Topological Vector Spaces and Algebras

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1 Topological Vector Spaces over $\mathbb{R}$ or $\mathbb{C}$

Recall that a topological vector space is a vector space with a T_0 topology such that addition and the field action are continuous. When the field is $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$, the field action is called **scalar multiplication**.

Examples:

- $\mathbb{R}^A$, such as sequences $\mathbb{R}^{\mathbb{N}}$, with pointwise convergence.
- Sequence spaces ℓ^p (real or complex) with topology generated by $B_r = \{ (a_n) : \sum_n \sqrt[p]{|a_n|^p} < r \}$, where $p > 0$.
- Lebesgue spaces $L^p(A)$ with $B_r = \{ f : A \to \mathbb{F},$ measurable, $\int \sqrt[p]{|f|^p} < r \}$ $(p > 0)$.
- Products and quotients by closed subspaces are again topological vector spaces.
  If $\pi_i : Y \to X_i$ are linear maps, then the vector space Y with the initial topology is a topological vector space, which is T_0 when the π_i are collectively 1-1.

The set of (continuous linear) morphisms is denoted by $B(X, Y)$. The morphisms $B(X, \mathbb{F})$ are called 'functionals'.

$+, *, \to$	**Finitely-Generated**	**Locally Bounded** Separable		**First countable**	
Top. Vec. Spaces	/////	L^p $0 < p < 1$	$\ell^p[0,1]$	$(\ell^p)^{\mathbb{N}}$	$(\ell^p)^{\mathbb{R}}$
Locally Convex	/////	L^p $p \geqslant 1$	L^∞	$\mathbb{R}^{\mathbb{N}}$, $C(\mathbb{R}^n)$	$\mathbb{R}^{\mathbb{R}}$ pointwise, ℓ^2_{weak}
Inner Product	/////	L^2	$\ell^2[0,1]$	/////	/////
Locally Compact	$\mathbb{R}^n$	/////	/////	/////	/////

1. A set is *balanced* when $|\lambda| \leqslant 1 \Rightarrow \lambda A \subseteq A$.
   (a) The image and pre-image of balanced sets are balanced.
   (b) The closure and interior are again balanced (if $A \in \mathcal{T}_0$; since $\lambda A^\circ = (\lambda A)^\circ \subseteq A^\circ$); as are the union, intersection, sum, scaling, and product $A \times B$ of balanced sets.

(c) Hence every set generates largest and smallest balanced sets,

$$\bigcup_{\substack{V\subseteq A\\ V\,\text{bal.}}} V =: \mathrm{bal}(A) \subseteq A \subseteq \mathrm{Bal}(A) := \bigcap_{\substack{V\supseteq A\\ V\,\text{bal.}}} V = \{\, \lambda a : |\lambda| \leqslant 1, a \in A \,\}$$

Bal(A) is open if A is, and $\bigcup_V V^\circ$ is balanced open in A; hence X has a topological base of balanced open sets.

(d) Balanced sets are star-shaped hence path-connected.

2. X is path-connected and locally connected. There are no open subspaces (clopen) except for X.

3. Connected open sets are path-connected (since a boundary point of a path-connected component would be surrounded by a balanced open set).

4. A **convex** set is one which contains every line segment joining any two of its points,

$$\begin{aligned} & 0 \leqslant t \leqslant 1 \ \Rightarrow\ (1-t)C + tC = C \\ \Leftrightarrow\quad & 0 \leqslant s, t \ \Rightarrow\ sC + tC = (s+t)C \end{aligned}$$

For example, subspaces.

(a) Convexity is preserved by linear images and pre-images.

(b) Convex sets are connected.

(c) $\mathrm{Convex}(A + \lambda B) = \mathrm{Convex}(A) + \lambda\mathrm{Convex}(B)$

(d) The closure, interior, sum, scaling, and product are convex (e.g. $tC^\circ + (1-t)C^\circ$ is open in C).

(e) The intersection of convex sets is again convex; hence every set generates its *convex hull*, the smallest convex set containing it,

$$\mathrm{Convex}(A) = \{\, t_1 a_1 + \cdots + t_n a_n : \sum_i t_i = 1, t_i \geqslant 0, a_i \in A \,\}.$$

(f) If A open or balanced, then so is Convex(A) (but $\mathrm{Convex}(A)^\circ \neq \mathrm{Convex}(A^\circ)$).
If A is convex, then so is bal(A) (not Bal(A)).
If K_1, K_2 are compact convex, then so is $\mathrm{Convex}(K_1 \cup K_2) = \bigcup_{t\in[0,1]}(1-t)K_1 + tK_2$ (as the continuous image of $[0,1] \times K_1 \times K_2$).

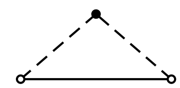
A *polyhedron* is Convex(F) of a finite set F; a *simplex* is when F is independent. A *cone* is $\mathrm{Convex}(A \cup \{\, x \,\})$ where A is of dimension $n-1$.

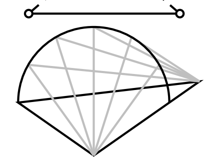
But the convex hull, even of a compact set, need not be closed (e.g. the compact set of sequences $x_n := (1, \ldots, \frac{1}{n}, 0, \ldots)$ and 0; then $\sum_{n=1}^N x_n/N \to (\frac{1}{n})$); convex sets ($\neq X$) may be dense in X (e.g. c_{00} in ℓ^1, $\{\, f \in C[0,1] : f(0) = 1 \,\}$ in $L^1[0,1]$).

5. An *extreme* subset A of a convex set C satisfies $\operatorname{Convex}(C \smallsetminus A) \subseteq C \smallsetminus A$; the intersection of extreme sets is extreme. In particular *extreme* points do not lie on proper line segments in C, $e \notin \operatorname{Convex}(C \smallsetminus e)$, equivalently, $e = \sum_i t_i a_i \;\Rightarrow\; \exists i, e = a_i$ ($\Rightarrow\; \forall i, e = a_i$).

   If $C = \operatorname{Convex}(E)$, then E is minimal $\Leftrightarrow$ E is the set of extreme points.

   There need not exist any extreme points, and the set of extreme points need not be closed.





6. Recall that a set is *bounded* when $\lambda A \to 0$ as $\lambda \to 0$, i.e.,

$$\forall U \in \mathcal{T}_0, \exists r > 0,\; B_r A \subseteq U.$$

   For a balanced set this is equivalent to $\exists \lambda,\; A \subseteq \lambda U$. The only bounded subspace is 0.

   Given a fixed open set $U \in \mathcal{T}_0$, the extent of a bounded set can be gauged by

$$N_U(A) := \inf\{\, r > 0 : A \subseteq rU \,\}$$

   (a) $N_U(\lambda A) = |\lambda|\, N_U(A)$ when U is balanced

   (b) $N_U(A + B) \leqslant N_U(A) + N_U(B)$ when U is convex

   (c) $N_U(A) = 0 \Leftrightarrow A = \{\, 0 \,\}$ when U is bounded

7. Any balanced convex open neighborhood of 0 generates a semi-norm $N_C(x)$ and conversely, $C = \{\, x \in X : N_C(x) < 1 \,\}$. (But there need not exist any non-trivial ones.)

8. If $T_i : X \to Y$ are morphisms such that $\bigcup_i T_i x$ is bounded for all $x \in K$ a non-meagre bounded convex subset, then $\bigcup_i T_i K$ is bounded.

   Proof: Let $A_c := \{\, x \in K : \forall i, T_i x \in c\bar{W} \,\}$ closed; then $K = \bigcup_c A_c$, so some A_c contains an interior point $x_0 + V$. But $K \subseteq x_0 + V/t$ for some $t < 1$, so $x_t := tx + (1 - t)x_0 \in K \cap (x_0 + V)$, then $tK \subseteq A_c + (1 - t)A_c$, so $tT_i K \subseteq c\bar{W} + c\bar{W} \subseteq cU$.

Dual Space

1. The *dual space* is $X^* := B(X, \mathbb{F})$. A linear map $\phi : X \to \mathbb{F}$ is continuous iff $\exists V \in \mathcal{T}_0$, $|\phi V| \leqslant 1$.

   For any balanced convex $C \in \mathcal{T}_0$, let $N_C(\phi) := \inf\{\, r > 0 : |\phi C| \leqslant r \,\} = \sup_{N_C(x)<1} |\phi(x)|$; then $|\phi(x)| \leqslant N_C(\phi) N_C(x)$.

   $(X \times Y)^* \cong X^* \times Y^*$ via $(\phi, \psi)(x, y) := \phi x + \psi y$.

   Note: When $\mathbb{F} = \mathbb{C}$, the real and imaginary parts of a functional are not independent: $\operatorname{Im} \phi(x) = -\operatorname{Re} \phi(ix)$.

2. Every linear map $Y \to \mathbb{F}$, which is bounded with respect to some semi-norm, $N_C(\phi) < \infty$, can be extended to all of X with $N_C(\tilde{\phi}) = N_C(\phi)$.

   Proof: ϕ can be extended from Y to $Y + [\![v]\!]$ by $\phi(y + \lambda v) = \phi(y) + \lambda c$ for some $c \in \mathbb{F}$. Given $|\phi(y)| \leqslant N(\phi)N(y)$; require a c such that $|\phi(y) + c| \leqslant N(\phi)N(y + v)$, which is possible when ϕ is real-valued since $\phi(y_1) - \phi(y_2) \leqslant N(\phi)(N(y_1 + v) + N(y_2 + v))$. For complex $\phi = \phi_1 + i\phi_2$, then $\phi_2(y) = -\phi_1(iy)$, so both can be extended. Let $\tilde{\phi}$ be a maximal extension of ϕ (exists by Hausdorff's maximality); its domain is X else can extend further by the above.

3. *Weak convergence*: Every pair $(x, \phi) \in X \times Y^*$ gives a functional on operators: $(x, \phi) \mapsto \phi T x$. Hence they induce a 'weak' convergence

   $$T_i \rightharpoonup T \Leftrightarrow \forall x \in X, \forall \phi \in Y^*,\ \phi T_i x \to \phi T x,$$

   In particular,

   $$\begin{aligned} x_i \rightharpoonup x &\Leftrightarrow \forall \phi \in X^*,\ \phi(x_i) \to \phi(x), \\ \phi_i \rightharpoonup \phi &\Leftrightarrow \forall x \in X,\ \phi_i(x) \to \phi(x) \qquad \text{(weak-*)}. \end{aligned}$$

   The topology induced by this convergence is generated from the sub-basic balanced convex open subsets $U_{r,x,\phi} := \{ T : |\phi T x| < r \}$, hence is locally convex but not necessarily T_0, nor locally bounded ($U_{r,x,\phi} \supseteq \ker \phi$) except when finite dimensional. However, X^* is a T_0 topological vector space since X separates points of X^*.

   Morphisms preserve weak convergence, $x_i \rightharpoonup x \ \Rightarrow\ T x_i \rightharpoonup T x$.

   Note that if $T_i \to T$ in Y^X, pointwise, i.e., $\forall x,\ T_i x \to T x$, then $T_i \rightharpoonup T$. $x_n \to x \Leftrightarrow x_n \rightharpoonup x$ AND $\{ x_n : n \in \mathbb{N} \}$ is totally bounded; $\bar{A} \subseteq \overline{A}^w$.

   Many properties of subsets have *weak* analogues e.g. weakly bounded when $\forall \phi \in X^*$, ϕA is bounded in $\mathbb{F}$ (A bounded $\Rightarrow$ A weakly bounded).

