & 14 & 4 & 2 & 0 \\
0 & -18 & 5 & -5 & 2 & 0 \\
0 & 0 & -1 & -8 & -4 & 9
\end{pmatrix}
$$

$(\mathrm{row}_1 + 14\mathrm{row}_3)/18$
$\Rightarrow$ $(\mathrm{row}_2 + 5\mathrm{row}_3)/9$
$$
\begin{pmatrix}
1 & 0 & 0 & -6 & -3 & 7 \\
0 & -2 & 0 & -5 & -2 & 5 \\
0 & 0 & -1 & -8 & -4 & 9
\end{pmatrix}
$$

$\Rightarrow$ $-\mathrm{row}_2/2$
$-\mathrm{row}_3$
$$
\begin{pmatrix}
1 & 0 & 0 & -6 & -3 & 7 \\
0 & 1 & 0 & 5/2 & 1 & -5/2 \\
0 & 0 & 1 & 8 & 4 & -9
\end{pmatrix}.
$$

3. Find the inverse of $\begin{pmatrix} -2 & -2 & 1 \\ 5 & 5 & -2 \\ 2 & 2 & 5 \end{pmatrix}$.

$$\begin{pmatrix} -2 & -2 & 1 & | & 1 & 0 & 0 \\ 5 & 5 & -2 & | & 0 & 1 & 0 \\ 2 & 2 & 5 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \qquad \begin{matrix} 2\text{row}_2 + 5\text{row}_1 \\ \text{row}_3 + \text{row}_1 \end{matrix} \quad \begin{pmatrix} -2 & -2 & 1 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 5 & 2 & 0 \\ 0 & 0 & 6 & | & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} (\text{row}_1 - \text{row}_2)/-2 \\ \Rightarrow \\ \text{row}_3 - 6\text{row}_2 \end{matrix} \quad \begin{pmatrix} 1 & 1 & 0 & | & 2 & 1 & 0 \\ 0 & 0 & 1 & | & 5 & 2 & 0 \\ 0 & 0 & 0 & | & -29 & -12 & 1 \end{pmatrix}.$$

We cannot reduce the left-hand matrix any further — it is in Gaussian echelon form. As this is not the identity matrix, it implies that the matrix is *not* invertible.

4. Find the inverse of $\begin{pmatrix} 0 & 1 & 4 \\ -1 & 1 & 1 \\ -4 & 2 & -3 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 & 4 & | & 1 & 0 & 0 \\ -1 & 1 & 1 & | & 0 & 1 & 0 \\ -4 & 2 & -3 & | & 0 & 0 & 1 \end{pmatrix}$$

We have to bring a non-zero number in the first pivot, so we switch the first with the second row.

$$\Rightarrow \qquad \text{row}_1 \leftrightarrow \text{row}_2 \quad \begin{pmatrix} -1 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 4 & | & 1 & 0 & 0 \\ -4 & 2 & -3 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \Rightarrow \\ \text{row}_3 - 4\text{row}_1 \end{matrix} \quad \begin{pmatrix} -1 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 4 & | & 1 & 0 & 0 \\ 0 & -2 & -7 & | & 0 & -4 & 1 \end{pmatrix}$$

$$\begin{matrix} \text{row}_1 - \text{row}_2 \\ \Rightarrow \\ \text{row}_3 + 2\text{row}_2 \end{matrix} \quad \begin{pmatrix} -1 & 0 & -3 & | & -1 & 1 & 0 \\ 0 & 1 & 4 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 2 & -4 & 1 \end{pmatrix}$$

$$\begin{matrix} (\text{row}_1 + 3\text{row}_3)/-1 \\ \Rightarrow \qquad \text{row}_2 - 4\text{row}_2 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 & | & -5 & 11 & -3 \\ 0 & 1 & 0 & | & -7 & 16 & -4 \\ 0 & 0 & 1 & | & 2 & -4 & 1 \end{pmatrix}.$$

2.7   Exercises

1. Find the inverses of the following matrices

$$\begin{pmatrix} 2 & -3 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{pmatrix}$$

2. Solve the following equations by first finding the inverse of a matrix,

$$\begin{aligned} 3y + 2x - z - 1 &= 0 \\ 3x + 2z + 5y - 8 &= 0 \\ 3z - x + 2y - 1 &= 0 \end{aligned}$$

3. A matrix $\mathbf{A}$ is such that $\mathbf{A}^\top = -\mathbf{A}$ and $\mathbf{I} + \mathbf{A}$ is invertible. Show that $(\mathbf{I} + \mathbf{A})^{-1\top} = (\mathbf{I} - \mathbf{A})^{-1}$.

