Ordered Sets

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1 September 2012

The *morphisms* on a **relation** (digraph) $\rightsquigarrow$ between points are those maps that preserve this relation:

$$x \rightsquigarrow y \;\Rightarrow\; f(x) \rightsquigarrow f(y)$$

Two points x, y are *indistinguishable* by the relation when (for all z)

$$x \rightsquigarrow z \Leftrightarrow y \rightsquigarrow z, \qquad z \rightsquigarrow x \Leftrightarrow z \rightsquigarrow y.$$

Digraphs are connective spaces (graphs): $\{x, y\}$ is connected when x, y are comparable, i.e., $x \rightsquigarrow y$ OR $y \rightsquigarrow x$.

The *product* relation on $A \times X$ is given by

$$(a, x) \rightsquigarrow (b, y) \text{ when } a \rightsquigarrow b \text{ AND } x \rightsquigarrow y$$

The *sum* $X \oplus Y$ is the disjoint union of X and Y, with elements of X and Y unrelated with each other.

The *exponential* set Y^X of functions into a digraph has the relation

$$f \rightsquigarrow g \Leftrightarrow \forall x \in X, f(x) \rightsquigarrow g(x)$$

A **pre-order** is a relation $\leqslant$ which is transitive and reflexive

$$x \leqslant y \text{ AND } y \leqslant z \;\Rightarrow\; x \leqslant z, \qquad x \leqslant x$$

The morphisms are those that preserve the order (*monotone* maps)

$$x \leqslant y \;\Rightarrow\; f(x) \leqslant f(y)$$

the monomorphisms are the 1-1 monotonic maps, epimorphisms are the onto monotonic maps; $\varnothing$ is the initial object; $\{*\}$ is the terminal object; (note that bijective morphisms need not be isomorphisms, and $X \subsetneqq Y$ AND $Y \subsetneqq X$ is possible without $X \cong Y$, e.g. $\mathbb{Q}$ and $\mathbb{Q}^*$). A pre-ordered set is itself a category in which the morphisms are $x \leqslant y$.

Points x, y are indistinguishable when $x \leqslant y \leqslant x$.

Every digraph gives rise to a pre-order: let $x \leqslant y$ mean that there exists a directed path from x to y: $x \rightsquigarrow \cdots \rightsquigarrow y$; in this case, the equivalence classes of indistinguishable points are called "strongly connected" components.

If x, y are incomparable, then one can extend the $\leqslant$ relation so that $x \leqslant' y$ becomes true: Define $a \leqslant' b$ to mean $a \leqslant b$ OR ($a \leqslant x$ AND $y \leqslant b$).

An **order** is a pre-order without indistinguishable points,

$$x \leqslant y \leqslant x \;\Rightarrow\; x = y$$

Any pre-order becomes an order by identifying its indistinguishable elements and defining $[x] \leqslant [y]$ when $x \leqslant y$. Subsets, $X \times Y$, $X \oplus Y$, X^A, are ordered spaces,

$$(x, a) \leqslant (y, b) \;\Leftrightarrow\; x \leqslant y \text{ AND } a \leqslant b$$

$$f \leqslant g \;\Leftrightarrow\; \forall x \in A, f(x) \leqslant g(x)$$

in particular the power set 2^X has the order $A \subseteq B$ on subsets.

Example:

1. Words with $x \leqslant y$ when the letters of x are in y in the same order, e.g. `ab` $\leqslant$ `caxb`

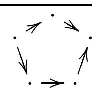
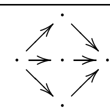
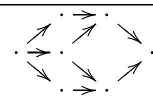
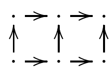
2. Given any collection $\mathcal{S}$ of subsets of X (which distinguish points), let $x \leqslant y$ when $\forall A \in \mathcal{S},\; x \in A \;\Rightarrow\; y \in A$.

3. Given functions $\pi_i : Y \to X_i$, with X_i ordered, then Y is also ordered by

$$y \leqslant z \;\Leftrightarrow\; \forall i, \pi_i(y) \leqslant \pi_i(z)$$

4. Decision problems (i.e., capable of yes/no answers) with $a \leqslant b$ when a can be transformed to b in polynomial number of steps (a pre-order called *reduction*).

A subset B is a *refinement* of another A when $\forall x \in A, \exists y \in B, \quad y \leqslant x$.

	Finite	**Finitely-Generated**	
Ordered Spaces $\leqslant$			Words
Lattices $\vee$, $\wedge$			Subgroups
Modular Lattices $x\leqslant y \Leftrightarrow x\vee(z\wedge y)=(x\vee z)\wedge y$			normal subgroups, submodules
Distributive Lattices $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$			$\mathbb{N}$ with $\mid$ closed subintervals of $[0,1]$
Heyting algebras $x \to y$	 eg divisors of n		bounded linear orders, open subsets
Boolean algebras $\neg\neg x = x$	2^n		2^A, clopen subsets eventually periodic $0,1$-sequences

Given a morphism $f : X \to X$, one can form sequences $x \leqslant f(x) \leqslant f(f(x)) \leqslant \cdots$, which may perhaps terminate at a fixed point $f(y) = y$. If this happens for each x, then the morphism $x \mapsto y$ is called a **closure** map: a morphism $x \mapsto \bar{x}$ such that

$$x \leqslant \bar{x} = \bar{\bar{x}}$$

(equivalently, $x \leqslant \bar{y} \Leftrightarrow \bar{x} \leqslant \bar{y}$.) Elements with $x = \bar{x}$ are called *closed*; then $\bar{x}$ is that unique smallest closed element larger than x (since $x \leqslant y \;\Rightarrow\; \bar{x} \leqslant \bar{y} = y$).

An *interior* map $x \mapsto \underline{x}$ is the dual: a morphism such that $x \geqslant \underline{x} = \underline{\underline{x}}$.

$f : X \to Y$ has a (right or upper) **adjoint** $f^* : Y \to X$ when

$$f(x) \leqslant y \Leftrightarrow x \leqslant f^*(y)$$

1. $x \leqslant f^* \circ f(x)$ and $f \circ f^*(y) \leqslant y$ (so f^* is unique)
2. f, f^* are morphisms (since $x \leqslant y \leqslant f^* \circ f(y)$)
3. $f \circ f^* \circ f = f$ and $f^* \circ f \circ f^* = f^*$

   Proof. $x \leqslant f^* \circ f(x)$, so $f(x) \leqslant f(f^*(f(x))) \leqslant f(x)$
4. $f^* \circ f$ is a closure map (x is closed iff $x = f^*(y)$), and $f \circ f^*$ is an interior map. (Every closure map arises this way.) So f and f^* are inverses on the closed elements.

5. When $y \leqslant f(x) \Leftrightarrow x \leqslant f^*(y)$, then the same identities hold except $y \leqslant f \circ f^*(y)$, $x \leqslant y \Rightarrow f(x) \geqslant f(y)$.

For example, given any relation $\rightsquigarrow \subseteq X \times Y$, the maps $f : 2^X \to 2^Y$ and $f^* : 2^Y \to 2^X$, (here $A \rightsquigarrow y$ means $\forall a \in A,\ a \rightsquigarrow y$),

$$f(A) := \{ y \in Y : A \rightsquigarrow y \}, \quad f^*(B) := \{ x \in X : x \rightsquigarrow B \}$$

satisfy $B \subseteq f(A) \Leftrightarrow A \subseteq f^*(B)$.

A set is *bounded below* when it has a **lowerbound**, namely an element $x \leqslant A$. An *infimum* of a set is a greatest lowerbound, denoted $\bigwedge A$ or $\inf A$ (necessarily unique *if* it exists),

$$z \leqslant A \Leftrightarrow z \leqslant \bigwedge A,$$

in particular, $x \wedge y := \bigwedge\{ x, y \}$,

$$z \leqslant x, y \Leftrightarrow z \leqslant x \wedge y$$

When the infima exist, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \wedge y = y \wedge x$, $x \wedge x = x$.

A *minimal* element $a \in A$ is one for which

$$\forall x \in A, \quad x \leqslant a \Rightarrow x = a$$

A *minimum* is a lowerbound which is an element of the set (hence the infimum and minimal). The dual concepts are: *bounded above*, *upperbound*, *supremum* (denoted $\sup A$ or $\bigvee A$ and $x \vee y$), *maximal* element, and *maximum*. The supremum and infimum certainly exist when the elements are comparable,

$$x \leqslant y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$$

The set of elements lower than a is denoted $\downarrow a := \{ x \in X : x \leqslant a \}$ and called the *principal lower-set* of a. More generally, the *lowerset* generated by A consists of its elements' lowerbounds,

$$\downarrow A := \{ x \in X : \exists a \in A,\ x \leqslant a \}$$

The dual concepts are $\uparrow a$ and $\uparrow A$. The maps $A \mapsto \uparrow A$ and $A \mapsto \downarrow A$ are closure operations; a set is *lower-closed* when $\downarrow A = A$; similarly for *upper-closed*.