4. If $T_i \rightharpoonup T$ and $S_i x \to S x, \forall x$ then $T_i S \rightharpoonup T S$; if $\phi S_i \to \phi S$ then $S_i T_i \rightharpoonup S T$.

5. There are links between a space and its dual, via the adjointly related *polar* of a subset in X and the *pre-polar* of a subset in X^*,

   $$\begin{aligned} A^{\oplus} &:= \{ \phi \in X^* : N_A(\phi) = \sup |\phi A| \leqslant 1 \} = \overline{\mathrm{ConvexBal}(A)}^{\oplus} \\ {}^{\oplus}\Phi &:= \{ x \in X : \sup |\Phi x| \leqslant 1 \} \end{aligned}$$

   $$\Phi \subseteq A^{\oplus} \Leftrightarrow |\Phi A| \leqslant 1 \Leftrightarrow A \subseteq {}^{\oplus}\Phi$$

   $A^{\oplus}$ is balanced, convex, and weak-closed in X^* (and ${}^{\oplus}\Phi$ in X).

   When $U \in \mathcal{T}_0(X)$, $U^{\oplus}$ is weak*-compact.

   Proof: $J : U^{\oplus} \to \overline{B_{\mathbb{F}}}^X$ (compact), $\phi \mapsto (\phi x)_{x \in X}$ is clearly an embedding. $J(U^{\oplus})$ is closed: $J(\phi_i) \to f \Leftrightarrow \forall x, \phi_i x \to f(x)$, hence f is linear with $\forall x \in U, |f(x)| \leqslant 1$, so $f \in U^{\oplus}$. Thus $U^{\oplus}$ is compact in X^*.

6. Similarly, *annihilator* and *pre-annihilator*

$$A^\perp := \{\,\phi \in X^* : \phi A = 0\,\} = \overline{[\![A]\!]}^\perp,$$
$$^\perp\Phi := \{\,x \in X : \Phi x = 0\,\} = {}^\perp\overline{[\![\Phi]\!]}, \qquad \Phi \subseteq A^\perp \Leftrightarrow \Phi A = 0 \Leftrightarrow A \subseteq {}^\perp\Phi$$
$$(A \cup B)^\perp = A^\perp \cap B^\perp, \qquad A^\perp + B^\perp \subseteq (A \cap B)^\perp.$$

They are weak-closed subspaces of X^* and X respectively. For A unbounded, $A^\oplus = A^\perp$.

7. Every morphism $T : X \to Y$ has an **adjoint** morphism $T^* : Y^* \to X^*$ defined by $T^*\phi := \phi \circ T$.

Then

$$TA \subseteq B \;\Rightarrow\; T^*B^\perp \subseteq A^\perp,$$
$$\ker T^* = (\operatorname{im} T)^\perp, \qquad (S + \lambda T)^* = S^* + \lambda T^*,$$
$$\operatorname{im} T^* \subseteq (\ker T)^\perp, \qquad (ST)^* = T^*S^*.$$

$T \mapsto T^*$ is not weakly continuous but $T_i^* \rightharpoonup T^* \;\Rightarrow\; T_i \rightharpoonup T$.

8. A continuous projection (idempotent) on a complete space decomposes it into the product of closed subspaces $X \cong M \times N$ ($M = \ker P$, $N = \operatorname{im} P = \ker(1 - P)$).

9. If M is a closed subspace of finite codimension, then $X \cong M \times N$ (using representatives $\pi_n x_n = e_n$).

Separability

The size of a space can be assessed by the minimum cardinality of a set A such that $X = \overline{[\![A]\!]}$.

1. X is separable $\Leftrightarrow$ A is countable.

   Proof: For any $x + U$, let $V + \cdots + V + W \subseteq U$, $\sum_{i=1}^n \lambda_i a_i \in x + W$; then $\exists \epsilon_i, B_{\epsilon_i} a_i \subseteq V$, and $\exists q_i \in \mathbb{Q} + i\mathbb{Q}$, $q_i \in \lambda_i + B_\epsilon$; thus $\sum_i q_i a_i \in \sum_i (\lambda_i + B_\epsilon) a_i \subseteq x + W + \sum_i V \subseteq x + U$.

2. A *topological basis* is a list of vectors e_n such that every $x = \sum_n \alpha_n e_n$ for some unique α_n. More strongly, e_n is a *Schauder basis* when $x \mapsto \alpha_n(x)$ are continuous. Such spaces are essentially sequence spaces $x \leftrightarrow (a_n)$. A functional is then of the form $\phi x = \sum_n b_n a_n$ (where $b_n = \phi e_n$).

3. For a separable vector space, $U^\oplus$ ($U \in \mathcal{T}_0$) is a compact metric space.

   Proof: If x_n are dense in U, then $\|\phi\|_w := \sum_n \frac{1}{2^n} |\phi x_n|$ is a metric on $U^\oplus$, with $\phi_i \rightharpoonup \phi \Leftrightarrow \|\phi_i - \phi\|_w \to 0$.

1.1 Quasi-Normed Spaces

are vector spaces with topology induced by a translation-invariant metric $d(x,y) = |x-y|$, equivalently, first countable; axiomatically, this quasi-norm satisfies

$$|x+y| \leqslant |x| + |y|, \qquad |-x| = |x|, \qquad |x| = 0 \Leftrightarrow x = 0$$
$$\lambda_n \to \lambda \text{ AND } x_n \to x \;\Rightarrow\; |\lambda_n x_n| \to |\lambda x|$$

This last condition can be achieved if, for example, $|\lambda x| \leqslant |\lambda||x|$. Note that by starting with a balanced local base, the quasi-norm can be chosen to also be balanced, i.e., $|\lambda| \leqslant 1 \;\Rightarrow\; |\lambda x| < |x|$ (see the construction of the norm in topological groups). As in groups, can be completed. A topological vector space may have more than one inequivalent quasi-norm.

- $\mathbb{R}^{\mathbb{N}}$. More generally, arrays of real numbers such that $|(a_{nm})| := \sum_n \frac{1}{2^n} \frac{|(a_{nm})|_1}{1+|(a_{nm})|_1}$, where $|(a_{nm})|_1 := \sum_m |a_{nm}|$ are finite.
- $L^0(A)$ with $|f|_E := \int_E (|f| \wedge 1)$, i.e., sub-basic open sets $V_{\epsilon,\delta} := \{\, f : \mu\{\, x : |f(x)| > \delta \,\} < \epsilon \,\}$.
- If $\pi_i : Y \to X_i$ are linear maps to a finite number of quasi-normed spaces (one of the π_i is 1-1), then the vector space Y can be given the quasi-norm $|y| := \sum_i |\pi_i y|$.
- Products have the quasi-norm $|(x,y)| = |x| + |y|$ (among others); for countable products can take $|x| = \sum_n \frac{1}{2^n} \frac{|x|_n}{1+|x|_n}$.
- Quotients have the quasi-norm $|x+M| = \inf_{a \in M} |x+a|$.

1. As in all normed groups, the quasi-norm is continuous and $B_r + B_s \subseteq B_{r+s}$. The norm *constant of concavity* is

   $$c := \sup \frac{|x+y|}{|x| \vee |y|} \leqslant 2.$$

   (But $x_i \rightharpoonup x \not\Rightarrow \|x_i\| \to \|x\|$.)

2. By continuity of scalar multiplication, $\forall r, \exists \epsilon, s,\ t < \epsilon \;\Rightarrow\; tB_s \subseteq B_r$.

3. The open mapping theorem of topological groups applies between complete quasi-normed spaces even if not separable: $TX = \bigcup_n nTB_r$, so $\overline{TB_r}$ contains some open ball; the remaining part of the proof remains valid.

   In particular, a bijective morphism is an isomorphism.

4. *Closed Graph Theorem*: A linear map is continuous iff its graph is closed in $X \times Y$, i.e., $x_n \to x$ AND $Tx_n \to y \;\Rightarrow\; y = Tx$.

   Proof: The graph is itself complete quasi-normed; the projection $\pi_X : G \to X$ is an isomorphism by the open mapping theorem, and $T = \pi_Y \circ \pi_X^{-1}$.

5. Isomorphism Theorems for complete spaces: $X/\ker T \cong \operatorname{im} T$ if $\operatorname{im} T$ is closed (via the continuous map $x + \ker T \mapsto Tx$).

   Hence $\frac{X+Y}{Y} \cong \frac{X}{X\cap Y}$, $\frac{X\times Y}{Y} \cong X$, $\frac{X/Z}{Y/Z} \cong \frac{X}{Y}$.

6. The totally bounded sets are the metrically bounded sets that are arbitrarily close to finite-dimensional subspaces.

   Proof: $K \subseteq F + B_\epsilon \subseteq [\![F]\!] + B_\epsilon$. Conversely, if $K \subseteq B_r$ and $K \subseteq Y + B_\epsilon$, then $K \subseteq Y \cap B_{r+\epsilon} + B_\epsilon \subseteq F + B_{2\epsilon}$ since in finite dimensions balls are totally bounded.

1.2 Locally Bounded Spaces

when there is a bounded open set; equivalently, a single (balanced bounded) set B generates the topology by translations and scalar multiplications, $x + \lambda B$ ($\lambda \neq 0$). Hence is first countable.

Examples:

- ℓ^p and $L^p(A)$ ($p > 0$).

Quotients are again locally bounded. An infinite product of topological vector spaces is not locally bounded.

1. $X = \mathbb{N}B = \bigcup_n nB$

2. There is a $c > 0$ such that $B + B \subseteq cB$; $rB + sB \subseteq c(r \vee s)B$.

   Proof: $V + V \subseteq B$, and $rB \subseteq V$, so $r(B + B) \subseteq B$.

3. There is an equivalent quasi-norm satisfying $|\lambda x| = |\lambda|^p |x|$ ($0 < p \leqslant 1$, $c^p = 2$).

   Proof: Let $|x| := \inf\{\sum_{i=1}^n \nu(x_i) : \sum_i x_i = x\}$, $\nu(x) := N_B(x)^p$, $\bar\nu(x) := 2^r \geqslant \nu(x)$. Note $\nu(x + y) \leqslant 2(\nu(x) \vee \nu(y))$. Claim: $\nu(\sum_{i=1}^n x_i) \leqslant 2\sum_i \bar\nu(x_i)$, since take $\nu(x_i)$ in decreasing order; if $\nu(x_j) \leqslant 2\nu(x_{j+1})$ then $\nu(x_j + x_{j+1}) \leqslant 2\nu(x_j) \leqslant \bar\nu(x_j) + \bar\nu(x_{j+1})$; if $2\nu(x_{i+1}) \leqslant \nu(x_i)$ for all i, then $\nu(x_1 + \cdots + x_n) \leqslant 2\nu(x_1) \vee 2^2\nu(x_2) \vee \cdots \vee 2^n\nu(x_n) = 2\nu(x_1) \leqslant 2\sum_i \bar\nu(x_i)$. Hence $\nu(\sum_i x_i) \leqslant 4\sum_i \nu(x_i)$ and $\frac{1}{4}\nu(x) \leqslant |x| \leqslant \nu(x)$.

4. A subset is bounded iff metrically bounded, i.e., covered by some $x + rB$.

5. Every vector has a magnitude and direction (unit vector): $x = |x|^{1/p} \frac{x}{|x|^{1/p}}$.

6. If e_n are bounded and $(a_n) \in \ell^p$ then $\sum_n a_n e_n$ converges absolutely.

7. A linear map is continuous iff

   (a) $\exists c > 0,\ TB_X \subseteq cB_Y$. It can be measured by $N(T) := N_{B_Y}(TB_X)$

   (b) T maps bounded sets to bounded sets ("bounded map").