2.8   LU Decomposition

Most of the time we do not need to reduce a matrix to its Gaussian form in order to solve a matrix equation $\mathbf{Ax} = \mathbf{b}$. It is enough to perform row operations that reduce $\mathbf{A}$ to an upper-triangular matrix $\mathbf{U}$. To take example 2.1.1 again,

$$\begin{pmatrix} 2 & -2 & 6 & | & 2 \\ 4 & -3 & 14 & | & -22 \\ -9 & 6 & -32 & | & 50 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 1 & -1 & 3 & | & 1 \\ 0 & 1 & 2 & | & -26 \\ 0 & 0 & 1 & | & -19 \end{pmatrix}.$$

The last equation tells us $z = -19$, the second is $y + 2z = -26$, so $y = -26 - 2(-19) = 12$, and the first $x - y + 3z = 1$ so $x = 1 + y - 3z = 1 + 12 - 3(-19) = 70$. We did not need to perform Gaussian elimination completely to solve the equation.

To summarize, starting from

$$\mathbf{Ax} = \mathbf{b}$$

we apply row operations

$$E_n \dots E_1 \mathbf{Ax} = E_n \dots E_1 \mathbf{b}$$

until the left-hand matrix is upper-triangular

$$\mathbf{Ux} = \mathbf{Eb}.$$

At this stage we solve for $\mathbf{x}$ by starting with the last row equation, and work our way to the first row equation.

If there are many equations of this sort to solve, it makes sense to work out **E** once, and apply it to **b** whenever needed. Recalling that $\mathbf{E} = E_n \ldots E_1 \mathbf{I}$, and using the trick we used for the inverse, we can find **E** as follows:

$$[\mathbf{A}|\mathbf{I}]$$

apply row operations

$$[E_r \ldots E_1 \mathbf{A}|E_r \ldots E_1 \mathbf{I}]$$

until the left-hand side is upper-triangular,

$$[\mathbf{U}|\mathbf{E}].$$

Then, whenever we need to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$, we can apply the matrix **E** on both sides to get $\mathbf{U}\mathbf{x} = \mathbf{E}\mathbf{b}$, which can easily be solved.

For the above example,

$$\left(\begin{array}{ccc|ccc} 2 & -2 & 6 & 1 & 0 & 0 \\ 4 & -3 & 14 & 0 & 1 & 0 \\ -9 & 6 & -32 & 0 & 0 & 1 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3/2 & 6 & 1 \end{array} \right)$$

It is not a coincidence that **E** is *lower-triangular*. If the diagonal of **A** has no zeros, there will not be any need to switch rows, so each of the row-operation matrices E_r that are used is lower-triangular. So their product $\mathbf{E} = E_n \ldots E_1$ must also be lower-triangular (convince yourself that the product of two lower-triangular matrices is again lower-triangular.)

If a diagonal coefficient of **U** is 0, then there are no (unique) solutions; we have already encountered such cases in the full Gaussian elimination method.

We can go one step further because it is quite straight-forward to find the inverse of a lower-triangular matrix. Suppose we try to find a matrix **L** such that $\mathbf{L}\mathbf{E} = \mathbf{I}$ for the above example. Let's find first the top right corner coefficient of **L**: since the last column of **E** is mostly zeros, we get $L_{13}E_{33} = 0$, so $L_{13} = 0$. Similarly going through the other coefficients in the upper half of the matrix **L** we find that they are all zero – **L** must be lower-triangular as well!

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To find the other coefficients we can start with the diagonal ones,

$$a/2 + 0 + 0 = 1, \ 0 + c + 0 = 1, \ 0 + 0 + f = 1,$$

then the sub-diagonal,

$$b/2 - 2c + 0 = 0, \ 0 + e + 6f = 0,$$

and finally

$$d/2 - 2e - 3f/2 = 0,$$

giving in sequence $a = 2$, $c = 1$, $f = 1$, $b = 4$, $e = -6$ and $d = -21$, so

$$\mathbf{L} = \mathbf{E}^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -21 & -6 & 1 \end{pmatrix}.$$

Note that since $\mathbf{EA} = E_n \dots E_1 \mathbf{A} = \mathbf{U}$, and multiplying by $\mathbf{L}$, we have

$$\mathbf{A} = \mathbf{LU}$$

called appropriately the *LU*-decomposition of the matrix $\mathbf{A}$ into a lower-triangular and an upper-triangular matrix.