A (closed) *interval* is $[a, b] := \{ x \in X : a \leqslant x \leqslant b \} = \uparrow a \cap \downarrow b$; also $[a, b[:= [a, b] \setminus \{ b \}$, etc. If $a \not\leqslant b$, then $[a, b] = \varnothing$. More generally, a set A is *convex* when $x, y \in A \Rightarrow [x, y] \subseteq A$, equivalently $A = \downarrow A \cap \uparrow A$ (e.g. intervals).

1. The intersection and union of lower-closed sets are lower-closed,

$$\downarrow(A \cup B) = \downarrow A \cup \downarrow B, \quad \downarrow(A \cap B) \subseteq \downarrow A \cap \downarrow B$$

2. The complement of a lower-closed set is upper-closed.
3. The map $x \mapsto {\downarrow}x$ is an embedding of X in 2^X. So every ordered space can be represented by a space of subsets.
4. Lowerbounds$(A) \subseteq {\downarrow}A$ is lower-closed, e.g. Lowerbounds$(x) = {\downarrow}x$.
5. Lowerbounds$(A) = \{ x \in X : x \leqslant A \}$, Upperbounds$(A) = \{ x \in X : A \leqslant x \}$, so Lowerbounds and Upperbounds are adjoints on 2^X,
$$A \subseteq \text{Lowerbounds}(B) \Leftrightarrow B \subseteq \text{Upperbounds}(A)$$
A 'closed' subset induces a *cut*: a pair of subsets A, B such that $A =$ Lowerbounds(B) and $B =$ Upperbounds(A).
6. The intersection of convex sets is convex, so every set A generates its *convex hull* Convex$(A) := \bigcap\{ B \subset X : \text{convex}, A \subseteq B \}$.
7. The kernel of a morphism consists of convex equivalence classes.

A set is **cofinal** when every element has a larger element $\forall x, \exists a \in A,\ x \leqslant a$; any cofinal subset contains the maximal elements of X. The dual concept is *co-initial*. The smallest cardinality of the cofinal subsets is called the *cofinality* of the ordered space; the space is called *regular* when the cofinality equals the cardinality of the space.

A set is (up) **directed** when every pair has an upperbound in the set
$$\forall x, y \in A, \exists z \in A, \quad x, y \leqslant z$$
(For example, an increasing sequence, $x_1 \leqslant x_2 \leqslant x_3 \cdots$.) Its lower-closure is another directed set, called an **ideal**, e.g. the *principal* ideal ${\downarrow}a$. Every proper ideal can be extended to a maximal proper ideal (using Zorn's lemma, since the union of a chain of ideals is an ideal). For example, uncomparable elements can be separated by a maximal ideal (that extends ${\downarrow}x$),
$$x \not\leqslant y \ \Rightarrow\ \exists I \text{ maximal ideal}, x \in I, y \notin I.$$
In a finite space, every ideal is principal (${\downarrow}\max I$).

A **filter** is the dual of an ideal, i.e., an upper-closed set such that every pair has a lowerbound $\forall x, y \in F, \exists z \in F,\ z \leqslant x, y$; e.g. the principal filter ${\uparrow}a$. If an ideal and a filter are set-complements, they are called *prime*.

A **net** is a map from a directed set. A net $(x_i)_{i \in I}$ is said to be *eventually* in A, here denoted by $x_i \twoheadrightarrow\!\in A$, when
$$\exists j \in I,\ i \geqslant j \ \Rightarrow\ x_i \in A.$$

The following properties hold:

$$\begin{aligned} x_i \twoheadrightarrow\!\!\in A \subseteq B &\Rightarrow x_i \twoheadrightarrow\!\!\in B, \\ x_i \twoheadrightarrow\!\!\in A \cap B &\Leftrightarrow x_i \twoheadrightarrow\!\!\in A \text{ AND } x_i \twoheadrightarrow\!\!\in B, \\ x_i \twoheadrightarrow\!\!\in A \text{ OR } x_i \twoheadrightarrow\!\!\in B &\Rightarrow x_i \twoheadrightarrow\!\!\in A \cup B, \\ x_i \twoheadrightarrow\!\!\in A^{\mathsf{c}} &\Rightarrow \text{NOT } (x_i \twoheadrightarrow\!\!\in A). \end{aligned}$$

A *subnet* is a composition $J \to I \to X$ such that $J \to I$ is increasing and $\forall a \in I, \exists j \in J, i_j \geqslant a$.

A net $(x_i)_{i \in I}$ *converges up in order* $x_i \nearrow x$ when x_i is increasing with supremum x. Similarly, $x_i \searrow x$ when x_i is decreasing with infimum x.

An element c is *compact* when for any ideal I, $c \leqslant \text{Upperbounds}(I) \Rightarrow c \in I$.

An order is **dense** when $\forall x < y, \exists z,\ x < z < y$. Otherwise, if $x < y$ have no other elements in between, then we say there is a *gap* between x and y (or that *y covers x*). An order is *locally finite* when every interval is finite.

0.0.1 Hausdorff Maximality Principle

Zorn's Lemma: In a non-empty ordered space, if every chain has an upperbound, then there is a maximal element.

Proof: Using the axiom of choice, map every chain C to an upperbound of it $g(C)$. Suppose there is no maximal element, so can map every element to $f(x) > x$. Let F : Ordinals $\to X$ be defined by $F(0) := x_0 \in X$, $F(n^+) := f \circ F(n)$, $F(\lim n) := f \circ g\{\, F(n)\,\}$. F is strictly increasing, hence 1-1; so there is an onto map $H : X \to$ Ordinals, contradicting that Ordinals is not a set.

Hausdorff's Maximality Principle: Every chain can be extended to a maximal chain.

Proof: Order the chains that contain c_0 by inclusion; every chain C of such chains has an upperbound, namely $\bigcup C$, which is a chain; hence there is a maximal chain.

0.0.2 DCC Orders

A **DCC** order is one in which every non-empty subset has a minimal element. (An *ACC* order is the dual: every non-empty subset has a maximal element.)

Equivalently, every descending sequence is finite (since if A has no minimal element, then it has an infinite descending sequence a_n by letting $a_{n+1} < a_n$ (using axiom of choice); conversely, suppose $x_1 \geqslant x_2 \geqslant x_3 \ldots$ with minimal element x_n; then $x_i = x_n$ for all $i \geqslant n$).

Transfinite Induction for DCC: If $(\forall y < x,\ y \in A) \Rightarrow x \in A$, then $A = X$ (since the set $X \smallsetminus A$ has no minimal element).

0.1 Bounded Orders

An ordered set is **bounded** when $\exists 0, 1,\ 0 \leqslant X \leqslant 1$; equivalently $0 := \bigvee \varnothing$, $1 := \bigwedge \varnothing$ exist; or when the map $X \to \{ * \}$ has an upper and lower adjoint. (Morphisms are now required to preserve $0, 1$.)

$$\begin{aligned} x \wedge 0 &= 0, & x \wedge 1 &= x, \\ x \vee 0 &= x, & x \vee 1 &= 1 \end{aligned}$$

An element a is an **atom** when it is minimal among the non-zero elements $x \leqslant a \ \Rightarrow\ x = a$ XOR $x = 0$ (i.e., $0 < a$ is a gap). The ordered space is called *atomic* when every non-zero element is greater than some atom. It is *atomistic* when every element is generated by atoms, $x = \bigvee_i a_i$ for some atoms a_i. An atom a is *independent* of a set A when $a \nleqslant \bigvee A$; a set A of atoms is *independent* when each atom a is independent of $A \smallsetminus \{ a \}$.

Two elements x, y are said to be **complementary** when they satisfy "excluded middle":

$$x \wedge y = 0, \quad x \vee y = 1$$

e.g. 0 and 1. A space is *complemented* when every element has a complement. In a complemented space, morphisms are required to preserve complements.

1. The only elements that compare with a non-$0, 1$ complementary pair are 0 and 1; so $[x, y] = \varnothing$, $\downarrow x \cap \downarrow y = 0$ and $\uparrow x \cup \uparrow y = 1$.

   Proof: If $z \leqslant x, y$, then $z \leqslant x \wedge y = 0$; if $x \leqslant z \leqslant y$ then $0 = x \wedge y = x$.

2. In a complemented space, prime ideals are maximal.

   Proof: If $P \subset Q$, then there is an $a \in Q \smallsetminus P$; its complement $b \in P$ since $a \wedge b = 0$; so $1 = a \vee b \in Q$ and $Q = X$.

An *implication* $\to$ is an operation such that $x \leqslant y \Leftrightarrow (x \to y) = 1$.

1 Lattices

A *semi-lattice* is an ordered space in which any finite subset has a least upper-bound, or *supremum*, in particular $0 = \bigvee \varnothing$ and $x \vee y = \bigvee \{ x, y \}$.

Equivalently, a set with an idempotent, commutative, associative operation $\vee$ (then $x \leqslant y \Leftrightarrow y = x \vee y$).

Equivalently, when the map $x \mapsto (x, x)$, $X \to X^2$, has an upper adjoint $(x, y) \mapsto x \vee y$.