$N(0) = 0$, $N(I) = 1$, $N(T^{-1}) \geqslant N(T)^{-1}$.

Proof: If $x_n \to 0$ then $Tx_n = |x_n|^{\frac{1}{p}} T \frac{x_n}{|x_n|^{1/p}} \to 0$.

8. For every proper closed subspace Y and $0 \leqslant c < 1$, there is a unit x such that $|x + Y| = c$. The cosets of Y up to a distance of 1 intersect the unit sphere.

   Proof: Let $|y + Y| = c$; the image of the map $z \mapsto |y + z|$, $Y \to \mathbb{R}$, contains $]c, \infty[$, hence some $|y + z| = 1$.

9. The boundary of B_r is $S_r := \{ x : |x| = r \}$, so $\overline{B}_r = \{ x : |x| \leqslant r \}$; moreover $\overline{S}_r^w = \overline{B}_r$ in infinite dimensions.

   Proof: Any neighborhood $\bigcap_{i=1}^n V_{\epsilon_i,\phi_i}$ of $x \in B$ contains the infinite dimensional subspace $Y := \bigcap_i \ker \phi_i$. So there is a unit $y \in S$ such that $y + Y = x + Y$.

10. Balls are not totally bounded except in finite dimensions. Infinite dimensional totally bounded sets have no interior.

    Proof: If $B \subseteq Y + \epsilon B$ and $Y \neq X$ then there is $x \in B$, $|x + Y| > \epsilon$.

2 Locally Convex Spaces

when there is a base of convex open sets (can be assumed balanced).

Examples:

- $\mathbb{R}^A$ with sub-base $V_{x,n} := \{ f : A \to \mathbb{R}, |f(x)| < \frac{1}{n} \}$.
- $C(\Omega)$ with $\Omega = \bigcup_n K_n$ a σ-compact topological space, and with the sub-base $V_{n,m} := \{ f \in C(\Omega) : |fK_n| < \frac{1}{m} \}$.
- $C^\infty(\Omega)$, with sub-base $V_{n,k,m} := \{ f \in C^\infty(\Omega) : |f^{(k)}K_n| < \frac{1}{m} \}$.
- $B(X, Y)$ for topological vector spaces, with weak topology (and indistinguishable morphisms identified). In particular, dual spaces X^*.

1. If A is bounded or totally bounded, then so is $\text{Convex}(A)$.

   Proof: $A \subseteq F + V$; $\text{Convex}(F) \subseteq F' + V$ as a compact set; so $\text{Convex}(A) \subseteq F' + V + V \subseteq F' + U$.

2. *Separating hyperplanes*: A compact convex set K and a disjoint closed convex set C can be separated by a real functional, $\phi K < \alpha < \phi C$. In particular X^* separates points from closed subspaces.

   Proof: A point x can be separated from an open convex set $U \in \mathcal{T}_0$ using an extension of the functional $\phi(\lambda x) := \lambda$; ϕ is continuous since $|\phi \text{bal}(U)| \leqslant 1$. K and C can be separated by $(K+V) \cap (C+V) = \varnothing$, V convex; let $x_0 \in K$,

$y_0 \in C$; $x_0 - y_0$ can be separated from the open convex neighborhood $U := (K - x_0 + V) - (C - y_0 + V)$. Hence $\phi(K+V) - \phi(C+V) = \phi U - 1 < 0$, so $\phi(K+V) < \phi(C+V)$.

3. A closed convex set is weakly closed (if $x \notin \bar{C}$ then can find ϕ that separates x from C).

   Hence, if $x_i \rightharpoonup x$ then $\exists y_i \in \mathrm{Convex}(x_i)$, $y_i \to x$.

4. ${}^{\oplus}(A^{\oplus}) = \overline{\mathrm{ConBal}(A)}$, ${}^{\perp}(A^{\perp}) = \overline{[\![A]\!]}$,
   $({}^{\oplus}\Phi)^{\oplus} = \overline{\mathrm{ConBal}(\Phi)}^{w}$, $({}^{\perp}\Phi)^{\perp} = \overline{[\![\Phi]\!]}^{w}$; hence $\overline{\operatorname{im} T^*}^{w} = (\ker T)^{\perp}$.

   Proof: If $x \notin \overline{CB(A)} =: F \ni 0$, it can be separated from it by a functional, $\phi F < \alpha < \phi x$; so $\psi := \phi/\alpha$ extended to $\mathbb{F}$, satisfies $|\psi F| < 1 < |\psi x|$ since F is balanced; so $\psi \in A^{\oplus}$ and $x \notin {}^{\oplus}(A^{\oplus})$.

5. Weakly bounded subsets iff bounded.

   Proof: $|x^{**}\phi| \leqslant c_\phi$ for each $x \in A$; for $\phi \in V^{\oplus}$ compact convex, $|V^{\oplus}x| = |x^{**}V^{\oplus}| \leqslant c$; $\therefore \frac{1}{c}A \subseteq {}^{\oplus}(V^{\oplus}) = \bar{V} \subseteq U$.

6. A functional achieves its largest value on a compact convex subset (as $|\phi|$ or $\operatorname{Re}\phi$) at an extreme point.

   Proof: If $|\phi|$ takes its max value α at b, and $x = sa + tb \in K$ then $\phi(x) \leqslant s\phi(a) + t\alpha$, so $\phi(a) = a = \phi(b)$.

7. A compact convex set has extreme points and they generate the set: $\overline{\mathrm{Convex}(E)} = K$.

   Proof: For any extreme set A (starting with K), as long as it has distinct points, can find $\phi \in X^*$ which separates them. Let ϕ achieve its maximum α on the closed set F; then F is an extreme subset. Hence can form a maximal nested chain of extreme closed sets; $\bigcap_i F_i$ is closed extreme and minimal, hence contains a single (extreme) point. If $x \in K \smallsetminus \overline{C(E)}$ then a functional separates them, $\phi(x) > \phi\overline{C(E)}$, so the max of ϕ contains an extreme point not in E.

8. Every finite dimensional subspace M induces a decomposition $X \cong M \times N$ (using the dual functionals δ_i).

9. A linear map $T : X \to Y$ is continuous when for any open convex $D \subseteq Y$, there is an open convex $U \subseteq X$, such that $N_V(TU) < \infty$.

10. X is embedded in X^{**}.

    Proof: $x \mapsto x^{**}$ is 1-1 since for $x \neq 0$, let $x \neq U$ convex, so separate x from U by a functional ϕ; $x^{**}(\phi) = \phi(x) \neq 0$, so $x^{**} \neq 0$.

11. $(\sum_i X_i)^* \cong \prod_i X_i^*$, via $(\phi_i) \mapsto \sum_i \phi_i$.

12. If there is a countable base of convex balanced sets C_n, then the space is quasi-normed by $|x| := \sum_n \frac{1}{2^n} \frac{N_{C_n}(x)}{1+N_{C_n}(x)}$.

13. Let K be a compact convex subset of X, and $T : K \to K$ is continuous and affine, then T has a fixed point $Tx = x$ (proof: let $T_n := (1+\ldots+T^{n-1})/n$, so T_nK is compact; so $\exists x \in K, \forall n, x \in T_nK$ ie $\exists x_n, x = T_nx_n$; so $x-Tx = (x_n - T^n x_n)/n \to 0$ since $x_n - T^n x_n \in K + K$ is compact). CHECK

14. If K convex compact and $f : K \to K$ continuous then f has a fixed point $f(x) = x$; (also, amenable locally compact T_2 groups acting continuously on a convex compact set has a fixed point $Gx = x$)

    Proof: $K \subseteq F+V \subseteq [\![F]\!]+V$; let $f_V := \pi_V \circ f : \overline{\mathrm{Convex}(F)} \to \overline{\mathrm{Convex}(F)}$. Then by Brouwer's fixed point theorem, $f_V(x_V) = x_V \in \overline{\mathrm{Convex}(F)}$. For some subsequence, $x_n \to x_*$, hence

    $$x_* - f(x_*) = x_* - x_n + f_{V_n}(x_n) - f(x_n) + f(x_n) - f(x_*) \in V + V + fV \subseteq U$$

3 Normed Spaces

have scale-homogeneous norms $\|\lambda x\| = |\lambda|\,\|x\|$; equivalently they are the locally convex locally bounded vector spaces (with norm $N_B(x)$). The unit ball B_X generates the topology via the convex bounded balls $B_r(x) = x + rB_X$. As in quasi-normed spaces, can be completed (called a Banach space).

Examples:

- ℓ^∞, the space of bounded sequences, with $\|(a_n)\|_\infty := \sup_n |a_n|$; its closed subspace c_0 of sequences that converge to 0.
- ℓ^1, the space of absolutely summable sequences, with $\|(a_n)\|_1 := \sum_n |a_n|$.
- $L^p(A)$, $p \geqslant 1$
- $L^\infty(A)$, and its closed subspace of bounded continuous functions $C_b(A)$.
- $C(K)$ with sup norm, K compact T_2. Every Banach space is embedded in some $C(K)$.

Quotients and finite products are also normed.

1. $T : X \to Y$ linear is continuous iff it is Lipschitz, $\|Tx\| \leqslant c\|x\|$.

2. $B(X,Y)$ is a normed space with $\|T\| = \sup_{\|x\|=1} \|Tx\|$,

    $$\|Tx\| \leqslant \|T\|\|x\|$$

    It is complete when Y is. In particular, X^* is also a complete normed space.

    $$\|S+T\| \leqslant \|S\| + \|T\|, \quad \|\lambda T\| = |\lambda|\|T\|, \quad \|I\| = 1, \quad \|ST\| \leqslant \|S\|\|T\|$$

    Proof: If T_n is Cauchy, then so are (T_nx).

3. X is isometrically embedded in $B(X)$: fix unit $a \in X$, $\phi \in X^*$, $\phi a = 1$, let $P_x := x\phi$; so $x = P_x a$, $TP_x = P_{Tx}$.

4. $\operatorname{im} T$ is closed $\Leftrightarrow$ $\operatorname{im} T^*$ is closed, in which case $\operatorname{im} T^* = (\ker T)^\perp$ (weak*-closed). So T invertible $\Rightarrow$ T^* invertible.

   Proof: If $\phi \in (\ker T)^\perp$, then can define $\psi(Tx) := \phi x$, extended to all of Y; $T^*\psi = \phi$. Conversely, let $\tilde{T} : X \to \overline{\operatorname{im} T}$, $\tilde{T}x := Tx$, so $\tilde{T}^*$ is 1-1. Separate $C := \overline{\tilde{T}B_X}$ from any other y by ψ, $|\psi\tilde{T}x| \leqslant r < |\psi y|$ for $x \in \overline{B}_X$; so $r < \|\psi\|\|y\| \leqslant \frac{1}{c}\|\tilde{T}^*\psi\|\|y\| \leqslant \frac{r}{c}\|y\|$, so $\|y\| > c$; hence $\tilde{T}B_X$ contains some open ball, so $\tilde{T}$ is onto, i.e., $\operatorname{im} T$ is closed.

5. T is onto $\Leftrightarrow$ $\|T^*\phi\| \geqslant c\|\phi\|$,
   T is an embedding $\Leftrightarrow$ $\|Tx\| \geqslant c\|x\|$.