Having reached this stage, it is a simple matter to find the inverse $\mathbf{A}^{-1}$,

$$\mathbf{A}^{-1} = (\mathbf{LU})^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} = \mathbf{U}^{-1}\mathbf{E}.$$

Although this is called *LU*-decomposition, in practice $\mathbf{L}$ need not be computed: only $\mathbf{E}$ is needed to solve $\mathbf{Ax} = \mathbf{b}$; and to find $\mathbf{A}^{-1}$ we need to calculate $\mathbf{U}^{-1}$ as well.

2.8.1   Example

Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ -3 & -1 & 2 \\ 1 & 4 & 2 \end{pmatrix}$ and solve the equation $\mathbf{Ax} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Performing row operations on the matrices $[\mathbf{A}|\mathbf{I}]$ to get $[\mathbf{U}|\mathbf{E}]$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -3 & -1 & 2 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 & 1 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & -1 & 0 \\ 0 & 0 & 1 & -11 & -4 & -1 \end{array} \right).$$

To find the inverse of $\mathbf{U}$, we let

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplying out the rows and columns, starting with the diagonal elements, gives $a = 1$, $d = 1$ and $f = 1$; then $b = 0$ and $d + e = 0$ so $e = -1$; and finally $-a + b + c = 0$, so $c = 1$, and

$$\mathbf{U}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
\Rightarrow \mathbf{A}^{-1} &= \mathbf{U}^{-1}\mathbf{E} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & 0 \\ -11 & -4 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -10 & -4 & -1 \\ 8 & 3 & 1 \\ -11 & -4 & -1 \end{pmatrix}
\end{aligned}
$$

To solve the equation (without using the inverse), is the same as solving $\mathbf{Ux} = \mathbf{Eb}$, i.e.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & 0 \\ -11 & -4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -16 \end{pmatrix},$$

which gives $z = -16$, $y + z = -4$ and $x - z = 1$, so that $x = -15$, $y = 12$, $z = -16$.

3   Determinants

We sometimes need to determine when a matrix is invertible or not. We will show in this chapter that the following definition of a determinant will do this job for us.

3.1   Definition

The **determinant** of an $n \times n$ matrix is defined recursively by

$$
\det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \dots & a_{nn} \end{pmatrix} - a_{21} \det \begin{pmatrix} a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}
$$

$$
+ \dots \pm a_{1n} \det \begin{pmatrix} a_{21} & \dots & a_{2n-1} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn-1} \end{pmatrix}
$$

where the signs in front of each sub-determinant alternates between $+$ and $-$.

In particular, for a 2×2 matrix,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

3.1.1   Example

The determinants of the following matrices are, using the definition,

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \times 4 - 2 \times 3 = -2$$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$
$$= (5 \times 9 - 6 \times 8) - 2(4 \times 9 - 6 \times 7) + 3(4 \times 8 - 5 \times 7)$$
$$= 0$$

To simplify the working of a determinant, we can make use of a number of properties:

3.2   Laplace's theorem

One can calculate the determinant by expanding along any row or column as long as we adopt the following signs for the sub-determinants.

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \end{pmatrix}$$

So in the previous example, we could have expanded along, say, the second column as follows

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = -2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 5 \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} - 8 \det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$
$$= -2(4 \times 9 - 6 \times 7) + 5(1 \times 9 - 3 \times 7) - 8(1 \times 6 - 3 \times 4)$$
$$= 0$$

One usually uses this property by expanding along a row/column that has a number of zeros.

3.3   Properties

The determinant of a triangular matrix is just the product of the diagonal elements,

$$\det \begin{pmatrix} a_{11} & & \cdots & \\ 0 & a_{22} & & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11} a_{22} \ldots a_{nn}$$

Example.

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} = 1 \times 4 \times 6 = 24$$

Let us consider, in particular, the elementary row matrices, that is, those matrices that are equivalent to the row operations of multiplying a row by a scalar, switching two rows, and adding a row to another:

$$\det E_I = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = \lambda$$

$$\det E_{II} = \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

$$\det E_{III} = \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

3.3.1   Transpose

$$\det \mathbf{A}^\top = \det \mathbf{A}$$

When we transfer the columns of a matrix to rows, its determinant is unchanged.