A semi-lattice morphism is one which preserves $\vee$ (hence monotonic)

$$f(x \vee y) = f(x) \vee f(y)$$

The map $x \mapsto x \vee y$ is a closure morphism and is the only such which is idempotent; more generally $\vee : X^2 \to X$ is a $\vee$-morphism. A map is a $\vee$-isomorphism $\Leftrightarrow$ it is a bijective $\vee$-morphism $\Leftrightarrow$ it is an order-isomorphism (since $f(z) = f(x) \vee f(y) \leqslant f(x \vee y)$, so $z \leqslant x \vee y$ and $x, y \leqslant z$, so $z = x \vee y$).

Images, Products are semi-lattices, $(x, a) \vee (y, b) := (x \vee y, a \vee b)$; also Functions Y^X when Y is a semi-lattice: $(f \vee g)(x) := f(x) \vee g(x)$.

An element a is said to be $\vee$-*irreducible* when

$$a = x \vee y \ \Rightarrow \ x = a \text{ OR } y = a$$

(e.g. atoms); in particular the $\vee$-*primes*:

$$a \leqslant x \vee y \ \Rightarrow \ a \leqslant x \text{ OR } a \leqslant y$$

(since if $a = x \vee y \geqslant x$ then $a \leqslant x$, say, so $a = x$).

1. A set A is upper-closed iff $x \vee A \subseteq A$ for any x.

2. $\uparrow(x \vee y) = \uparrow x \cap \uparrow y$.

3. For any closure map, $\bar{x} \vee \bar{y} \leqslant \overline{x \vee y}$. The set of closed elements form a $\vee$-semi-lattice (and also a $\wedge$-semi-lattice: $x \wedge y \leqslant \overline{x \wedge y} \leqslant \bar{x}, \bar{y}$).

4. I is an ideal iff it is a lower-closed semi-lattice,

$$x \vee y \in I \Leftrightarrow x \in I \text{ AND } y \in I.$$

5. A filter is prime iff $x \vee y \in F \Leftrightarrow x \in F$ OR $y \in F$.

   e.g. $\uparrow x$ is a prime filter $\Leftrightarrow$ x is $\vee$-prime.

6. If $f : X \to Y$ is a $\vee$-morphism, then its kernel $f^{-1}0$ is an ideal.

   Conversely, when I is an ideal, the map $x \mapsto \begin{cases} 0 & x \in I \\ 1 & x \notin I \end{cases}$ is a $\vee$-morphism (with kernel I).

7. The intersection of ideals is an ideal. So every set generates a unique ideal, namely the smallest ideal containing it,

$$(A] := \downarrow\{\, a_1 \vee \cdots \vee a_n : a_i \in A, n \in \mathbb{N}\,\}$$

The map $A \mapsto (A]$ is a closure map in 2^X.

8. An element c is compact when for any subset A,

$$c \leqslant \text{Upperbounds}(A) \;\Rightarrow\; \exists n \in \mathbb{N},\; a_i \in A,\; c \leqslant a_1 \vee \cdots \vee a_n.$$

The set of compact elements of a semi-lattice is a semi-lattice.

9. If a semi-lattice is generated by a set A, then the compact $\vee$-irreducibles are in A (for example, in an atomistic semi-lattice, the compact irreducibles are the atoms).

Proof. If $x = \bigvee_i a_i$, then $x \leqslant a_1 \vee \cdots \vee a_n \leqslant \bigvee_i a_i = x$ (x compact); so $x = a_i$ (x irreducible).

10. In an atomic semi-lattice, an *essential* element b is an upperbound of the set of atoms, equivalently

$$b \wedge x = 0 \;\Rightarrow\; x = 0.$$

Proof: For any $x \neq 0$, there is an atom a, $0 < a \leqslant x, b$, so $0 < a = a \wedge b \leqslant x \wedge b$. Conversely, for any atom $a \neq 0$, $0 < a \wedge b$, so $a \wedge b = a$.

The dual notions are a *superfluous* element b, which is a lowerbound of the coatoms, equivalent to $b \vee x = 1 \;\Rightarrow\; x = 1$.

Examples of ideals: the set of finite subsets of a set; the set of bounded subsets of a topological space; the compact elements of any lattice.

A **lattice** has both suprema $x \vee y$ and infima $x \wedge y$.

Equivalently, a set with two operations that are associative, commutative, idempotent, and absorptive $x \wedge (x \vee y) = x = x \vee (x \wedge y)$; in this case $x \leqslant y \Leftrightarrow x \wedge y = x \;\Leftrightarrow\; x \vee y = y$. Equivalently the map $x \mapsto (x, x)$ has both an upper and lower adjoint.

The morphisms are now those maps that preserve $\vee$, $\wedge$. Images, products, and exponentials are lattices; so are ideals, filters, and intervals $[a, b]$.

All properties for semi-lattices apply for both $\vee$ and $\wedge$ (in dual form).

1. $(x \wedge y) \vee (x \wedge z) \leqslant x \wedge (y \vee z)$, $x \vee (y \wedge z) \leqslant (x \vee y) \wedge (x \vee z)$ (but strict distributivity need not hold). In particular, $x \vee (z \wedge y) \leqslant (x \vee z) \wedge y \Leftrightarrow x \leqslant y$.

More generally, the *minimax* principle states that maxmin$\leqslant$minmax,

$$(x_1 \wedge a_1) \vee \cdots \vee (x_1 \wedge a_n) \vee \cdots \vee (x_m \wedge a_n) \leqslant (x_1 \vee \cdots x_m) \wedge (a_1 \vee \cdots a_n)$$

2. Complements are preserved by morphisms: $f(x) \vee f(y) = f(x \vee y) = 1$, $f(x) \wedge f(y) = f(x \wedge y) = 0$.

3. A congruent partition of X is one such that $[x] \wedge [y] := [x \wedge y]$, $[x] \vee [y] := [x \vee y]$ are well-defined. Then $X/\sim$ is a lattice, and $f : X \to X/\sim$, $f(x) := [x]$ is an onto morphism.

   Conversely, every morphism gives rise to a congruent partition $\ker f$ of equivalence classes $f^{-1}(y)$, with $f^{-1}0$ an ideal and $f^{-1}1$ a filter. The first isomorphism theorem holds: $X/\ker f \cong fX$.

4. Adjoints f, f^* are semi-lattice morphisms:

   $$f \text{ preserves } \vee \text{ and } f^* \text{ preserves } \wedge$$

   Proof. $f(x),\ f(y) \leqslant z \Leftrightarrow x \vee y \leqslant f^*(z) \Leftrightarrow f(x \vee y) \leqslant z$.

5. A sublattice A is convex iff for all $x \in X$, $a, b \in A$, $a \vee (x \wedge b) \in A$.

A **complete lattice** is an order in which every subset has a supremum and an infimum (it is enough for all sets to have suprema since $\bigwedge A = \bigvee \text{Lowerbounds}(A)$). *Scott*-morphisms are those that preserve suprema and infima $f(\bigvee A) = \bigvee fA$.

An order-isomorphism between complete lattices is Scott-continuous. Proof: $f(x) = \bigvee fA \geqslant f(a)$ for all $a \in A$ implies $a \leqslant x$, hence $f(\bigvee A) \leqslant \bigvee fA$. Also $a \leqslant \bigvee A$ implies $f(a) \leqslant f(\bigvee A)$, so $\bigvee fA \leqslant f(\bigvee A)$.

Examples: 2^X, $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$; finite $\vee$-semi-lattices; the upper-closed subsets of an order.

Proposition 1

> **Let X be a complete lattice.**
> **If Y is a complete lattice in X, then**
>
> $$x \mapsto \bar{x} := \bigwedge\{\, y \in Y : x \leqslant y \,\}$$
>
> **is a closure map, with Y as its closed elements.**
> **Given a closure map on X, the set of closed elements is a complete lattice.**

PROOF: $x \leqslant y \Rightarrow \bar{x} \leqslant \bar{y}$ and $x \leqslant \bar{x}$ are obvious. Moreover, for $y \in Y$, $x \leqslant y \Leftrightarrow \bar{x} \leqslant y$, so $\bar{\bar{x}} = \bar{x}$. Since Y is complete, $\bar{x} \in Y$.

Let A consist of closed elements. Its least upper bound is $\overline{\bigvee A}$ since it is an upperbound for A and any other closed upperbound is larger $A \leqslant x = \bar{x} \Rightarrow \overline{\bigvee A} \leqslant x$.

□

Every order can be embedded in a complete lattice; the smallest such lattice is called its *completion*:

Proposition 2

(Dedekind-MacNeille)

Every ordered space has a completion.

PROOF: The maps $L :=$ Lowerbounds and $U :=$ Upperbounds are adjoints on 2^X, with closure LU. So their 'closed' sets (cuts) form a complete subspace of 2^X. The map $x \mapsto L\{x\} = \downarrow x$ embeds X into this complete lattice.

□

Proposition 3

(Tarski)

Every monotonic function on a complete lattice has a fixed point.

PROOF: The non-empty set $A = \{ x \in X : f(x) \leqslant x \}$ has an infimum a; thus $x \in A \Rightarrow a \leqslant x \Rightarrow f(a) \leqslant f(x) \leqslant x$, so that $f(a) \leqslant a$; also $x \in A \Rightarrow f(x) \in A$; but then $a, f(a) \in A$ and $f(a) \leqslant a \leqslant f(a)$.

□

The set of fixed points of an order-morphism on a complete lattice is itself a complete lattice.