   Proof: T is onto implies $\operatorname{im}(T^*)$ is closed and T^* 1-1, hence by the open mapping theorem, $\|T^*\phi\| \geqslant c\|\phi\|$.

6. $\overline{B}_{X^*} = B_X^{\oplus}$, hence weak*-closed bounded subsets of X^* are weak*-compact. (So X^* is meagre when infinite dimensional.)
   Similarly, $\overline{B}_X = {}^{\oplus}B_X$; its weak topology is metrizable when X^* is separable (using $\|x\|_w := \sum_n \frac{1}{2^n}|\phi_n x|$).

7. (Krein) If K is weakly compact, then so is $\overline{\operatorname{Convex}(K)}$.
   (Eberlein-Shmulian) Weakly compact iff every sequence has a weakly convergent subsequence.

8. Every Banach space is embedded in $C(K)$ for some compact T_2 space K (take $K = \overline{B}_{X^*}$) and hence embedded in some $\ell^\infty(A)$; and covered by some $\ell^1(A)$ (via $(a_i)_{i\in A} \mapsto \sum_i a_i x_i$, x_i dense in $\overline{B}$). For example, separable Banach spaces are embedded in $C(2^{\mathbb{N}})$ (Cantor space) and ℓ^∞, and covered by ℓ^1.

9. X^* is not separable if X isn't.

   Proof: If ϕ_n is dense in X^*, then $|\phi_n x_n| \geqslant (\|\phi_n\| - \epsilon)$ for some unit x_n. If $M := \overline{[\![x_n]\!]} \neq X$, then $\psi M = 0$ with $\|\psi - \phi_n\| < \epsilon$, so $|\phi_n x_n| = |(\psi - \phi_n)x_n| \leqslant \epsilon$.

10. $\|x + \ker\phi\| = |\phi x|/\|\phi\|$ (since $\|\phi\| = \sup_{a\in\ker\phi} |\lambda||\phi x|/\|\lambda x + a\|$).
    $\|x\| = \sup_{\|\phi\|=1} |\phi x| = \|x^{**}\|$, hence X is isometrically embedded in X^{**}.
    T^{**} extends T.

$$\|T\| = \sup_{\substack{\|\phi\|=1 \\ \|x\|=1}} |\phi T x| = \|T^*\|$$

11. If Y is a closed subspace, then $(X/Y)^* \cong Y^\perp$ (via $\phi(x+Y) := \phi x$) and $X^*/Y^\perp \cong Y^*$ (via $\phi \mapsto \phi|_Y$).

12. If T_i satisfy $\|T_i x\| \leqslant c_x$ then T_i are equicontinuous, hence $\|T_i\| \leqslant c$.

13. If T_i are weakly bounded, $|\phi T_i x| \leqslant c_{\phi,x}$, then T_i are bounded, $\|T_i\| \leqslant c$. In particular if $T_n \rightharpoonup T$ then $\|T\| \leqslant \liminf \|T_n\|$.

    Proof: $\|T\| = \sup |\phi Tx| = \sup \lim_{n\to\infty} |\phi T_n x| \leqslant \lim_{n\to\infty} \|T_n\|$.

14. A morphism is called a **compact** operator when it maps bounded sets to totally bounded sets; equivalently, if x_n is a bounded sequence in X, then Tx_n has a Cauchy subsequence; or $x_n \rightharpoonup x \;\Rightarrow\; Tx_n \to Tx$.

    (a) The space of compact operators forms a closed *-ideal in $B(X, Y)$.

    (b) $\operatorname{im} T$ is separable.

    Proof: $TB \subseteq T_n B + (T - T_n)B \subseteq F + \epsilon B + \epsilon B$. $\operatorname{im} T = T \bigcup_n nB = \bigcup_n nTB$ separable.

    Examples include finite rank operators $T : X \to \mathbb{F}^N$: they are the only compact operators with closed range (by open mapping theorem, TB is open and totally bounded in $\operatorname{im} T$).

15. A *Fredholm* operator is a morphism whose kernel is finite dimensional and image is finite co-dimensional. Its *index* is

    $$\operatorname{index}(T) := \dim \ker T - \dim(\operatorname{im} T)^\perp$$

    $T : X \xrightarrow{\pi} X/\ker T \xrightarrow{R} \operatorname{im} T \xrightarrow{\iota} Y$ with R an isomorphism.

    The product and adjoint are again Fredholm,

    $$\operatorname{index}(ST) = \operatorname{index}(S) + \operatorname{index}(T), \qquad \operatorname{index}(T^*) = -\operatorname{index}(T).$$

    T is Fredholm $\Leftrightarrow$ it is invertible up to compact operators (since $TR^{-1} = I$, $R^{-1}T = I - P$).

    If $\operatorname{index}(T) = 0$ then T is 1-1 $\Leftrightarrow$ T is onto.

16. In a space with a Schauder basis, the coefficients depend continuously on x.

    Proof: Let $|||x||| := \sup_n \|\sum_{i=1}^n \alpha_i e_i\| \geqslant \|x\|$, complete; hence $I : X_{|||\,|||} \to X_{\|\,\|}$ has continuous inverse and $|\alpha_n(x)| = \|\sum^n \alpha_i e_i - \sum^{n-1} \alpha_i e_i\| \leqslant 2|||x||| \leqslant c\|x\|$.

$T^\top$ is defined on the space $B = \{\, \phi : \phi \circ T \text{continuous} \,\} \subseteq Y^*$; when B is dense in Y^*, then T and $T^\top$ are closed, $T^{\top\top} = T$; if T is 1-1 and densely onto, then $T^\top$ is 1-1 and $T^{\top -1} = T^{-1\top}$;

3.1 Reflexive Banach Spaces

are spaces for which $x \mapsto x^{**}$ is an isomorphism $X^{**} \cong X$.

Example: Arrays of numbers with $a_{ij} = 0$ for $j > i$ and $\|(a_{ij})\| := \sqrt{\sum_j (\sum_i |a_{ij}|)^2} < \infty$.

Closed subspaces, the dual space X^*, quotients, countable products with $\|(x_n)\| := \sqrt{\sum_n \|x_n\|^2_{X_n}} < \infty$ are again reflexive.

1. $A^\perp$ can be identified with ${}^\perp A$; and T^{**} with T, since $T^{**}x^{**} = (Tx)^{**}$. $\overline{\operatorname{im} T^*} = (\ker T)^\perp$.

2. X reflexive iff X^* reflexive. (The weak and weak-* topologies of X^* coincide.)

   Proof: If $\phi^{**} \in X^\perp$ then $\phi x = \phi^{**}(x^{**}) = 0$, so $\phi = 0$.

3. Weakly closed bounded subsets are weakly compact.

   Proof: $\overline{B}_X = \overline{B}_{X^{**}}$ is weak*-compact in X^{**}, hence weak compact in X.

4. $\overline{S}^w = \overline{B}$ using sequences.

   Proof: Let $v_n \in S$, $\|v_n - v_m\| \geqslant \frac{1}{2}$. Then $\exists v_n \rightharpoonup v$; $y_n := v_{n+1} - v_n \rightharpoonup 0$, $x_n := x + \frac{\lambda_n}{\|y_n\|} y_n$ ($\lambda_n \leqslant 2$) such that $\|x_n\| = 1$; then $x_n \rightharpoonup x$.

5. Any functional attains its norm somewhere on S.

   Proof: Let $|\phi x_n| \to \|\phi\|$, $x_n \in \overline{B}$; then for a subsequence, $x_n \rightharpoonup x$, so $\phi x_n \to \phi x$ and $|\phi x| = \|\phi\|$; $\|x\| = 1$.

6. A weakly closed subset has a closest point to any other point.

   Proof: Let $\|y_n - x\| \to d := \inf\{\, \|y - x\| : y \in F \,\}$; y_n bounded, so $\exists y_n \rightharpoonup y$; $\therefore |\phi(y - x)| = \lim_{n\to\infty} |\phi(y_n - x)| \leqslant d\|\phi\|$ and $\|y - x\| \leqslant d$.

7. X is weakly complete, i.e., every weakly Cauchy sequences converges weakly (let $\Psi(\phi) := \lim_i \phi x_i$, so $\Psi = x^{**}$; then $x_i \rightharpoonup x$).

8. $T_i \rightharpoonup T \;\Rightarrow\; T_i^* \rightharpoonup T^*$.

3.2 Uniformly Convex Banach Spaces

are Banach spaces such that $\|x + y\|/2 \to 1 \;\Rightarrow\; \|x - y\| \to 0$ uniformly on unit vectors,

$$\forall \epsilon > 0\ \exists \delta > 0,\ \forall x, y \in \overline{B}_X,\ 1 - \delta < \left\|\frac{x+y}{2}\right\| \;\Rightarrow\; \|x - y\| < \epsilon$$

Example:

- ℓ^p and $L^p(A)$ $1 < p$.

1. The set of extreme points of a closed ball is its sphere.

2. $x_n \to x \Leftrightarrow x_n \rightharpoonup x$ AND $\|x_n\| \to \|x\|$.

   Proof: $y_n := \frac{x_n}{\|x_n\|} \rightharpoonup \frac{x}{\|x\|} =: y$; let $\phi y = 1 = \|\phi\|$. Then $1 \geqslant |\phi(\frac{y_n + y_m}{2})| \to 1$, so $\left\|\frac{y_n+y_m}{2}\right\| \to 1$, $\|y_n - y_m\| \to 0$, and $y_n \to y$. Hence $x_n = \|x_n\| y_n \to \|x\| y = x$.

3. For any closed convex set, the point closest to x is unique.

   Proof: $y_n \rightharpoonup y$, $\|y\| = d$; so $y_n \to y$. If v is another closest point then $1 \leqslant \left\|\frac{y+v}{2d}\right\| \leftarrow \frac{1}{2}\|\hat{y}_n + \hat{v}_n\| \leqslant 1$; hence $\|\hat{y}_n - \hat{v}_n\| \to 0$ and $y = v$.

4. X is reflexive.

   Proof: Given unit $\Psi \in X^{**}$; let $\|\phi_k\| = 1$, $\Psi(\phi_k) \to 1$. $\overline{B}_X$ is dense in $\overline{B}_{X^{**}}$, so $\exists x_n$, unit, $\phi(x_n) \to \Psi(\phi)$. Then $1 \geqslant |\phi(\frac{x_n+x_m}{2})| \to 1$, so $\left\|\frac{x_n+x_m}{2}\right\| \to 1$, $\|x_n - x_m\| \to 0$, $x_n \to x$.

3.3 Inner Product Spaces

have a norm induced by an inner product, $\|x\| = \sqrt{\langle x, x\rangle}$, where

$$\begin{aligned} \langle x, y+z\rangle &= \langle x, y\rangle + \langle x, z\rangle, & \langle y, x\rangle &= \overline{\langle x, y\rangle}, \\ \langle x, \lambda y\rangle &= \lambda\langle x, y\rangle, & \langle x, x\rangle = 0 &\Leftrightarrow x = 0, \\ & & \langle x, x\rangle &\geqslant 0. \end{aligned}$$

Equivalently, a normed space that satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Can be completed by taking $\langle [x_n], [y_n]\rangle := \lim_{n\to\infty} \langle x_n, y_n\rangle$ (called a *Hilbert* space).

Isometric morphisms preserve the inner product, $\langle Px, Py\rangle = \langle x, y\rangle$. *Unitary* morphisms are the automorphisms, i.e., invertible isometries. *Conformal* morphisms preserve orthogonality $\langle x, y\rangle = 0 \ \Rightarrow\ \langle Tx, Ty\rangle = 0$; hence are multiples of isometries.