3.3.2   Products

$$\det \mathbf{A}\mathbf{B} = \det \mathbf{A} \det \mathbf{B}$$

More generally, the determinant of a product of any number of matrices, can be found by multiplying out the determinants of the matrices.

In particular, we can deduce the following sub-properties.

- $$\det E_I \mathbf{A} = \lambda \det \mathbf{A}$$

  If we find a common scalar in a whole row (or column), then that scalar can be taken outside the determinant.

- $$\det E_{II} \mathbf{A} = -\det \mathbf{A}$$

  Switching two rows of a matrix changes the sign of its determinant.

- $$\det E_{III} \mathbf{A} = \det \mathbf{A}$$

  Adding a row to another row does not change the determinant at all.

Note that the second property implies that if two rows of a matrix are identical, then its determinant is zero, since switching the two identical rows does not change the matrix, but reverses the sign of the determinant.

$$E_{II}\mathbf{A} = \mathbf{A}$$

$$-\det \mathbf{A} = \det \mathbf{A}$$

$$\det \mathbf{A} = 0$$

Any of these properties can be used to work out a determinant. We illustrate this by working out a previous example, this time making full use of these properties.

3.3.3   Example

$$
\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 3 \det \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 3 \end{pmatrix} \text{ taking 3 outside the 3rd column}
$$

$$
= 3 \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{pmatrix} \begin{array}{l} \text{row}_2 - 4\text{row}_1 \\ \text{row}_3 - 7\text{row}_1 \end{array}
$$

$$
= 6 \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{pmatrix} \text{ taking out -1 and -2 from 2nd and 3rd rows}
$$

$$
= 6 \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \text{row}_3 - \text{row}_2
$$

$$
= 1 \times 3 \times 0
$$

$$
= 0
$$

3.3.4 Example

Similarly,

$$
\begin{aligned}
\det \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{pmatrix} &= 2 \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{pmatrix} \\
&= 2 \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -5 & -11 \end{pmatrix} \begin{matrix} \\ \text{row}_2 - 4\text{row}_1 \\ \text{row}_3 - 3\text{row}_1 \end{matrix} \\
&= 6 \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 5 & 11 \end{pmatrix} \\
&= 6 \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ \text{row}_3 - 5\text{row}_2 \end{matrix} \\
&= 6
\end{aligned}
$$

These properties are especially useful when some of the elements of the matrix are variables.

3.3.5 Example

$$
\begin{aligned}
\det \begin{pmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{pmatrix} &= \det \begin{pmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \text{row}_1 + \text{row}_2 \\ \\ \end{matrix} \\
&= (a+b+c) \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{pmatrix} \\
&= 0
\end{aligned}
$$

since there are two equal rows.

3.3.6   Example

$$
\det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = \det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}
$$

$$
= \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & (b+a)(b-a) \\ 0 & c-a & (c+a)(c-a) \end{pmatrix}
$$

$$
= (b-a)(c-a) \det \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{pmatrix}
$$

$$
= (b-a)(c-a) \det \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & c-b \end{pmatrix}
$$

$$
= (b-a)(c-a)(c-b)
$$

since the last matrix is triangular.

3.4   Proposition

A matrix $\mathbf{A}$ is invertible exactly when $\det \mathbf{A} \neq 0$.

Proof. Suppose $\mathbf{A}$ is invertible. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, and so taking the determinant on both sides we get

$$
\det \mathbf{A}^{-1} \det \mathbf{A} = \det \mathbf{I} = 1
$$

Hence $\det \mathbf{A}$ must be non-zero for the product to give 1.

Conversely suppose $\mathbf{A}$ is not invertible. From Chapter 2, we know that this happens when, after performing row operations on it, we end up with a row of zeros.

$$
E_k \ldots E_1 \mathbf{A} = \begin{pmatrix} & \cdots & \\ 0 & \cdots & 0 \end{pmatrix}
$$

which implies that

$$
\det E_k \ldots \det E_1 \det \mathbf{A} = \det \begin{pmatrix} & \cdots & \\ 0 & \cdots & 0 \end{pmatrix} = 0
$$

But all the row operation matrices E_i have non-zero determinant, which implies that $\det \mathbf{A} = 0$.