In a complete atomic lattice, the *socle* is $\bigvee\{ a : \text{atom} \} = \bigwedge\{ b : \text{essential} \}$. Dually, the *radical* is $\bigwedge\{ a : \text{coatom} \} = \bigvee\{ b : \text{superfluous} \}$.

1.0.1 Algebraic Lattices

An *algebraic lattice* is a complete lattice that is generated by compact elements.

1. Atoms are compact.

2. In an algebraic lattice, any two elements $a < b$ contain a gap.

   Proof: If $a < b$ then there is a compact element c such that $c \leqslant b$, $c \not\leqslant a$. Let $A := \{ x \in [a, b] : c \not\leqslant x \}$, contains a. Any chain in A has sup in A, so A contains a maximal element d, by Zorn's lemma; thus $a \leqslant d < c \vee d \leqslant b$.

Examples:

- Given a closure operation on 2^X, if

$$\bar{A} = \bigcup\{ \bar{B} : B \subseteq A, B \text{ finite} \}$$

then the set of closed subsets is an algebraic lattice, and the compact elements are the closures of finite subsets.

Proof: Let X be the lattice of closed subsets. For any subset $B \subseteq A$, if $\bar{B} \leqslant \bigvee_i A_i$ with A_i closed, then $B \subseteq \bigvee_i A_i = \bigcup_i A_i = \bigcup\{\,\bar{F} : F \subseteq \bigcup_i A_i, F \text{ finite}\,\}$. If B is finite then $B \subseteq A_1 \cup \cdots \cup A_n$ and $\bar{B} \subseteq A_1 \cup \cdots \cup A_n$; thus $\bar{B}$ is compact.

- A universal algebra X has a closure map $A \mapsto [\![A]\!]$ in 2^X with the above property. Hence the subalgebras ('closed' subsets) form an algebraic lattice, with $M \wedge N = M \cap N$, $M \vee N = [\![M \cup N]\!]$; the compact elements are the finitely generated subalgebras. (And every algebraic lattice is induced from some universal algebra.)
- The lattice of ideals of a semi-lattice, with $\bigwedge_i I_i = \bigcap_i I_i$, $\bigvee_i I_i = (\bigcup_i I_i]$. Its compact elements are $\downarrow x$. X is embedded in it via $x \mapsto \downarrow x$.

1.0.2 ACC Lattices

1. An ACC semi-lattice is a complete lattice.

   Proof. The set $\{\,a_1 \vee \cdots \vee a_n : a_i \in A, n \in \mathbb{N}\,\} \supseteq A$ has a maximal element b, so $\bigvee A = b$.

2. Every element of an ACC lattice is compact.

   Proof. If $x \leqslant \bigvee A = \bigvee\{\,a_1 \vee \cdots \vee a_n : a_i \in A, n \in \mathbb{N}\,\}$, then x is less than a maximal element; conversely, if $x_1 < x_2 < \cdots$ is an ascending chain, then $\bigvee_i x_i \leqslant x_1 \vee \cdots \vee x_n = x_n$.

3. Every non-zero element of an ACC lattice can be written as $x = a \wedge \cdots \wedge b$ for some $\wedge$-irreducibles.

   Proof. Let x be a maximal element without such a decomposition; then x is not irreducible, i.e., $x = a \wedge b$ with $x < a, b$, so a, b have such decompositions.

1.1 Complements

A *dual* is a self-adjoint map $\star : X \to X$,

$$x \leqslant y^\star \Leftrightarrow y \leqslant x^\star$$

$a \leqslant b^\star$ can be denoted by $a \perp b$. Then Products have duals $(x, y)^\star := (x^\star, y^\star)$; as do functions $f^\star(x) := f(x)^\star$.

1. $\star$ satisfies all the properties of adjoints: uniqueness of $\star$, $x \leqslant y \Rightarrow y^\star \leqslant x^\star$, $x \leqslant x^{\star\star}$, $x^{\star\star\star} = x^\star$, $x \mapsto x^{\star\star}$ is a 'closure' map, $0^\star = 1$.

2. $x^\star \vee y^\star \leqslant (x \wedge y)^\star$ and $(x \vee y)^\star \leqslant x^\star \wedge y^\star$.

   Proof. Apply $\star$ to reverse the inequalities in $x \wedge y \leqslant x, y \leqslant x \vee y$.

A **quasi-complement** of x is a 'closed' dual, $x^{\star\star} = x$. Products are quasi-complemented.

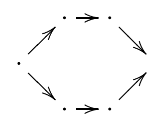
Example: Fuzzy logic $[0,1]$ with max, min, $x^\star := 1 - x$.

1. $x^\star = 0 \Leftrightarrow x = 1$, $x^\star = y \Leftrightarrow x = y^\star$.
2. *de Morgan's laws*: $(x \wedge y)^\star = x^\star \vee y^\star$, $\quad (x \vee y)^\star = x^\star \wedge y^\star$.

   Proof. Since $\star$ is a dual, $x \wedge y \leqslant (x^\star \vee y^\star)^\star \leqslant x^{\star\star} \wedge y^{\star\star} = x \wedge y$.
3. When the lattice is complete, $(\bigvee_i x_i)^\star = \bigwedge_i x_i^\star$ and $(\bigwedge_i x_i)^\star = \bigvee_i x_i^\star$ (by the same reasoning).
4. If A is lower-closed then the set of its quasi-complements $A^\star$ is upper-closed.

An **ortho-complement** of x is a complement quasi-complement $x^\perp$,

$$x \vee x^\perp = 1, \quad x \wedge x^\perp = 0, \quad x^{\perp\perp} = x, \quad x \leqslant y \Rightarrow y^\perp \leqslant x^\perp$$



Define $x \equiv y := (x \wedge y) \vee (x^\perp \wedge y^\perp)$, and let x COMM y to mean $x = (x \wedge y) \vee (x \wedge y^\perp)$ (or $(x \vee y^\perp) \wedge y = x \wedge y$).

One dual is a **pseudo-complement** of x: an element $\neg x$ such that

$$y \leqslant \neg x \Leftrightarrow x \wedge y = 0$$

i.e., the largest element such that $x \wedge \neg x = 0$. Products and Functions have pseudo-complements.

1. $\wedge$-isomorphisms preserve pseudo-complements $f(\neg x) = \neg f(x)$.

   Proof. $f(x) \wedge f(\neg x) = f(x \wedge \neg x) = 0$. If $0 = f(x) \wedge f(y) = f(x \wedge y)$ then $x \wedge y = 0$, so $y \leqslant \neg x$ and $f(y) \leqslant f(\neg x)$.
2. For any atom a, either $a \leqslant x$ or $a \leqslant \neg x$ (since $a \wedge x = a$ or 0).
3. A quasi-pseudo-complement is a complement (in fact the lattice must be Boolean). So the 'closure' of a pseudo-complemented lattice is Boolean.

The dual pseudo-complement is an element $x_\star$ such that $x_\star \leqslant y \Leftrightarrow x \vee y = 1$.

Proposition 4

Ultrafilters

$$\boxed{\textbf{A filter is maximal} \Leftrightarrow \forall x,\ x \in F \text{ XOR } \neg x \in F}$$

PROOF: Let F be maximal. If $x \notin F$, then the filter generated by F and x equals X, so for some $y \in F$, $0 = y \wedge x$, hence $y \leqslant \neg x$ and $\neg x \in F$. If $x, \neg x \in F$ then $0 = x \wedge \neg x \in F$, so $F = X$. For the converse, suppose G is a filter that contains F and $x \in G \setminus F$; then $\neg x \in F$, so $0 = x \wedge \neg x \in G$.

□

1.1.1 Ortho-modular Lattices

An **ortho-modular** lattice is an ortho-complemented lattice such that

$$x \equiv y = 1 \Leftrightarrow x = y$$

Equivalently, any of the following statements hold:

$$x \leqslant y \Leftrightarrow x \vee (x^{\perp} \wedge y) = y,$$
$$x \perp y \Leftrightarrow x^{\perp} \wedge (x \vee y) = y,$$
$$x \vee y = x \vee ((x \vee y) \wedge x^{\perp}),$$
$$x \leqslant y \Leftrightarrow x \leqslant (x^{\perp} \vee y),$$
$$x, y \text{ are complements such that } y \leqslant x^{\perp} \;\Rightarrow\; y = x^{\perp},$$
$$x \text{ COMM } y \Leftrightarrow y \text{ COMM } x (\Leftrightarrow x \text{ COMM } y^{\perp})$$

Example: *Quantum Logic*, the (atomistic) lattice of the closed linear subspaces of a Hilbert space.

1. There are various "quantum implications" $x \to y$: any of $x^{\perp} \vee y$, $x^{\perp} \vee (x \wedge y)$, $x \vee (x^{\perp} \wedge y^{\perp})$, $(x \wedge y) \vee (x^{\perp} \wedge y) \vee (x^{\perp} \wedge y^{\perp})$. For any of these,

$$x \vee y = (x \to y) \to (((x \to y) \to (y \to x)) \to x))$$

2. A *state* is a function $f : X \to [0,1]$ such that $x \perp y \;\Rightarrow\; f(x \vee y) = f(x) + f(y)$ and $f(1) = 1$. Hence f is an order-morphism and $f(0) = 0$.