Example: ℓ^2 and $L^2(A)$.

Subspaces, products have inner products:

$$\langle (x_1, y_1), (x_2, y_2)\rangle_{X\times Y} := \langle x_1, x_2\rangle_X + \langle y_1, y_2\rangle_Y$$

For a 'complexified' real inner product space, $X + iX$, $\langle x, y\rangle = g(x, y) + i\omega(x, y)$ with g, ω real bilinear non-degenerate forms on X^2, but g is symmetric and ω skew-symmetric.

1. (a) $\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$.
   (b) $\langle x, y\rangle = \frac{1}{4}(\|y+x\|^2 + i\|y+ix\|^2 - \|y-x\|^2 - i\|y-ix\|^2)$.
   (c) $|\langle x, y\rangle| \leqslant \|x\|\|y\|$, so the inner product is continuous (but not necessarily weakly continuous). (Take $x = \frac{\langle y, x\rangle}{\langle y, y\rangle}y + z$ with $\langle z, y\rangle = 0$.)
   (d) Uniformly convex (since for $x, y \in \overline{B}$, $\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 = 1$).

2. $X^* \cong X$ via $x \mapsto \langle x, \cdot\rangle$ (onto since $\phi(x)y - \phi(y)x \in \ker\phi = x^\perp$).

   Hence $A^\perp = \{\, x \in X : \langle a, x\rangle = 0, \forall a \in A \,\}$; $A \cap A^\perp \subseteq 0$.

   T^* acts on X as $\langle T^*x, y\rangle = \langle x, Ty\rangle$; $(\lambda T)^* = \bar{\lambda}T^*$.

3. There are linear orthogonal projections onto closed subspaces, so closed subspaces are complemented, $X \cong Y \times Y^\perp$.

   If M, N are complete orthogonal subspaces, then so is $M + N \cong M \times N$.

   To find the best approximate solution for $Tx = y$ in x, solve $T^*Tx = T^*y$ (since $y - Tx \in (\operatorname{im} T)^\perp$).

4. T^*T has kernel $\ker T$, closed image $\overline{\operatorname{im} T^*}$ and norm $\|T\|^2$.

5. A *frame* is a set of (unit) vectors e_i such that the norm $\|\langle e_i, x\rangle\|_{\ell^2(I)}$ is equivalent to $\|x\|$. Then $\overline{[\![e_i]\!]} = X$.

   The associated *Fourier series* operator $F : X \to \ell^2(I)$, $x \mapsto (\langle e_i, x\rangle)_{i\in I}$ is 1-1; its adjoint is $F^*(a_i) = \sum_i a_i e_i$; $F^*F \geqslant c > 0$ hence has a continuous inverse.

   Each frame has a dual 'biorthogonal' frame $\tilde{e}_i := (F^*F)^{-1}e_i$, with an associated Fourier operator $\tilde{F} = F(F^*F)^{-1}$, and $\langle e_i, \tilde{e}_j\rangle = \delta_{ij}$

   $$\forall x \in X,\ x = \sum_i \langle e_i, x\rangle \tilde{e}_i = \sum_i \langle \tilde{e}_i, x\rangle e_i.$$

   $\tilde{F}F^*$ is an orthogonal projection onto $\operatorname{im} F \subseteq \ell^2$, so among all $\sum_i \alpha_i e_i = x$, $\|\tilde{F}x\|_{\ell^2} \leqslant \|(\alpha_i)\|_{\ell^2}$.

   Proof: $\langle \tilde{e}_i, x\rangle = \langle e_i, (F^*F)^{-1}x\rangle = F(F^*F)^{-1}x$. $F = \tilde{F}F^*F$, so $\operatorname{im} F = \operatorname{im} \tilde{F}$.

   A *Riesz frame* is a linearly independent frame (equivalent to an unconditional Schauder basis)

6. An **orthonormal basis** is a maximal set of orthonormal vectors e_i, $\langle e_i, e_j\rangle = \delta_{ij}$ (exists). Hence $\overline{[\![E]\!]} = X$ (since $E^\perp = 0$).

   $\sum_i a_i e_i$ converges $\Leftrightarrow$ $(a_i) \in \ell^2$ $\Leftrightarrow$ $\sum_i a_i e_i$ converges weakly; hence e_i is a self-dual frame and F is an isomorphism:

   $$x = \sum_i \langle e_i, x\rangle e_i, \qquad \langle x, y\rangle = \langle Fx, Fy\rangle_{\ell^2}$$

   Hence every Hilbert space is isomorphic to some $\ell^2(I)$, via $x \mapsto Fx$; the separable Hilbert spaces are ℓ^2 and $\mathbb{F}^n$.

7. Any compact operator is diagonalizable $T = VDU^*$, $X \overset{U^*}{\mapsto} \ell^2 \overset{D}{\mapsto} \ell^2 \overset{V}{\mapsto} Y$; $Tu_n = \lambda_n v_n$, $T^*v_n = \lambda_n u_n$. Thus, any compact operator can be approximated by a matrix.

   Proof: T^*T and TT^* share the same non-zero (positive) eigenvalues $\lambda_n^2 \to 0$, with orthonormal eigenvectors u_n; $v_n := Tu_n$ are also orthonormal.

   Any solution of $Tx = y$ is given by $\langle u_n, x\rangle = \langle v_n, y\rangle/\lambda_n$, assuming the latter coefficients are in ℓ^2.

3.4 Symplectic Spaces

are vector spaces with a *symplectic* form $\omega : X^2 \to \mathbb{R}$ such that

$$\begin{array}{ll} \omega(x,y+z) = \omega(x,y) + \omega(x,z), & \omega(y,x) = -\omega(x,y), \\ \omega(x,\lambda y) = \lambda\omega(x,y), & \forall y,\ \omega(x,y) = 0 \Leftrightarrow x = 0. \end{array}$$

The symplectic morphisms preserve this form

$$\omega(Tx,Ty) = \omega(x,y)$$

1. Every symplectic space is isomorphic to some $V \times V^*$ with $\omega((u,\phi),(v,\psi)) := \psi(u) - \phi(v)$.

2. $A^\perp := \{\, x : \omega(a,x) = 0\ \forall a \in A \,\}$. $A \subseteq B^\perp \Leftrightarrow B \subseteq A^\perp$, so $A \subseteq A^{\perp\perp}$.

   Y is *isotropic* when $Y \subseteq Y^\perp$; in this case, $Y^\perp/Y$ is also symplectic. It can be extended to a *Lagrangian* subspace, $Y = Y^\perp$.

3. Y is a symplectic subspace of X iff $Y \cap Y^\perp = 0$.

4 Finite Dimensional Spaces, $\mathbb{R}^N$

They are the locally compact topological vector spaces; equivalently, a totally bounded open set exists.

Proof: Let K be a compact (bounded) balanced neighborhood of 0; then $K \subseteq F + \frac{1}{2}K$ for some finite F with $M := [\![F]\!]$; so $K \subseteq \frac{1}{2}K + M \subseteq \frac{1}{2^r}K + M$, so $K \subseteq \bigcap_r (M + \frac{1}{2^r}K) = M$ and $X = \bigcup_r 2^r K \subseteq M$.

X is isomorphic to *Euclidean* space $\mathbb{F}^N$ with the inner product $\langle x,y \rangle = \sum_{n=1}^N \bar{a}_n b_n$. In particular, all norms are equivalent and complete.

Proof: $T : \mathbb{F}^N \to X$, $(a_k) \mapsto \sum_{k=1}^N a_k e_k$ is continuous, since $(a_k) \mapsto a_i \mapsto a_i e_i$ is continuous. Conversely, let $f(v) := \|Tv\|$ continuous; then $0 \notin fS$ compact, where S is the unit sphere of $\mathbb{F}^N$, i.e., $[0,c[\subseteq fS$, $c \leqslant \|Tv\|/\|v\|$.

1. Totally bounded $\Leftrightarrow$ bounded
   Compact $\Leftrightarrow$ closed and bounded
   $x_n \to x \Leftrightarrow x_n \rightharpoonup x$
   T linear are compact and Fredholm.

2. If K is compact then so is Convex(K).

   Proof: Let $x = \sum_i t_i v_i$; the matrix $\begin{pmatrix} 1 & \cdots & 1 \\ v_1 & \cdots & v_k \end{pmatrix}$ has a null vector if $k > n+1$, i.e., $\exists \alpha_i$, $\sum_i \alpha_i = 0$, $\sum_i \alpha_i v_i = 0$; $\beta := \min t_i/|\alpha_i|$; then $\sum_i (t_i - \beta\alpha_i) = 1$, $\sum_i (t_i - \beta\alpha_i) v_i = x$ but has less terms.

3. $A^* = \bar{A}^\top$. Unitary matrices have orthonormal columns.

4. The Hausdorff measure satisfies $\mu_\alpha(\lambda E) = |\lambda|^\alpha \mu_\alpha(E)$. Also $\mu_{\alpha+\beta}(E \times F) \geqslant c_{\alpha,\beta}\mu_\alpha(E)\mu_\beta(F)$. Borel sets are μ_α-measurable; countable sets are μ_α-null.

   Normalized μ_n ($n \in \mathbb{N}$) are called Lebesgue measures: cardinality, length, area, volume, etc..

5. The *dimension* of E is $\dim(E) := \inf\{\, \alpha : \mu_\alpha(E) = 0 \,\}$.

   $\dim(A \cup B) = \max(\dim A, \dim B)$,
   $A \subseteq B \;\Rightarrow\; \dim A \leqslant \dim B$,
   $\dim(E \times F) \leqslant \dim E + \dim F$.

5 Topological Algebras over $\mathbb{R}$ or $\mathbb{C}$

A **topological algebra** is a topological ring $+, \lambda, \cdot$ that contains $\mathbb{F}$ in its center. Thus it is a topological vector space with continuous $+, \lambda, \cdot$.

The morphisms are those maps which preserve $+, \lambda, \cdot$,

$$\phi(x+y) = \phi(x) + \phi(y),\ \phi(\lambda x) = \lambda\phi(x),\ \phi(xy) = \phi(x)\phi(y)$$

must be continuous with $\|\phi\| = 1$ (the automorphisms form a closed Lie subgroup of $GL(X)$ with Lie algebra $\mathrm{Der}(X)$). The morphisms $X \to \mathbb{C}$ (if there are any) are called *characters*; they form the set $\widehat{X}$.

Examples:

- $\mathbb{R}^A$ with $fg(x) := f(x)g(x)$.
- $B(X)$ for X a topological vector space.

Products are again a topological algebra.

1. 1

6 Normed Algebras

A normed algebra is a topological algebra with a norm such that

$$\|x+y\| \leqslant \|x\| + \|y\|, \quad \|\lambda x\| = |\lambda|\|x\|,$$

$$\|xy\| \leqslant \|x\|\|y\|, \quad \|1\| = 1$$

Can be completed so that $[x_n][y_n] = [x_n y_n]$; it is then called a *Banach algebra*. If $\|xy\| \leqslant c\|x\|\|y\|$ then there is an equivalent norm with $c = 1$.

Examples:

1. $C(K)$ with K compact.
2. $L^1(G)$ with convolution; in particular, $\ell^1 = L^1(\mathbb{Z})$.
3. $\mathbb{C}^n$ with convolution and 1-norm.
4. $B(X)$ for X a Banach space; contains the closed ideal of compact operators. Every normed algebra is embedded in some $B(X)$ via $a \mapsto L_a$, $L_a(x) := ax$.
5. $\mathbb{H}$ quaternions, with absolute value as norm.