3.5 Exercises

1. Evaluate the determinants of the following matrices:

$$\begin{pmatrix} 2 & 0 & -1 \\ 3 & 0 & 5 \\ 1 & 8 & 2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & -4 \\ 1 & 0 & -2 \\ -2 & 3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

2. Prove the following identities:

(a)

$$\det \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

(b)

$$\det \begin{pmatrix} \cos\phi & \sin\phi\cos\theta & \sin\phi\sin\theta \\ -\sin\phi & \cos\phi\cos\theta & \cos\phi\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = 1$$

(c)

$$\det \begin{pmatrix} \sin^2 x & \sin 2x & \cos 2x \\ \sin^2 y & \sin 2y & \cos 2y \\ \sin^2 z & \sin 2z & \cos 2z \end{pmatrix} = -2\sin(x-y)\sin(y-z)\sin(z-x)$$

3. Solve the equation

$$\det \begin{pmatrix} z & z & z \\ 2 & z^2+2 & 2 \\ z & z+1 & z^2+z \end{pmatrix} = 3i$$

4 Diagonalization

Suppose we are asked to calculate the following matrix

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^{100}$$

The straightforward way is to multiply out a hundred copies of the matrix. This section will introduce a way of doing such calculations much quicker. It is based on the fact that it is easy to calculate

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{100} = \begin{pmatrix} a^{100} & 0 \\ 0 & b^{100} \end{pmatrix}.$$

The idea is to decompose a matrix into three matrices, the inner one being a **diagonal** matrix.

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

For example, we will learn how to decompose the above matrix into

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}$$

Why is this better than before? Let us see what happens when we calculate $\mathbf{A}^3$ using the decomposition.

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{PDP}^{-1}\,\mathbf{PDP}^{-1}\,\mathbf{PDP}^{-1} \\ &= \mathbf{PD}^3\mathbf{P}^{-1} \end{aligned}$$

More generally,

$$\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$$

It is much easier to work with diagonal matrices than with matrices in general. In fact, the addition and multiplication of diagonal matrices are other diagonal matrices.

$$\begin{pmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \\ \vdots & & \end{pmatrix} + \begin{pmatrix} b_1 & 0 & \cdots \\ 0 & b_2 & \\ \vdots & & \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & 0 & \cdots \\ 0 & a_2 + b_2 & \\ \vdots & & \end{pmatrix}$$

$$\begin{pmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \\ \vdots & & \end{pmatrix} \cdot \begin{pmatrix} b_1 & 0 & \cdots \\ 0 & b_2 & \\ \vdots & & \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 & \cdots \\ 0 & a_2 b_2 & \\ \vdots & & \end{pmatrix}$$

So, to turn back to the original problem,

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^{100} &= \mathbf{PD}^{100}\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^{100} & 0 \\ 0 & 5^{100} \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix} \end{aligned}$$

which involves only three matrix multiplications.

This idea can be generalized even further to include any algebraic expression that involves $\mathbf{A}$ only.

$$\begin{aligned} \mathbf{A}^3 - 2\mathbf{A}^2 + 3\mathbf{A} + \mathbf{I} &= \mathbf{PD}^3\mathbf{P}^{-1} - 2\mathbf{PD}^2\mathbf{P}^{-1} + 3\mathbf{PDP}^{-1} + \mathbf{PIP}^{-1} \\ &= \mathbf{P}(\mathbf{D}^3 - 2\mathbf{D}^2 + 3\mathbf{D} + \mathbf{I})\mathbf{P}^{-1} \end{aligned}$$

Just as before, it is quite easy to calculate the inner expression which involves a sum of diagonal matrices. We then need to multiply it with $\mathbf{P}$ in front and $\mathbf{P}^{-1}$ at the back to get the required answer.

4.1   Eigenvalues and Eigenvectors

Definition   The numbers in the diagonal matrix $\mathbf{D}$ are called the **eigenvalues** of $\mathbf{A}$. The columns of the matrix $\mathbf{P}$ are called the **eigenvectors** of $\mathbf{A}$.

An eigenvalue, λ, and its corresponding eigenvector, $\mathbf{v}$, have the special property that they satisfy the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Exercise: Check that this is true for the matrix in the example above.