3. Let $C(A) := \{\, x \in X : \forall a \in A,\ x \text{ COMM } a \,\}$. Then C is self-adjoint: $A \subseteq C(B) \Leftrightarrow B \subseteq C(A)$; hence, $A \subseteq C(C(A))$, $C(C(C(A))) = C(A)$.

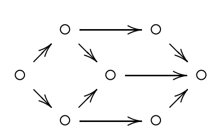
   (Foulis-Holland) If x, y, z commute with each other, then distributivity holds for x, y, z.

4. An atomic orthomodular lattice is atomistic.

   Proof. Let $y := \bigvee\{\, a \leqslant x : a \text{ atom} \,\}$; if $y < x$, then $x \wedge y^{\perp} \neq 0$, is greater than some atom a, i.e., $a \leqslant x$, $a \not\leqslant y$, a contradiction; hence $y = x$.

1.2 Semi-Modular Lattices

A lattice is *semi-modular* when $x \wedge y < x$ is a gap $\Rightarrow$ $y < x \vee y$ is a gap. Example:



1. For any atoms a, b, c, $c \leqslant a \vee b \;\Rightarrow\; a \vee b \vee c = b \vee c = a \vee c$ (for any two atoms, $a < a \vee b$ is a gap, so $b < b \vee c \leqslant a \vee b$).

2. If $0 < a_1 < \cdots < a_n < 1$ and $0 < b_1 < \cdots < b_m < 1$ are chains of gaps, then $n = m$ (called the height of X) and there is a permutation which maps one chain to the other such that $[a_i, a_{i+1}] \cong [b_j, b_{j+1}]$.

   Proof. For $n = 1$, follows by definition of semi-modularity. If $a_1 = b_1$, then follows by induction on the rest of the chains. Otherwise let k be the largest integer such that $a_1 \not\leqslant b_k$, so $a_1 \leqslant b_{k+1}$. Then $a_1 < a_1 \vee b_1 < \cdots < a_1 \vee b_k = a_1 \vee b_{k+1} = b_{k+1} < b_{k+2} < \cdots < b_n < 1$. By induction, $[a_i, a_{i+1}] \cong [a_1 \vee b_j, a_1 \vee b_{j+1}] \cong [b_j, b_{j+1}]$.

3. A lattice with a morphism $\delta : X \to \mathbb{N}$ such that $\delta(0) = 0$, $x < y$ gap $\Rightarrow \delta(y) = \delta(x) + 1$, and $\delta(x \vee y) + \delta(x \wedge y) \leqslant \delta(x) + \delta(y)$ must be semi-modular.

   Proof. Let $x \wedge y < x$ be a gap; then $\delta(x \vee y) > \delta(y)$ else $x \leqslant y$. So $\delta(x \wedge y) + \delta(x \vee y) \leqslant \delta(x \wedge y) + \delta(y) + 1$, hence $\delta(y) < \delta(x \vee y) = \delta(y) + 1$, and $y < x \vee y$ is a gap.

A **geometric** lattice is an algebraic semi-modular atomistic lattice such that every chain is finite.

Subintervals are again geometric lattices; so there is a *rank* function $\delta : X \to \mathbb{N}$ which gives the length of any maximal chain from 0 to x.

1. Every basis of a geometric lattice have the same number of atoms (since $0 < a_1 < a_1 \vee a_2 < \cdots < a_1 \vee \cdots \vee a_n = 1$).

2. Geometric lattices are complemented.

   Proof. For any x, if $x \wedge a = 0$ but $x \vee a < 1$, then there is an atom $b \not\leqslant x \vee a$; so $x \vee a < x \vee a \vee b$; there can be no atom $c \leqslant x \wedge (a \vee b)$, otherwise $x \vee a \vee b = x \vee a \vee c = x \vee a$; so $x \wedge (a \vee b) = 0$; hence one can pick a sequence of atoms a_i such that $x \wedge (a_1 \vee \cdots \vee a_i) = 0$ and $x \vee a_1 \vee \cdots \vee a_{i+1} > x \vee a_1 \vee \cdots \vee a_i$; as this chain is finite we find $x \wedge y = 0$, $x \vee y = 1$.

Examples:

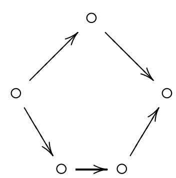
- The set of equivalence relations on a set, with $\rho \leqslant \sigma$ when ρ is finer than σ (i.e., $x\rho y \Rightarrow x\sigma y$). ($x(\rho \vee \sigma)y$ holds when there is a path of relations $x\tau_1 \cdots \tau_k y$ where $\tau_i = \rho$ or σ.) Every lattice can be embedded in some lattice of equivalence relations (hence embedded in the lattice of subgroups: $a \mapsto \{\, g \in S(X) : \forall x \in X,\ g.x\rho_a x \,\}$).
- Projective Geometry: subspaces of a vector space.
- Incidence Geometry: elements are points and lines.

1.3 Modular Lattices

A lattice is **modular** when $x \leqslant y \Leftrightarrow x \vee (z \wedge y) = (x \vee z) \wedge y$, or equivalently,

$$(x \vee y) \wedge (x \vee z) = x \vee (y \wedge (x \vee z))$$

Equivalently, any lattice which does not contain this embedded lattice (otherwise $y \wedge z < x < y < x \vee z$ and $y \wedge z < z < x \vee z$):



For example, subspaces of a vector space.
Images, sublattices and products are again modular.

Proposition 5

Diamond Isomorphism theorem

The intervals $[a, a \vee b]$ and $[a \wedge b, b]$ are order-isomorphic, via the maps $x \mapsto x \vee a$ and $x \mapsto x \wedge b$.

Proof: If $a \leqslant x \leqslant a \vee b$ then $(x \wedge b) \vee a = x \wedge (b \vee a) = x$.
If $a \wedge b \leqslant x \leqslant b$ then $(x \vee a) \wedge b = x \vee (a \wedge b) = x$.
Hence the monotonic maps $x \mapsto x \vee a$ and $x \mapsto x \wedge b$ are isomorphisms.
□

1. Modular lattices are thus semi-modular. By duality, its rank function satisfies $\delta(x \wedge y) + \delta(x \vee y) = \delta(x) + \delta(y)$.

2. Minimal (irredundant) decompositions into $\wedge$-irreducibles $a = r_1 \wedge \cdots \wedge r_n$ have unique lengths.

   Proof. Suppose $a = r_1 \wedge r = s_1 \wedge s$ with r_1, s_1 irreducibles. Then $[a, r] = [r_1 \wedge r, r] \cong [r_1, r_1 \vee r]$. But $a = r \wedge s_1 \wedge s = (r \wedge s_1) \wedge (r \wedge s)$, with $a \leqslant r \wedge s_1 \leqslant r$. Hence $r_1 = J(r \wedge s_1) \wedge J(r \wedge s)$, so $r_1 = J(r \wedge s_1)$ say, i.e., $a = r \wedge s_1$. Repeating for the other irreducibles gives $a = s_1 \wedge \cdots \wedge s_m \wedge r_{m+1} \wedge r_n = a \wedge \tilde{r}$ (if $n > m$) which implies $a \leqslant \tilde{r}$ impossible; similarly $n < m$ can't hold.

3. The lattice generated by $\downarrow\{x, y\}$ is isomorphic to $\downarrow x \times \downarrow y$.

1.4 Distributive Lattices

A lattice is distributive when the map $x \mapsto a \wedge x$ (or $x \mapsto a \vee x$) is a lattice morphism (onto $\downarrow a$, resp. $\uparrow a$), that is,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

($\Leftrightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$) so modular.

Equivalently, any of these identities holds:

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x),$$
$$x \vee (z \wedge y) \leqslant (x \vee z) \wedge y,$$
$$\text{cancelation } (x \vee z = y \vee z \text{ AND } x \wedge z = y \wedge z) \Leftrightarrow x = y.$$

Equivalently, a modular lattice which does not contain the diamond sublattice (since suppose a, b, c do not obey distributivity; let $p := (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$, $q := (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$, $x := (a \wedge q) \vee p$, $y := (b \wedge q) \vee p$, $z := (c \wedge q) \vee p$, then $p < x < q$, $p < y < q$, $p < z < q$ form a diamond):

$$\begin{tikzcd}
 & \circ \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[r] & \circ \\
 & \circ \arrow[ur] &
\end{tikzcd}$$

Equivalently,

$$a \geqslant x \wedge y \;\Rightarrow\; \exists u \geqslant x,\ v \geqslant y, \quad a = u \wedge v$$
$$a \leqslant x \vee y \;\Rightarrow\; \exists u \leqslant x,\ v \leqslant y, \quad x = u \vee v$$

$$\begin{tikzcd}
 & x \arrow[rr] & & u \\
x \wedge y \arrow[ur] \arrow[rr] \arrow[dr] & & a \arrow[ur] \arrow[dr] & \\
 & y \arrow[rr] & & v
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[rr] \arrow[dr] & & x \arrow[dr] & \\
 & a \arrow[rr] & & x \vee y \\
v \arrow[rr] \arrow[ur] & & y \arrow[ur] &
\end{tikzcd}$$

Equivalently, when the lattice is embedded in some (Boolean) lattice 2^A.