Products are again normed algebras (with ∞-norm).

1. The **state space** is $\mathcal{S}(X) := \{\, \phi \in X^* : \phi 1 = 1 = \|\phi\| \,\}$, a weak*-compact convex set.

   $$\mathcal{S}(x+y) \subseteq \mathcal{S}x + \mathcal{S}y, \quad \mathcal{S}(x+\lambda) = \mathcal{S}(x) + \lambda, \quad \mathcal{S}(\lambda x) = \lambda \mathcal{S}x, \quad \mathcal{S}1 = \{\,1\,\}$$

   Proof: $\mathcal{S}$ is weak*-closed in the weak*-compact $\overline{B}_{X^*}$

2. The **spectrum** of an element is $\sigma(x) := \{\, \lambda \in \mathbb{C} : x - \lambda \text{ is not invertible} \,\}$. It is a non-empty compact subset of $\mathbb{C}$, with largest extent $\rho(x)$ and smallest extent $\rho(x^{-1})^{-1}$ (or 0). It depends continuously on x:

   Proof: $\sigma(x)^{\mathsf{c}} = f^{-1}GL(X)$ open; if $|\lambda| > \rho(x)$ then $\rho(x/\lambda) < 1$, so $x - \lambda = -\lambda(1 - x/\lambda)$ is invertible. If $x_n \to x$, then $\sigma(x_n)$ is eventually in $\sigma(x) + \epsilon B$.

   $\|(x-\lambda)^{-1}\| \geqslant 1/d(\lambda, \sigma(x))$. When an algebra is enlarged, the interior of $\sigma(x)$ decreases, and its boundary increases; ultimately, the result is the 'singular spectrum' of $x - \lambda$ that are topological divisors of zero.

3. The character set $\widehat{X}$ is weak*-compact in $\mathcal{S}$,

   $$\widehat{X}(x+y) \subseteq \widehat{X}x + \widehat{X}y, \quad \widehat{X}(xy) \subseteq (\widehat{X}x)(\widehat{X}y), \quad \widehat{X}1 = \{\,1\,\}$$

   $$\widehat{X}x \subseteq \sigma(x) \subseteq \mathcal{S}x \subseteq \|x\|\bar{B}$$

   Proof: $\widehat{X}$ is weak*-closed. If $y := x - \lambda$ is not invertible, then $1 \notin [\![y]\!]$, so there is a $\phi \in \mathcal{S}$, $\phi[\![y]\!] = 0$, i.e., $\phi x = \lambda$. If $\phi \in \widehat{X}$ and y is invertible, then $\phi x - \lambda = \phi y \neq 0$.

4. The extreme points of $\mathcal{S}$ are called *pure* states, $\mathcal{S}_E$, and their weak*-closure $\bar{W}$. They generate the state space

   $$\mathcal{S} = \overline{\mathrm{Convex}(\mathcal{S}_E)}^{w}, \qquad \mathcal{S}x = \overline{\mathrm{Convex}(\mathcal{S}_E x)}$$

   Thus the largest value of $\mathcal{S}x$ is achieved by a pure state.

5. Except for $X = \mathbb{C}$, there are non-zero topological divisors of zero (else as $\sigma(x)$ has non-empty boundary, $x = \lambda \in \mathbb{C}$).

6. a is a quasi-nilpotent (or radical element), i.e., $1 - xa$ is invertible for all x, iff $\rho(xa) = 0$, $\forall x$. Then $\sigma(x+a) = \sigma(x)$.

   Proof: $y + a = y(1 + y^{-1}a)$ is invertible since $\rho(y^{-1}a) = 0$, so $\lambda \notin \sigma(x + a) \Leftrightarrow 0 \notin \sigma(x - \lambda)$.

7. If f is analytic on an open set around $\sigma(x)$, then define

   $$f(x) := \frac{1}{2\pi i} \oint f(z)(z-x)^{-1}\,\mathrm{d}z$$

   (a) $ax = xb \;\Rightarrow\; f(a)x = xf(b)$, so $f(x^{-1}ax) = x^{-1}f(a)x$

   (b) $xy = yx \;\Rightarrow\; f(x)g(y) = g(y)f(x)$.

(c) The map $f \mapsto f(x)$ is a Banach-algebra-morphism $C^\omega(\sigma(x)) \to X$.

(d) $\sigma(f(x)) = f(\sigma(x))$; for $\psi \in \widehat{X}$, $\psi f(x) = f(\psi x)$.

Proof: If $d(\lambda, f\sigma(x)) > 0$, then $(f(z)-\lambda)^{-1}$ is analytic. If $f(x) - f(\lambda)$ has an inverse y, then $(x-\lambda)F(x)y = 1 = yF(x)(x-\lambda)$, where $F(z) = (f(z) - f(\lambda))/(z-\lambda)$.

If x satisfies $f(x) = 0$, then $\sigma(x) \subseteq \{\, \lambda : f(\lambda) = 0 \,\}$. For example, idempotents have spectrum $\{\, 0, 1 \,\}$; nilpotents $\{\, 0 \,\}$.

8. If f is analytic on an open annulus Rr then it is a Laurent series with coefficients $a_n = \frac{1}{2\pi i}\oint f(z) z^{-1-n}\,\mathrm{d}z$ (so $|a_n| \leqslant \frac{\|f\|_\infty}{R^n}$ for $n \in \mathbb{N}$).
For $\sigma(x) \subset Rr$,
$$f(x) = \sum_{n=-\infty}^{\infty} a_n x^n$$
Proof: $(z-x)^{-1} = \sum_n x^n / z^{1+n}$.

9. If $\sigma(x) = \sigma_1 \cup \cdots \cup \sigma_n$, each enclosed by a simple curve, then there are idempotents $e_i := 1_{\sigma_i}(x)$, such that $1 = e_1 + \cdots + e_n$, $\sigma(xe_i) = \sigma_i$.

10. *Exponential function*
$$e^x := 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \lim_{n\to\infty}(1 + \frac{x}{n})^n$$
(a) $e^0 = 1$, $(e^x)^{-1} = e^{-x}$, $e^{nx} = (e^x)^n$, $\frac{\mathrm{d}}{\mathrm{d}t} e^{tx} = e^{tx}x$.

(b) $e^{x+y} = \lim_{n\to\infty}((1 + \frac{x}{2n})(1 + \frac{y}{2n}))^n$; $e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\cdots}$;
if $xy = yx$ then $e^{x+y} = e^x e^y$.

(c) $e^x = \cosh x + \sinh x$, even/odd parts. $\tanh x := \sinh x (\cosh x)^{-1}$.

(d) The exponential function is periodic with purely imaginary period τi; $\pi := \tau/2$. Then
$$e^{i\pi} + 1 = 0$$
(e) $e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x)$, so $\sin(x+y) = \sin x \cos y + \cos x \sin y$, $\cos(x+y) = \cos x \cos y - \sin x \sin y$;

11. For any continuous derivative D, e^{tD} is an automorphism of X; in particular $e^{tD_x} y = e^{tx} y e^{-tx}$.
Proof: $e^{tD}(xy) = \sum_n \frac{1}{n!} t^n (D^n x y + \cdots + x D^n y) = \sum_n \frac{1}{n!} t^n D^n x \sum_m \frac{1}{m!} t^m D^m y$.

12. *Logarithm function* For $\rho(x) < 1$, let $\ln(1+x) := x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n+1}}{n} x^n + \cdots$.
Then $e^{n \ln(1+x)} = (1+x)^n$, so let $(1+x)^p := e^{p \ln(1+x)}$ ($p \in \mathbb{C}$), then
$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \cdots + \binom{p}{n} x^n + \cdots$$

More generally, given any simple path "branch cut" from 0 to ∞ (typically $-\mathbb{R}^+$), let $\ln z := \int_1^z \frac{1}{w}\,dw$ (along a path that does not intersect the branch cut). Then $e^{\ln x} = x = \ln e^x$, $x^p := e^{p\ln x}$

13. *Gelfand Transform*: $\mathcal{F} : X \to C(\widehat{X})$, where $\mathcal{F}(x) = \hat{x}$, $\hat{x}(\psi) := \psi x \in \sigma(x)$, is a morphism,

$$\widehat{x+y} = \hat{x} + \hat{y}, \quad \widehat{\lambda x} = \lambda\hat{x}, \quad \widehat{xy} = \hat{x}\hat{y}, \quad \widehat{1} = 1, \quad \widehat{f(x)} = f \circ \hat{x}.$$

The kernel of $\mathcal{F}$ contains all elements with $\rho(x) = 0$ and all commutators.

6.1 $B(X)$

1. An morphism $J : B(X) \to B(Y)$ induces a morphism $L : X \to Y$; if J is an isomorphism, then so is L, with $J(T) = LTL^{-1}$. Hence all automorphisms of $B(X)$ are inner; they form the Lie group $GL(X)$.

   Proof: $X \precsim B(X)$ via $x \mapsto P_x$. $J(P_a) = b\psi = P_b$ for some unit b, ψ, $\psi b = 1$, since they have the same kernel and image. Hence $J(P_x) = J(P_xP_a) = J(P_x)P_b = P_{J(P_x)b}$; $L(x) := J(P_x)b$; invertible when J is.

2. The center of $B(X)$ is $\mathbb{F}$.

   Proof: $T(x\phi) = (x\phi)T$, so $Tx = \lambda x$.

3. There are no proper radical elements: For every $T \neq 0$ there is $S := x\phi$ such that $(1 - ST)x = 0$, so $1 \in \sigma(ST)$.

4. There are no characters unless $X = \mathbb{C}$.

   Proof: Let M be a two-dimensional (complemented) subspace, and E_{ij} a basis for $B(M)$. Then $E_{ii}E_{jj} = 0$, $E_{ii}E_{ij} = E_{ij}$, $E_{jj} = E_{ij}E_{ji}$, so $\psi E_{ij} = 0$, $\forall i, j$.

5. The spectrum of $T \in B(X)$ splits into the

   - *eigenvalues* when $T - \lambda$ is not 1-1 (a left divisor of zero);
   - the *continuous spectrum* with $T - \lambda$ 1-1 and dense (a left topological divisor of zero);
   - the *residual spectrum* (otherwise; a right divisor of zero).

   It includes *approximate eigenvalues*, i.e., $(T - \lambda)x_n \to 0$ for some unit x_n (i.e., $T - \lambda$ is a left topological divisor of zero).

6. Distinct eigenvalues have linearly independent eigenspaces.

   Proof: If $v := \sum_n \alpha_n e_n = 0$ then $0 = \prod_{n \neq k}(T - \lambda_n)v = \alpha_k \prod_{n\neq k}(\lambda_k - \lambda_n)e_k$.

7. $\sigma(T^*) = \sigma(T)$, $\sigma_r(T) \subseteq \sigma_p(T^*) \subseteq \sigma_p(T) \cup \sigma_r(T)$, $\sigma_c(T^*) \subseteq \sigma_c(T)$.

   When X is reflexive, $\sigma_r(T^*) \subseteq \sigma_p(T)$ and $\sigma_c(T^*) = \sigma_c(T)$.

8. Recall that if $T \in B(X)$ has finite ascent and descent (see Universal Algebras) then every $x \in X$ can be represented uniquely by some $T^n y$, modulo $\ker T^n$, i.e., $X = \ker T^n \oplus \operatorname{im} T^n$.