Once we know what the eigenvalues and eigenvectors of a matrix are, we can form the matrix $\mathbf{D}$ from the eigenvalues, and the matrix $\mathbf{P}$ from the eigenvectors, and hence get the decomposition $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

4.1.1   Finding the Eigenvalues

The eigenvalues are the roots of the *characteristic equation*

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This follows from the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, which is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$. We require the vector $\mathbf{v}$ not to be the zero vector, so it must be the case that $\mathbf{A} - \lambda\mathbf{I}$ is *not* invertible; if it were we would get $\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\mathbf{0} = \mathbf{0}$.

So, let us show that the eigenvalues of $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ are -1 and 5. Substituting the matrix into the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, we get

$$\det\left(\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{pmatrix} = 0.$$

$$\Rightarrow \quad (1 - \lambda)(3 - \lambda) - 8 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

That is, the eigenvalues are precisely -1 and 5.

4.1.2   Finding the Eigenvectors

We must use the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ again, this time to find $\mathbf{v}$. Given $\mathbf{A}$ and λ, this reduces to the type of problem we tackled in the second chapter.

Returning to our example, let us find the eigenvector corresponding to -1. We need to solve

$$\left(\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the method of chapter 2,

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

As expected there is a row of zeros at the bottom, since the determinant of the matrix $\mathbf{A} - \lambda\mathbf{I}$ is zero. Therefore there is only one effective equation $x + y = 0$. Picking $x = 1$ forces $y = -1$, so that we get the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Of course, we could have picked any value for x, with $y = -x$, and this would give an equally valid eigenvector (except $x = 0 = y$ which would give the zero vector).

Similarly, to find an eigenvector for $\lambda = 5$, we use the same procedure. We have to solve the equation

$$\left(\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{v} = \mathbf{0}$$

$$\Rightarrow \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Once again, the bottom row of zeros is a sign that the working was correct. There is only one equation $-2x + y = 0$, as it should be. Picking $x = 1$ forces $y = 2$, so that we get the eigenvector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

4.2   Example

Let us work out in full an example of a 3×3 matrix,

$$\begin{pmatrix} 10 & -2 & -10 \\ 3 & 1 & -3 \\ 5 & -1 & -5 \end{pmatrix}$$

First we find its eigenvalues, according to the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

$$\det\begin{pmatrix} 10 - \lambda & -2 & -10 \\ 3 & 1 - \lambda & -3 \\ 5 & -1 & -5 - \lambda \end{pmatrix} = 0$$

$$\Rightarrow \qquad (10-\lambda)\left((1-\lambda)(-5-\lambda)-3\right)+2\left(3(-5-\lambda)+15\right)-10\left(-3-5(1-\lambda)\right)=0$$

$$\Rightarrow \qquad -\lambda^3+6\lambda^2-8\lambda=0$$

$$\Rightarrow \qquad -\lambda(\lambda-2)(\lambda-4)=0$$

Its eigenvalues turn out to be 0, 2 and 4.

Next we find an eigenvector for each of the three eigenvalues.

For $\lambda=0$, we have to solve $(\mathbf{A}-0\mathbf{I})\mathbf{v}=\mathbf{0}$, i.e.,

$$\left(\begin{array}{ccc|c} 10 & -2 & -10 & 0 \\ 3 & 1 & -3 & 0 \\ 5 & -1 & -5 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 5 & -1 & -5 & 0 \\ 3 & 1 & -3 & 0 \\ 5 & -1 & -5 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 5 & -1 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This yields two equations $x+z=0$ and $y=0$. Therefore one possible eigenvector has components $x=1$, $y=0$ and $z=-x=-1$, i.e.

$$\mathbf{v}=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda=2$, we solve $(\mathbf{A}-2\mathbf{I})\mathbf{v}=\mathbf{0}$,

$$\left(\begin{array}{ccc|c} 8 & -2 & -10 & 0 \\ 3 & -1 & -3 & 0 \\ 5 & -1 & -7 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 4 & -1 & -5 & 0 \\ 3 & -1 & -1 & 0 \\ 5 & -1 & -7 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 4 & -1 & 5 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

With the final equations being $2x+z=0$ and $-y+3z=0$, we get a solution $x=1$, $z=-2x=-2$ and $y=3z=-6$, so that one eigenvector is $\mathbf{v}=\begin{pmatrix} 1 \\ -2 \\ -6 \end{pmatrix}$.