Images, sublattices, and products are distributive.

1. If $a \leqslant x \vee b$ and $a \wedge x \leqslant b$ then $a \leqslant b$ (since $a = a \wedge (x \vee b) \leqslant b$).

2. $\{ x : a \vee x = 1 \}$ is a filter, and $\{ x : a \wedge x = 0 \}$ is an ideal.

3. Complements are pseudo-complements and quasi-complements, hence unique and denoted x':

$$x'' = x, \qquad x \leqslant y \Leftrightarrow x' \vee y = 1 \Leftrightarrow x \wedge y' = 0$$

   Proof. If x, y and x, z are complementary, then $y = y \wedge (x \vee z) = y \wedge z$, so $y \leqslant z$; similarly $z \leqslant y$. So $x'' = x$ since both are complementary to x'. If $x' \wedge y = 0$ then $y = (x \vee x') \wedge y = x \wedge y$, so $y \leqslant x$.

4. $$x \triangle y = (x \vee y) \wedge (x' \vee y') = (x \wedge y') \vee (x' \wedge y),$$
$$x \equiv y = (x \wedge y) \vee (x' \wedge y') = (x \vee y') \wedge (x' \vee y)$$

5. Pseudo-complements satisfy a de Morgan law: $\neg(x \vee y) = \neg x \wedge \neg y$.

   Proof. $(x \vee y) \wedge (\neg x \wedge \neg y) = 0$, so $\neg x \wedge \neg y \leqslant \neg(x \vee y)$.

   Also, $\neg(x \vee \neg x) = 0$ but $\neg x = 0 \not\Rightarrow x = 1$.

6. The *boundary* of an element is $\partial x := x \vee \neg x$; so
$$\partial\partial x = \partial x, \qquad \partial(x \vee y) = (\partial x \vee y) \wedge (x \vee \partial y)$$
   $\neg x$ is the complement of x when $\partial x = 1$.

7. Irreducibles are primes. (Proof. If x is irreducible and $x \leqslant a \vee b$, then $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$ so $x = x \wedge a \leqslant a$ or $x = x \wedge b \leqslant b$.)

8. The *spectrum* of an element a is $\sigma(a) := \{ x \leqslant a : x \text{ is } \vee\text{-irreducible} \}$. Then $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$ and $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$; $\sigma(a') = S \smallsetminus \sigma(a)$.

9. The set of lower-closed sets of an ordered space form a complete distributive lattice (with $\cup$, $\cap$). For a complete lattice, a lower-closed set is $\vee$-irreducible $\Leftrightarrow$ it is principal.

10. The set of ideals form a complete distributive pseudo-complemented lattice, with $\neg I := \{ x : x \wedge I = 0 \}$, $I \vee J = \{ x \vee y : x \in I, y \in J \}$ (since if $a \in I \vee J$ then there are $x \in I$, $y \in J$ such that $a \leqslant x \vee y$, so $a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$).

11. Maximal ideals (and filters) are prime.

    Proof. Suppose I is a maximal ideal and $a \wedge b \in I$, $a \notin I$. The ideal generated by I and a must be X, so that $b \leqslant x \vee a$ for some $x \in I$. Then $b \leqslant (x \vee a) \wedge b = (x \wedge b) \vee (a \wedge b) \in I$, hence $b \in I$.

12. If $I \vee J$, $I \wedge J$ are principal ideals, then so are I, J.

    Proof. If $I \vee J = {\downarrow} a$, $I \cap J = {\downarrow} b$, then $a = x \vee y$, and $b \vee x \in I$, $b \vee y \in J$, so if there is $z \in I$ with $z > b \vee x$, then $a, b, b \vee x, b \vee y, z$ would form a pentagon. Similarly, $J = {\downarrow} b \vee y$.

13. Distributive lattices can be embedded into 2^S, where $S = \mathrm{Spec}(X)$ is the set of prime ideals, via the map $x \mapsto \{ I \in \mathrm{Spec}(X) : x \notin I \}$ (or via the map $x \mapsto \sigma(x)$).

14. A result is true for all distributive lattices iff it is true for $2 := \{ 0, 1 \}$. (Proof. If a result is true for 2, then it is true for 2^S, hence for its sublattices.)

15. In a finite distributive lattice, every element has a unique irredundant decomposition into $\vee$-irreducibles, $x = a_1 \vee \cdots \vee a_n$.

    Proof. The irreducibles are the maximal elements of $\sigma(x)$.

2 Heyting Algebras

are bounded lattices in which $\wedge$ residuates $\vee$, i.e.,

$$x \wedge w \leqslant y \Leftrightarrow w \leqslant (x \to y)$$

That is, $x \to y$ is the largest element satisfying $x \wedge (x \to y) \leqslant y$.

Equivalently, a lattice with an operation $\to$ satisfying

$$x \to x = 1, \quad x \wedge (x \to y) = x \wedge y, \quad (x \to y) \wedge y = y$$

$$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$$

Heyting-morphisms must preserve $0, 1, \vee, \wedge$ and $\to$. Products are Heyting, with $(x_1, y_1) \to (x_2, y_2) := (x_1 \to x_2, y_1 \to y_2)$; for functions, $(f \to g)(x) = f(x) \to g(x)$.

Complete distributive lattices

$$x \wedge \bigvee A = \bigvee \{ x \wedge a : a \in A \}$$

are (complete) Heyting (also called *frames*), with $x \to y := \bigvee \{ w : x \wedge w \leqslant y \}$. In particular a *completely distributive lattice*, which satisfies $\bigwedge \bigvee A = \bigvee \bigwedge A$.

Example: open subsets in a topological space, with $A \to B = (A^{\mathsf{c}} \cup B)^{\circ}$, $\neg A = \bar{A}^{\mathsf{c}} = \mathrm{ext}(A)$.

1. Must be a distributive lattice with a pseudo-complement $\neg x := x \to 0$, $\to$ is an implication.

   Proof: If $x \wedge y \leqslant (x \wedge y) \vee (x \wedge z) =: w$ then $y, z \leqslant x \to w$, so $y \vee z \leqslant x \to w$ and $x \wedge (y \vee z) \leqslant x \wedge (x \to w) \leqslant w$.

2. $x \leqslant y \Leftrightarrow (x \to y) = 1$,
   $x \to (y \to z) = (x \wedge y) \to z \leqslant (x \to y) \to (x \to z)$,
   $x \leqslant (y \to x)$,
   $x \leqslant (y \to x \wedge y)$,
   $(x \to z) \leqslant ((y \to z) \to (x \vee y \to z))$.

3. A filter induces a quotient Heyting algebra X/F defined by the equivalence relation $x \to y \in F$ AND $y \to x \in F$. First isomorphism theorem holds.

4. If $x \vee \neg x = 1$ then x is *regular*, $\neg\neg x = x$.

   (since $\neg\neg x = \neg\neg x \wedge (x \vee \neg x) = \neg\neg x \wedge x$)

5. The set of complemented elements form a Boolean subalgebra of X, while the *regular* elements form a Boolean algebra (but its $\vee$ may be different: $x \vee y := \neg(\neg x \wedge \neg y)$), and $x \mapsto \neg\neg x$ is a morphism from the Heyting algebra to this Boolean algebra.

Note: the dual notion is an adjoint for $\vee$,

$$w \vee y \geqslant x \Leftrightarrow w \geqslant x - y$$

It has its own pseudo-complement, namely $x_* := 1 - x$; so $x \vee x_* = 1$, $0_* = 1$, $1_* = 0$; and its "boundary" map $x \mapsto x \wedge x_*$.

2.1 Boolean Algebras

are Heyting algebras with $x'' = x$.

Equivalently

- a complemented distributive lattice; or an orthomodular lattice in which all elements commute.
- a complete and completely distributive lattice (with $x \to y := \bigvee\{ z : z \wedge x \leqslant y \}$).
- a ring with $0, 1$ such that $x^2 = x$ ('Proof'. Let $x \vee y := x + y + xy, x \wedge y := xy$; conversely $x + y = x \triangle y := (x \wedge y') \vee (x' \wedge y), xy := x \wedge y$; its characteristic is 2 i.e., $-x = x$; note $+$ is not preserved by $\leqslant$; $x' = 1 - x$; $x \to y = 1 - x + xy$.)
- a set with an associative commutative operation $x \vee y$ and x' and 0, such that

  $$x'' = x,\ x \vee 0 = x,\ x \vee x = x,\ x \vee x' = 1 := 0',$$
  $$(x \vee (y \wedge z))' = (x \vee y)' \vee (x \vee z)'$$

  (Then $x \wedge y := (x' \vee y')'$.)
- a set with an operation $x - y$ and 1, such that

  $$x - 0 = x,\ x - (y - x) = x,\ \neg\neg x = x,\ (x - y) - z = (x - z) - y,$$
  $$(x - (y - z))' = (x - y)' - (z - x')$$

  where $x' := 1 - x$, $0 := \neg 1$, $x \wedge y := x - y'$, $x \vee y := (x' \wedge y')'$.

Products are Boolean. The completion is a Boolean algebra.