9. The compact operators form a closed ideal, so $B(X)/\mathcal{K}$ is a Banach algebra; contains the ideal $F(X)$ of finite-rank operators.

10. If K is a compact operator, then $1 + K$ is Fredholm of finite ascent and descent, its spectrum is a countable set of eigenvalues whose only possible limit point is 0, and each non-zero eigenvalue has a finite dimensional extended eigenspace.

    Proof: If $1+K$ has infinite ascent/descent, then can choose separated unit $x_n \in \ker(1+K)^n$ or $\operatorname{im}(1+K)^n$, so Kx_n is not Cauchy. $T-\lambda = \lambda(1-T/\lambda)$. Similarly, can choose separated unit eigenvectors, so $Te_n = \lambda_n e_n \to \lambda e_n$ has no Cauchy subsequence unless $\lambda = 0$. $(T - \lambda)^n$ is still Fredholm.

    T^* has the same non-zero eigenvalues and eigenspace dimensions as T, $\ker(S^*) = \operatorname{im}(S)^\perp \cong Y/\operatorname{im} S \cong \ker S$.

6.2 Commutative Banach algebras

Example: $Z(Z(x))$ for any $x \in X$.

1. The only simple commutative Banach algebra is $\mathbb{C}$ (the closed ideal Xa is 0 or contains 1).

2. The radical consists of elements with zero spectrum, $\rho(x) = 0$ (since $\rho(xy) \leqslant \rho(x)\rho(y)$).

3. Any maximal ideal is the kernel of some character; so $\widehat{X} \neq \varnothing$.

   Proof: $I = \ker \pi$ for $\pi : X \to X/I$; if I is maximal, X/I is simple, i.e., $\mathbb{C}$.

4. $\sigma(x + y) \subseteq \sigma(x) + \sigma(y)$, $\sigma(xy) \subseteq \sigma(x)\sigma(y)$ (in $Z(Z(x, y))$).

5. $X/\mathcal{J}$ is embedded in $C(\widehat{X})$, since $\ker \mathcal{F} = \mathcal{J}$.

   $$\operatorname{im} \widehat{x} = \widehat{X}x = \sigma(x), \quad \|\widehat{x}\|_{C(\widehat{X})} = \sup |\widehat{X}x| = \rho(x), \quad \widehat{x^{-1}} = \widehat{x}^{-1}.$$

   Proof: If $\lambda \in \sigma(x)$ then $x - \lambda \in I = \ker \phi$ maximal, $\phi x = \lambda$.

6. The Banach algebras that are embedded in some $C(K)$ are those that satisfy $\|x\|^2 \leqslant c\|x^2\|$ for all x. In particular, they are commutative and have trivial $\mathcal{J}$.

   Proof: $\|x\| \leqslant c\|x^{2^n}\|^{2^{-n}} \to c\rho(x) = c\|\widehat{x}\|$, so $\mathcal{J} = 0$; $\|xy\| \leqslant c\rho(yx) \leqslant c\|yx\|$; let $F(z) := e^{-zx}ae^{zx}$, analytic, then $\|F(z)\| \leqslant c\|a\|$, hence $F(z) = a$, i.e., $xa = ax$.

   Those that are isometrically embedded in $C(\widehat{X})$ are the commutative semi-simple Banach algebras, equivalently $\|x^2\| = \|x\|^2$.

7. $De^x = e^x$, $D\cosh x = \sinh x$, $D\sinh x = \cosh x$, $D\cos x = -\sin x$, $D\sin x = -\cos x$.

7 Involution algebras

are the normed algebras with an **involution** $*: X \to X$,

$$\begin{aligned} &x^{**} = x, \\ &(x+y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad i^* = -i, \\ &\|x^*\| = \|x\| \end{aligned}$$

So $*$ is a (continuous) anti-automorphism. A complete involution algebra is called a C^*-algebra. The $*$-morphisms preserve involution $\phi(x^*) = \phi(x)^*$.

Example: $C_b(\mathbb{R})$ with $f^*(t) := f(-t)$. Products are again involutive with $(x,y)^* = (x^*, y^*)$.

A $*$-sub-algebra/ideal has to be closed under involution.

An element is called **normal** when $x^*x = xx^*$, i.e., $x^* \in Z(x)$; e.g. $x + e^{i\theta}x^*$. It is called **self-adjoint** when $a^* = a$; e.g. x^*x, $x + x^*$, $i(x - x^*)$. It is **unitary** when $u^* = u^{-1}$; e.g. x^*x^{-1} when x is normal, in particular e^{ia} when a is self-adjoint.

1. $1^* = 1^*1 = (1^*1)^* = 1$, so the involution on $\mathbb{C}$ is conjugation.

2. $(x^{-1})^* = (x^*)^{-1}$, $\sigma(x^*) = \sigma(x)^*$.
   If x is nilpotent, radical, divisor of zero, or topological divisor of zero, then so is x^*.
   If x^*x and xx^* are both invertible then so is x: $x^{-1} = (x^*x)^{-1}x^* = x^*(xx^*)^{-1}$.

3. Any element can be written as $a + ib$, with a, b self-adjoint, called the real and imaginary parts; $\|a\|, \|b\| \leqslant \|x\|$.
   $x^* = a - ib, x^*x = (a^2+b^2) + i[a,b], xx^* = (a^2+b^2) - i[a,b]$;
   x is normal $\Leftrightarrow$ $ab = ba$, unitary $\Leftrightarrow$ $ab = ba$ AND $a^2 + b^2 = 1$.

4. *Polarization identity*: For $\omega := e^{2\pi i/N}$,

$$x^*y = \frac{1}{N}\sum_{n=1}^{N} \omega^n (x + \omega^n y)^*(x + \omega^n y)$$

$$x^*x + y^*y = \frac{1}{N}\sum_{n=1}^{N} (x + \omega^n y)^*(x + \omega^n y)$$

5. (a) The closed $*$-sub-algebra generated by x is $\overline{\mathbb{C}[x, x^*]}$ (non-commuting polynomials).

(b) $Z(A^*) = Z(A)^*$, so $Z(A)$ is a closed $*$-sub-algebra when $A^* = A$.

6. The kernel of a $*$-morphism and the radical $\mathcal{J}$ are closed $*$-ideals.

7. The normal elements form a closed subset containing $\mathbb{C}$: if x is normal, so are x^*, αx, $x+\alpha$, $x^{\pm n}$.

   $Z(x^*) = Z(x)$. If $q \in Z(x)$ is a quasi-nilpotent, then $x+q$ is not normal unless $q = 0$.

   Proof: For $y \in Z(x^*)$, let $\alpha x = a+ib$, $F(\alpha) := e^{-\alpha x} y e^{\alpha x} = e^{-a-ib} y e^{a+ib} = e^{-2ib} y e^{2ib}$ is bounded $\|F(z)\| \leqslant \|y\|$, so constant; i.e., $e^{\bar{\alpha}x^*} y = y e^{\bar{\alpha}x^*}$.

8. The self-adjoints form a real closed sub-space (Jordan algebra) containing $\mathbb{R}$: $a+b$, $(ab+ba)/2$ (e.g. $b \in \mathbb{R}$), $a^{\pm n}$, $i[a,b]$, are again self-adjoint.

9. The unitaries form a closed sub-group of the invertible elements $\mathcal{G}(X)$ (closed under $*$ but not a normal sub-group), containing $e^{i\mathbb{R}}$.

8 C^*-algebras

are $*$-algebras such that $\|x^*x\| = \|x\|^2$.

1. For normal elements, $\|x^2\| = \sqrt{\|x^*xx^*x\|} = \|x\|^2$, so $\rho(x) = \|x\|$.
   $\mathcal{S}x = \overline{\text{Convex}(\sigma(x))}$. The only normal quasi-nilpotent is 0.

   Proof: If $\lambda \notin \overline{\text{Convex}(\sigma(x))}$ then can separate by a ball $z + r\overline{B}$. So $|\phi x - z| = |\phi(x-z)| \leqslant \|x-z\| < |\lambda - z|$ for $\phi \in \mathcal{S}$.

2. $\|x\| = \sqrt{\rho(x^*x)}$, so the norm is unique. The involution is also unique.

3. *Semi-simple*: There are no radical elements, as $\|q\| = \sqrt{\rho(q^*q)} = 0$.

4. $\mathcal{S}$ preserves involution, $\phi(x^*) = \phi(x)^*$, $\|\phi\| \leqslant 1$, and separates points.
   $\mathcal{S}x^* = (\mathcal{S}x)^*$.

   Proof: If $a^* = a$ and $\phi(a) = \alpha + i\beta$, then $|\beta + t| \leqslant |\phi(a+it)| \leqslant \|a+it\| = \rho(a+it) = \sqrt{\|a\|^2 + t^2}$, so $(2t+\beta)\beta \leqslant \|a\|^2$ and $\beta = 0$. $\phi(x^*) = \phi(a-ib) = \phi(x)^*$. $\sigma(a) \subseteq \mathcal{S}(a) = 0 \Rightarrow a = 0$. $\|\phi x\|^2 = \rho(\phi(x^*x)) \leqslant \rho(x^*x) = \|x\|^2$.

5. The Gelfand transform preserves involution: $\widehat{x^*} = \widehat{x}^*$.

6. If x is normal, $\overline{\mathbb{C}[x, x^*]} \equiv C(\sigma(x))$, via $\mathcal{F} : p(x, x^*) \mapsto p(\widehat{x}, \widehat{x}^*)$.

   In particular, can define $f(x)$ for any $f \in C(\sigma(x))$ via $f(x) := \mathcal{F}^{-1} f \mathcal{F} x$. Then $f^*(x) = f(x)^*$, $\sigma(f(x)) = f(\sigma(x))$, and if $xy = yx$ then $f(x)g(y) = g(y)f(x)$. For example, $|x|$.

7. The self-adjoints are the normal elements with $\mathcal{S}a \subseteq \mathbb{R}$ (since $\phi(a^* - a) = 0$).

   Let $a \leqslant b$ when $\mathcal{S}(b-a) \geqslant 0$. Then

(a) $\alpha \leqslant a \leqslant \beta \Leftrightarrow \mathcal{S}a \subseteq [\alpha, \beta]$

(b) $a + c \leqslant b + c$; if $a, b \geqslant 0$ commute, then $ab \geqslant 0$.

(c) $a = a_+ + a_-$, $|a| = a_+ - a_-$, $a_+a_- = 0$, $a_- \leqslant a \leqslant a_+ \leqslant |a| \leqslant \|a\|$.

(d) $a \vee b = a + (b-a)_+$, $a \wedge b = a - (a-b)_+$; hence a $(+, \vee)$-group lattice.

(e) $a \leqslant b \Rightarrow x^*ax \leqslant x^*bx$, in particular $x^*x \geqslant 0$.

(f) For $\phi \in \mathcal{S}$, $\phi(x^*y)$ is a semi-inner product, $\phi(x^*ax) \leqslant \phi(x^*x)\|a\|$ and $|\phi(x)|^2 \leqslant \phi(x^*x)$ (since $a \leqslant \|a\|$).

(g) If $\phi \leqslant \psi$, $\phi \in \mathcal{S}$, $\psi \in \widehat{X}$, then $\phi = \psi$.

(h) $\widehat{X}$ is part of the extreme points of $\mathcal{S}$.