Finally for $\lambda=4$, we have to solve $(\mathbf{A}-4\mathbf{I})\mathbf{v}=\mathbf{0}$,

$$\left(\begin{array}{ccc|c} 6 & -2 & -10 & 0 \\ 3 & -3 & -3 & 0 \\ 5 & -1 & -9 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 3 & -1 & -5 & 0 \\ 1 & -1 & -1 & 0 \\ 5 & -1 & -9 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & -1 & -5 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

which gives the equations $x - 2z = 0$ and $-y + z = 0$. Picking $z = 1$ gives $x = 2z = 2$ and $y = z = 1$ to get the eigenvector

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Combining these results we can form the matrices $\mathbf{D}$ and $\mathbf{P}$,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ -1 & -6 & 1 \end{pmatrix}$$

Care must be taken to place the eigenvalues in the diagonal of $\mathbf{D}$ in the same order as the corresponding eigenvectors are placed in the columns of $\mathbf{P}$.

In fact, we can also calculate the inverse of $\mathbf{P}$ as in chapter 2 to get (after some working),

$$\mathbf{P}^{-1} = \begin{pmatrix} -4 & 13 & -5 \\ 1 & -3 & 1 \\ 2 & -5 & 2 \end{pmatrix}$$

We can now firmly state that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$,

$$\begin{pmatrix} 10 & -2 & -10 \\ 3 & 1 & -3 \\ 5 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ -1 & -6 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -4 & 13 & -5 \\ 1 & -3 & 1 \\ 2 & -5 & 2 \end{pmatrix}$$

4.3   What can go wrong

There are two reasons why things may go awry in this diagonalization process.

- The characteristic equation may have complex roots, and not just real roots. In actual fact, this is not exactly a problem. We can continue in the same way, except we have to allow for complex eigenvalues and complex eigenvectors. That is, the presence of complex roots complicates the actual calculation but the method remains valid. But to avoid these complications we will not have examples like this.

- An eigenvalue may be repeated. In this case there may or may not be a problem. When we try to find the eigenvector for the repeated

eigenvalue, we may find fewer eigenvectors than are needed to make the matrix $\mathbf{P}$. Let us illustrate this by an example of a 2×2 matrix,

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

We first find its eigenvalues, using the characteristic equation.

$$\det \begin{pmatrix} 3 - \lambda & -1 \\ 1 & 1 - \lambda \end{pmatrix} = 0$$

$$\Rightarrow (3 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

The eigenvalues are therefore $\lambda = 2$ repeated twice.

To find its corresponding eigenvector(s), we solve $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$,

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array} \right) \Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

As expected, there is a row of zeros, with only one remaining equation $x - y = 0$, so that we can pick as eigenvector, say,

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

But this way we end up with just a single eigenvector, which is not enough to fill the 2×2 matrix $\mathbf{P}$. We cannot do anything about this situation and must conclude that the original matrix is **not** diagonalizable.

We should not get the impression that this always happens; in many cases a repeated eigenvalue gives enough eigenvectors to fill out the matrix $\mathbf{P}$. The following example is one such,

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

The eigenvalues are found from the characteristic equation, which yields the roots $\lambda = 1, 2, 2$ (check!).

The eigenvalue $\lambda = 1$ yields an eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ (again check this out!) The eigenvalue $\lambda = 2$ has corresponding eigenvectors that

satisfy the equation $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$.

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

This gives just *one* equation $y = 0$, with *both* x and z that can vary independently of each other. So we can pick two eigenvectors for $\lambda = 2$,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In total we now have three eigenvectors, with the matrices $\mathbf{D}$ and $\mathbf{P}$ given by

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Note that in this case, when an eigenvalue λ has more than one eigenvector, it does not matter which order the eigenvectors of λ are placed as columns of $\mathbf{P}$, as long as they are not placed under the column "belonging" to the other eigenvalues.

4.4   Exercises

1. Find the eigenvalues and eigenvectors of the following matrices,

$$\begin{pmatrix} -4 & 2 & 14 \\ 6 & 0 & -14 \\ -3 & 1 & 9 \end{pmatrix}, \qquad \begin{pmatrix} 11 & 8 & 4 \\ -12 & -9 & -4 \\ 4 & 4 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -7 & 6 & -3 \\ 0 & -1 & 1 \\ 12 & -12 & 7 \end{pmatrix}.$$

2. For the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 7 & -2 \\ -4 & -6 & 2 \\ 0 & -1 & 1 \end{pmatrix},$$

find $\mathbf{A}^{10} - 5\mathbf{A}$.