The subset $\downarrow a$ (or $\uparrow a$) is a Boolean algebra with the inherited operations and $x' := \neg x \wedge a$ (or $\neg x \vee a$). Then $\downarrow a \cong \uparrow \neg a$ via the map $x \mapsto x \vee \neg a$.

Examples include 2^A, ring ideals, the divisors of a square-free integer with $x \leqslant y$ meaning $x|y$, the finite/cofinite subsets of any set, the algebra generated (as a ring) from any collection of points.

1. $x \to y = x' \vee y$. Lattice morphisms automatically preserve $\to, '$.
2. For any atom a, either $a \leqslant x$ or $x \leqslant a'$. Hence $X \cong \uparrow a \times 2$. $X \cong \downarrow x \times \uparrow x$ for any x.
3. A Boolean algebra either has atoms or is dense (since $x < y$ is a gap $\Leftrightarrow$ $0 < x \triangle y$ is a gap).
4. A finite distributive lattice is Boolean iff atomic iff geometric.
5. I is an ideal $\Leftrightarrow$ $\neg I$ is a filter; I is a prime ideal $\Leftrightarrow$ $\neg I$ is maximal $\Leftrightarrow$ I is maximal (similarly for filters).

6. The space of order-morphisms $X \to 2$ from an ordered space form a Boolean lattice X^*. X^* distinguishes points in X, i.e., $x \neq y \Rightarrow \exists f \in X^*,\ f(x) \neq f(y)$ (e.g. take $f(z) := \begin{cases} 0 & z \leqslant x \\ 1 & z \nleqslant x \end{cases}$). Any order X is order-embedded in X^{**}.
7. A state is a probability distribution.

Proposition 6

Birkhoff-Stone Embedding Theorem

$$\boxed{\textbf{Every Boolean algebra can be embedded in some } 2^K.}$$

PROOF: Let $M := \{\, \phi : X \to 2,\ \text{morphism} \,\}$ (a totally disconnected compact T_2 space), and let $J : x \mapsto \{\, \phi \in M : \phi(x) = 1 \,\}$, $X \to 2^M$.

Then

$$\begin{aligned}
J(x \vee y) &= \{\, \phi \in M : 1 = \phi(x \vee y) = \phi(x) \vee \phi(y) \,\} \\
&= \{\, \phi \in M : 1 = \phi(x) \text{ OR } 1 = \phi(y) \,\} = J(x) \cup J(y) \\
J(x \wedge y) &= \{\, \phi \in M : 1 = \phi(x \wedge y) = \phi(x) \wedge \phi(y) \,\} \\
&= \{\, \phi \in M : 1 = \phi(x) \text{ AND } 1 = \phi(y) \,\} = J(x) \cup J(y) \\
J(x') &= \{\, \phi \in M : 1 = \phi(x') = \phi(x)' \,\} = J(x)' \\
J(0) &= \{\, \phi \in M : 1 = \phi(0) = 0 \,\} = \varnothing
\end{aligned}$$

J is 1-1: If $x \neq 0$ then x' is contained in some maximal ideal I, so $x \notin I$, so $\chi_I(x) = 1$ and $\chi_I \in J(x)$. Thus $J(x) = \varnothing \Rightarrow x = 0$.

$$J(x) = J(y) \Rightarrow \varnothing = J(x) \cap J(y)' = J(x \wedge y') \Rightarrow x \wedge y' = 0$$

and similarly $x' \wedge y = 0$, hence $x' = y'$.

$\square$

In particular, the free Boolean algebra with n generators is isomorphic to the set of monotonic functions $2^n \to 2$; each element is of the form $(x_1 \wedge \cdots \wedge x_n) \vee \cdots$ where $x_i = a_i$ or a_i'.

A finite Boolean algebra has 2^n elements for some n and is unique (since an atom splits X into two separate but isomorphic Boolean algebras $\uparrow a$ and $\downarrow a'$; by induction, such finite Boolean algebras are unique for each n). The Boolean algebra generated by n elements has 2^n atoms ($x_i \wedge \cdots \wedge x_j,\ i, j \leqslant n$), so 2^{2^n} elements.

There is only one countable dense Boolean algebra.

A **σ-algebra** is a Boolean algebra which is closed under countable suprema: for any sequence, $\bigvee_n x_n$ exists. For example, the algebra generated by a collection of points. The intersection of σ-algebras is again a σ-algebra, so a σ-algebra can be generated from any collection of points (can be larger than the generated Boolean algebra); e.g. the *Borel* σ-algebra generated by the basic open sets in a topological space.

2.2 Linearly Ordered Spaces

$$x \leqslant y \text{ OR } y \leqslant x$$

Equivalently, a maximally ordered space, or a lattice such that $x \wedge y$ and $x \vee y$ are both one of x, y. (The monotone maps preserve $\wedge, \vee$.) A bijective morphism is an isomorphism.

Linearly ordered spaces are Heyting algebras with $x \to y = \begin{cases} 1 & x \leqslant y \\ y & x > y \end{cases}$ and $\neg x = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$; satisfy both de Morgan's laws (but only $0, 1$ have quasi-complements, so the only Boolean linearly ordered spaces are 0 and $\{0, 1\}$).

Can define the *sum* $+$ and *multiplication* . of linearly ordered sets by

$$\sum_i X_i, \qquad x < y \Leftrightarrow \exists i < j, x \in X_i, y \in X_j \text{ OR } \exists i, x, y \in X_i, x < y$$

$$\prod_i X_i, \qquad x < y \Leftrightarrow J := \{ i : x_i \neq y_i \} \neq \varnothing \text{ AND } x_{i_0} < y_{i_0}$$

where i_0 is the least element of J (note that this is not the categorical order product defined previously).

1. Any non-empty bounded set has four possible relations with its upper bounds (and dually its lower bounds):

		Set max	
		Y	N
UB min	Y	$-\bullet\,\bullet-$	$-\circ\,\bullet-$
	N	$-\bullet\,\circ-$	$-\circ\,\circ-$

   The diagonal possibilities are called a *gap* and a (two-sided) *cut* respectively.

   A linear order is complete iff it has no cuts.

2. A finite linear order is isomorphic to some $\boldsymbol{n} := (0 < 1 < \cdots < n-1)$, hence complete (and bounded). A morphism $\boldsymbol{m} \to \boldsymbol{n}$ is completely described by the sub-intervals of $\boldsymbol{m}$ where it is not 1-1, and of $\boldsymbol{n}$ where it is not onto.

3. $\mathbb{N}^*$ is the smallest infinite bounded linear order, in the sense that $\mathbb{N}$ or $-\mathbb{N}$ is embedded in every infinite linear order. The smallest linear order without upper/lowerbounds is $\mathbb{Z}$.

4. The only linear orders in which every cut is given by a gap are $\boldsymbol{n}$ ($n \in \mathbb{N}$), $\mathbb{N}$, $-\mathbb{N}$, and $\mathbb{Z}$ (depending on existence of upper/lower bounds).

5. The countably infinite orders are sums of $\boldsymbol{n}$, $\mathbb{N}$, and $-\mathbb{N}$, where of course, $\boldsymbol{n}+\mathbb{N} = \mathbb{N}$, $-\mathbb{N}+\boldsymbol{n} = -\mathbb{N}$, and $-\mathbb{N}+\mathbb{N} = \mathbb{Z}$. Their number is $2^{\mathbb{N}}$: From a binary sequence, say 00101..., create the order $\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+1+\mathbb{Z}+\mathbb{Z}+1+\cdots$; each such order is not isomorphic to any other.

6. The smallest dense bounded linear order is $\mathbb{Q}^*$ (apart from 0). It is countably infinite and incomplete. Every countable linear order is embedded in it.

   Proof. Let $X = \{x_1, x_2, \dots\}$. Define $f : X \to \mathbb{Q}$ inductively by finding, at each step n (take $f(x_1) := 0$, say), the immediate neighbors $x_i < x_n < x_j$ with $i, j < n$ and letting $f(x_n)$ be any rational number with $f(x_i) < f(x_n) < f(x_j)$. Then $f : X \to \mathbb{Q}$ is an embedding. If X is dense bounded, f can be modified to an isomorphism $X \to \mathbb{Q}^*$: let $\mathbb{Q} = \{q_1, q_2, \dots\}$ and alternate in the same way between finding $f(x_n)$ and $f^{-1}(q_n)$; start by $f(0) = -\infty$ and $f(1) = +\infty$.

   For example, Words with alphabetical order is a dense countably infinite linear order.

7. For any countable linear order α, $\mathbb{Q}\alpha \cong \mathbb{Q}$ (hence $\mathbb{Q}^n \cong \mathbb{Q}$).

8. The order-completion of $\mathbb{Q}$ is called $\mathbb{R}^*$. It is the smallest complete dense bounded linear order; and $\mathbb{Q}$ is dense in it.

   Proof. Let C be a countable dense subset of X, hence isomorphic to $\mathbb{Q}$; extend the isomorphism $f : \mathbb{Q} \to C$ to $\mathbb{R} \to X$ by $f(\sup_n q_n) := \sup_n f(q_n)$.

9. But (Suslin) whether $\mathbb{R}^*$ is the only complete dense bounded linear order such that every set of disjoint intervals is countable, is undecidable; it is not, with the axiom of constructibility.

10. Hausdorff's maximality principle: Every chain in an ordered set X has a maximal chain. (If the maximal chain M has an upperbound $x \in X$, then $x \in M$ else M can be extended further by x; so x is a maximal element in X.)

11. Every finite ordered space can be extended to a linear order.

2.2.1 Well-ordered spaces

are ordered spaces for which every non-empty subset has a minimum. They are linearly ordered (since $\{ x, y \}$ has a minimum), complete (since the set of upperbounds has a minimum), and DCCs.

Subspaces, images, sums and products (as linear orders) are again well-ordered.

1. Every element x has an immediate *successor* x^+ except for the largest element if there is one, $x^+ = \min X \smallsetminus [0, x]$. The successor function is a 1-1 morphism with $x < x^+$. $[0, x^+[= [0, x]$. Each $x < x^+$ is a gap, and the only dense subsets of X are trivial.

2. Every element is either a successor or a *limit* point:

$$\exists y,\ x = y^+ \text{ XOR } x = \sup [0, x[= \bigvee_{y<x} y.$$

   (Let $z := \sup [0, x[$, then $z < x$ or $z = x$.)

3. Every lower-closed set is an *initial segment* $[0, x[$ for some $x \in X$, or X itself.

   Proof. Either $A = \varnothing$ when $x = 0$, or A is unbounded when $A = X$, or $y := \sup A$ exists; then either $y \in A$ when $x = y^+$, or $y \notin A$ when $x = y$.

4. *Transfinite induction*: If $(\forall y < x,\ y \in A) \Rightarrow x \in A$ then $\forall x,\ x \in A$, i.e., if $\forall x,\ [0, x[\subseteq A \Rightarrow x \in A$ then $A = X$. This is usually split into cases (i) $0 \in A$, (ii) successors $x \in A \Rightarrow x^+ \in A$, and (iii) limit points $(\forall y < x,\ y \in A) \Rightarrow x \in A$). (Note: $[0, 0[= \varnothing \subseteq A$, so $0 \in A$.)

5. A 1-1 morphism $f : X \to X$ satisfies $x \leqslant f(x)$ (since $\{ x \in X : f(x) < x \}$ has no minimum). So the only automorphism on X is the identity, and any isomorphism $X \to Y$ is unique (take $g^{-1} f$).

6. Either $X \precsim Y$ or $Y \precsim X$.

   Proof. The set $A := \{ x \in X : \exists y \in Y,\ \exists f_x : [0, x[\to [0, y[\subset Y,\ \text{isomorphism} \}$ contains 0 and is lower-closed. Define $f := \bigcup_{x \in A} f_x$, a 1-1 morphism. So either $A = X$ (in which case $f : X \to Y$ is an embedding); or $A = [0, \alpha[$; if fA is bounded, define $f(\alpha) := \bigvee_{x<\alpha} f(x)$ to make $[0, \alpha] \cong [0, f(\alpha)]$, so $\alpha \in A$, a contradiction. Hence $fA = Y$ and $f^{-1} : Y \to X$ is an embedding.

7. Every set has a well-order ("well-ordering principle", equivalent to the axiom of choice). Conversely, a set with a well-order has min as a choice function.

   Proof: Let $X \neq \varnothing$ and consider the well-ordered subsets of X, ordered by $A \leqslant B$ iff A is initial in B. There is a maximal subset A; if $x \notin A$, then $A \leqslant A \cup \{ x \}$; so $A = X$.

8. An ordered space in which every non-empty subset has a max and a min must be finite: $\{0, 0^+, 0^{++}, \dots\}$ has a maximum in the set.

Classification: An **ordinal number** is a set X in which $\overset{\in}{=}$ is a well-order and *transitive*, $y \in x \in X \Rightarrow y \in X$; hence $x \in X \Rightarrow x \subset X$. For ordinals,

1. For every $x \in X$, $x = [0, x[\subset X$ and $x^+ = [0, x]$. Hence

$$0 = \varnothing, \quad 0^+ = \{0\} = 1, \quad 1^+ = [0,1] = \{0,1\} = 2, \dots.$$

2. Elements of an ordinal number are themselves ordinal; conversely, any proper transitive subset is an element.

   Proof. $a \in X \Rightarrow a \subseteq X$, so $\in$ is a well-order for a; moreover if $x \in y \in z \in X$ then $x, y \in X$, so $x \in z$. Conversely, transitive subsets are lower closed, so of the type $[0, x[= x$ if a proper subset.

3. Either $X \subseteq Y$ or $Y \subseteq X$. Distinct ordinals are non-isomorphic.

   Proof. Let X, Y be ordinals, then $X \cap Y \subseteq X, Y$ is transitive; this gives the contradiction $X \cap Y \in X, Y$ unless $X \cap Y = X$ or $X \cap Y = Y$. Note that if $X \subset Y$ then $X = [0, X[$. If X is isomorphic to Y, then $X \subseteq Y \subseteq X$.

4. The class of ordinal numbers is itself an ordinal (so not a set), with well-order $\subseteq$, $X^+ = X \cup \{X\}$, and $\bigvee A = \bigcup A$.

   Proof. If $x \in X \in$ Ordinals then x is an ordinal. Any non-empty class of ordinals has the minimum $\bigcap A$ which is an ordinal because it is a transitive subset of any $X \in A$.

5. Every well-ordered space is isomorphic to a unique ordinal. Hence every set is numerically equivalent to some ordinal; the (least) ordinals can serve as representative cardinal numbers.

   Proof. Existence by transfinite induction: $[0, 0[= \varnothing$ (the 0 ordinal); if $[0, x[\cong O_x$ (ordinal), then $[0, x^+[= [0, x] \cong O_x \cup \{O_x\} = O_x^+$; if $[0, y[\cong O_y$ for every $y < x$, then $x = \sup [0, x[\cong \bigcup_{y<x} O_y$ an ordinal. Hence $X = \sup_x [0, x[\cong \bigcup_x O_x$.

6. Transfinite Induction: if $A \subseteq$ Ordinals, and $\forall a, a \subset A \Rightarrow a \in A$ then $A =$ Ordinals.

   Proof. Suppose that $A \neq$ Ordinals, and let $a := \min(\text{Ordinals} \setminus A)$. Then $a \notin A \Rightarrow a \not\subset A$ as well as $A \in \text{Ordinals} \setminus A \Rightarrow a \subseteq A$.

7. The sum $(+)$ of ordinal numbers satisfies

$$A + 0 = A, \quad A + B^+ = (A + B)^+, \quad A + B = \bigcup_{C \in B} (A + C)$$

   with zero 0, associative, preserves $\leqslant$.

(a) B is a limit ordinal $\Leftrightarrow$ $A+B$ is as well (unless $B=0$);

(b) If $n\in\mathbb{N}$ and A an infinite ordinal then $n+A=A$;

(c) Ordinal numbers are of the type $A+n$ where A is a limit ordinal (perhaps 0) and n a natural number.

8. Multiplication satisfies

$$A0=0,\quad AB^+=AB+A,\quad AB=\bigcup_{C\in B}AC$$

with unity 1, associative, left-distributive, preserves $\leqslant$, allows left-cancelation;

(a) AB is a limit ordinal $\Leftrightarrow$ A or B is also; in particular $nB=B$;

(b) $\forall A,B\neq 0, A=BC+R$ with $R\subset B$.

(c) Multiplication is not right-distributive, e.g. $(1+1)\omega\neq\omega+\omega$

9. Can define powers (not the same as set exponentiation) by

$$A^0:=1,\quad A^{B^+}:=A^BA,\quad A^B:=\bigcup_{C\in B}A^C(\text{except } 0^B:=0)$$

Then $A^{B+C}=A^BA^C$, $A^{BC}=(A^B)^C$.

10. The class of ordinal numbers starts

$$\begin{array}{llll}
0, & 1, & 2, & \ldots\\
\omega:=\mathbb{N}, & \omega+1, & \omega+2, & \ldots,\\
\omega2=\omega+\omega, & \omega2+1, & \ldots, & \\
\omega3, & \ldots, & & \\
\ldots & & & \\
\omega^2=\omega\omega, & \ldots, & & \\
\omega^3, & \ldots, & & \\
\ldots & & & \\
\omega^\omega, & \ldots, & & \\
\omega^{\omega^\omega}, & \ldots, & & \\
\ldots & & & \\
\epsilon_0:=\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}, & \ldots, & & \\
\epsilon_1:=\epsilon_0^{\epsilon_0^{\cdot^{\cdot^{\cdot}}}}, & \ldots & & \\
\ldots & & &
\end{array}$$

These are all countable ordinals; the first uncountable ordinal is $\omega_1:=\sup\{\,x : x \text{ is a countable ordinal}\,\}$ (it's uncountable else $\omega_1^+\in\omega_1$).

3 Finite Orders

Size	Orders	Lattices	Distributive Lattices	Boolean Lattices
1	1	1	1	1
2	2	1	1	1
3	5	1	1	0
4	16	2	2	1
5	63	5	3	0
6	318	15	5	0
7	2045	53	8	0
8	16999	222	15	1
9	183231	1078	26	0
10	2567284	5994	47	0