Proof: $x^*x = a_+ + a_-$, so $(xa_-)^*(xa_-) = a_-^3 \leqslant 0$; let $xa_- = b + ic$, then $0 \leqslant 2(b^2+c^2) = (xa_-)^*(xa_-)+(xa_-)(xa_-)^* \leqslant 0$ and $xa_- = 0$; hence $a_-^3 = (xa_-)^*(xa_-) = 0$, and $x^*x = a_+ \geqslant 0$. $a \geqslant 0 \Rightarrow x^*ax = (\sqrt{a}x)^*(\sqrt{a}x)$. If $\phi \leqslant \psi$ then $|\phi(x)|^2 \leqslant \phi(x^*x) \leqslant |\psi(x)|^2$, so $\ker\psi \subseteq \ker\phi$ and $\psi = \phi$. If $\psi = \frac{1}{2}(\phi_1 + \phi_2) \in \widehat{X}$, then $|\phi_1(x)|^2 + |\phi_2(x)|^2 \leqslant \phi_1(x^*x) + \phi_2(x^*x) = 2\psi(x^*x) = \frac{1}{2}|\phi_1(x)+\phi_2(x)|^2$, hence $|\phi_1(x)-\phi_2(x)|^2 = 0$ and $\phi_1 = \phi_2 = \psi$.

For example, $0 \leqslant a \leqslant b \Rightarrow b^{-\frac{1}{4}}a^{\frac{1}{2}}b^{-\frac{1}{2}}a^{\frac{1}{2}}b^{-\frac{1}{4}} \leqslant 1 \Rightarrow 0 \leqslant b^{-\frac{1}{4}}a^{\frac{1}{2}}b^{-\frac{1}{4}} \leqslant 1 \Rightarrow 0 \leqslant a^{\frac{1}{2}} \leqslant b^{\frac{1}{2}}$. A map which preserves $+$ and $*$ automatically preserves $\leqslant$ (since $a \leqslant b \Leftrightarrow b - a = x^*x$). A bijective $*$-morphism is an isomorphism.

8. For unitary u,

   (a) $\|u\| = 1$, $\|ux\| = \|x\| = \|xu\|$.

   (b) They are the normal elements with $\sigma(u) \subseteq e^{i\mathbb{R}}$.

   (c) The inner automorphism by αu is a $*$-automorphism.

   Proof: $\sigma(u^{-1}) = \sigma(u^*) = \sigma(u)^*$

9. A normal element is idempotent iff self-adjoint with $\sigma(e) \subseteq \{0, 1\}$.

10. *Polar decomposition*: Every invertible element can be written uniquely as $x = ur$, where $r = \sqrt{x^*x} \geqslant 0$, $u := xr^{-1}$ unitary.

11. Every C^*-algebra is embedded in some $B(H)$.

    Proof: Map $a \in X$ to $J_a : (x_\phi)_{\phi\in\mathcal{S}} \mapsto (ax_\phi)_{\phi\in\mathcal{S}}$, where x_ϕ is a coset of $M_\phi := \{x : \phi(x^*x) = 0\}$. Hence X embeds in $B(\ell^2(X/M_\phi))$. Note $\langle xy, z\rangle = \langle y, x^*z\rangle$.

    A state ψ is pure iff for any state ϕ, $0 \leqslant \lambda\phi \leqslant \psi \Rightarrow \phi = \alpha\psi$.

    Proof. If $\psi = t\psi_1 + (1-t)\psi_2$, then $0 \leqslant t\psi 1 \leqslant \psi$, so $t\psi_1 = \lambda\psi$ so $\psi_1 = \psi = \psi_2$.

    Conversely, if $0 \leqslant \phi \leqslant \psi$ then $0 \leqslant \phi 1 \leqslant 1$; if $\phi 1 = 0$ then $|\phi T| \leqslant \phi\|T\| = 0$ so $\phi = 0$; if $\phi 1 = 1$ then $(\psi - \phi)1 = 0$ so $\psi - \phi = 0$; if $0 < \phi 1 < 1$ then $\psi = (1 - \phi 1)\frac{\psi-\phi}{1-\phi 1} + \phi 1\frac{\phi}{\phi 1}$, so $\phi/\phi 1 = \psi$.

12. A *tensor algebra* is the free (unital) algebra generated by a vector space V, so that any morphism from V extends to tensors on it.

    (a) Every element decomposes into sub-components of different *grades* $x = \alpha + v + v_2 + \cdots +$ with $\alpha \in \mathbb{F}$, $v \in V$, $v_2 \in V \otimes V$, etc. The grade-0 part is called its *real* part: $\mathrm{Re}(x) := \alpha$; $\mathrm{Re}(xy) = \mathrm{Re}(yx)$.

    (b) Exterior product: $v_1 \wedge \cdots \wedge v_n := \frac{1}{n!} \sum_\sigma \mathrm{sgn}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(n)})$
    $$w \wedge v = \frac{wv - vw}{2} = -v \wedge w, \ v \wedge v = 0$$
    $T(v_1 \wedge \cdots \wedge v_n) := Tv_1 \wedge \cdots \wedge Tv_n$ (in finite dimensions $T\omega = \det(T)\omega$).

    (c) Inversion (an involution) ($v_r^* = (-1)^{r(r-1)/2} v_r$)
    $$(\alpha + v + v_2 + \cdots)^* := \alpha + v - v_2 - v_3 + \cdots$$

    (d) The algebra splits in two parts $X^+ \oplus X^-$, i.e., the even and odd grades: $x = \frac{x+n(x)}{2} + \frac{x-n(x)}{2}$, where $n : v \mapsto -v$. A product of r vectors gives an element in $X^\pm$ depending on whether r is even/odd, so X^+ is a sub-algebra.

    (e) The *symmetric* algebra is the commutative algebra of the quotient of tensors by the ideal generated by the commutators; it is isomorphic to $\mathbb{F}[V]$.

13. Conjecture: The only closed $*$-sub-algebra that separates extreme points of $\mathcal{S}$ is X

8.1 $B(H)$

1. A $*$-automorphism is of type $T \mapsto LTL^{-1}$ where L is non-zero multiple of a Hilbert space isomorphism. The isometric ones are the unitary operators.

2. Distinct eigenvalues in $\sigma(T)$ and $\sigma(T^*)^*$ have orthogonal eigenspaces.

   Proof: $(\lambda - \mu)\langle x, y \rangle = \langle x, Ty \rangle - \langle T^* x, y \rangle = 0$.

3. The *mean value* of T in the direction x is $\langle x, Tx \rangle$ (it minimizes $\|Tx - \lambda x\|$; a functional on T). The *numerical range* $W(T)$ is the set of mean values of T. $W(I) = \{1\}$, $W(\lambda T + z) = \lambda W(T) + z$, $W(T^*) = W(T)^*$, $W(S+T) \subseteq W(S) + W(T)$.

   $W(T)$ is a convex subset of $\mathbb{C}$ satisfying
   $$\sigma(T) \subseteq \overline{W(T)} \subseteq \mathcal{S}(T).$$
   Proof: Let $0 < \alpha := d(\lambda, W(T)) \leqslant \|(T - \lambda)x\|$, so $T - \lambda$ is 1-1 with closed image; as is $T^* - \lambda^*$; so $T - \lambda$ is invertible.

4. *Uncertainty principle*: For a fixed unit x, there is a semi-inner-product,

$$\mathrm{Cov}(S,T) := \langle Sx, Tx\rangle - \langle Sx, x\rangle\langle x, Tx\rangle$$

and semi-norm $\sigma_T := \sqrt{\mathrm{Cov}(T,T)}$, then

$$|\mathrm{Cov}(S,T)| \leqslant \sigma_S\sigma_T$$

$\sigma_T \leqslant \frac{1}{2}\mathrm{diam}(\sigma(T))$, $\sigma_T = 0 \Leftrightarrow x$ is an eigenvector of T.

5. *Normal operators*:
   (a) $\|T^*x\| = \|Tx\|$
   (b) $\ker T^* = \ker T = \ker T^2$ are T and T^* invariant.
   (c) $\operatorname{im} T$ is dense $\Leftrightarrow$ T is 1-1
   (d) T is an embedding $\Leftrightarrow$ invertible
   (e) $\mathcal{S}(T) = \overline{W(T)} = \overline{\mathrm{Convex}(\sigma(T))}$
   (f) $\sigma(T)$ has no residual spectrum, and isolated points are eigenvalues.
   (g) Eigenvalues of T and T^* are conjugate; no extended eigenvectors.

6. *Self-adjoint*: $S \leqslant T \Leftrightarrow \langle x, Sx\rangle \leqslant \langle x, Tx\rangle,\ \forall x$.

7. *Polar decomposition*: Every $T = UR$, where $R = \sqrt{T^*T}$ and $U(Rx) := Tx$ is an isometry on $\operatorname{im} T$. Then $T^* = RU^* = U^*TU^*$, $\|R\| = \|T\|$. T is normal $\Leftrightarrow$ $R = TU^*$, unitary $\Leftrightarrow$ $T = U$ invertible.

   Hence ideals are automatically $*$-ideals since $T^* = U^*TU^*$.

8. *Unitaries*: Every unitary is of the type e^{iA} with A self-adjoint.
   ($U = B + iC$, $C = V|C|$, $A := V \arccos(B)$)

   $U_n \rightharpoonup U \Leftrightarrow U_n x \to Ux$ (since $\|U_n x - Ux\|^2 = \|U_n x\|^2 + \|Ux\|^2 - 2\,\mathrm{Re}\,\langle Ux, U_n x\rangle \to 2\|x\|^2 - 2\,\mathrm{Re}\,\|Ux\|^2 = 0$).

   (Stone): any one-parameter group of normal operators which is weakly continuous in t must be of the type e^{tT} with T normal and $\mathrm{Re}(\sigma(T))$ bounded above; for unitary operators, e^{itA}; more generally any unitary representation of a locally compact T_2 abelian group which is weakly continuous in t is of the form $U_x = \int \chi(x)dE_\chi$).

9. Ergodic theorem: If T normal, $\|T\| = 1$, then $T^n x \to y$ (Cesaro) such that $Ty = y$.

10. *Compact operators*
    (a) $B(H)$ contains the closed subalgebra $\mathbb{C} \oplus \mathcal{K}$.
    (b) Every ideal contains the simple ideal $\mathcal{K}_F$ of finite-rank operators.

(c) The compact operators form the closed ideal $\mathcal{K} = \overline{\mathcal{K}_F}$; so $B(X)/\mathcal{K}$ is simple (its invertible elements are the Fredholm operators). It is maximal when $X \cong \ell^2$.

(d) T has a matrix consisting of blocks of type

$$\begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$

Compact normal operators are diagonalizable.

(e) $Tx = y$, if $y \in (\ker T^*)^\perp$ and $\langle e_\sigma, y\rangle/\sigma \in \ell^2$, then the solutions are $x = \sum_\sigma \frac{1}{\sigma}\langle e_\sigma, y\rangle e_\sigma + \ker T$, else no solutions.

Proof: Given $T \in \mathcal{I}$ and $Ta = b$ unit; let $E_{xy} := xy^*$ for any unit y. Then $E_{xy} = E_{xb}TE_{ay} \in \mathcal{I}$. As a compact operator, on each finite dimensional eigenspace, $T = \lambda + (T - \lambda)$. As kernel basis for the nilpotent $A := T - \lambda$ pick $u, Au, \ldots, A^{n-1}u$, etc.

11. There are various closed ideals contained in $\mathcal{K}$: Let the *trace* of an operator be defined by $\operatorname{tr}(T) := \sum_i \langle e_i, Te_i\rangle$; it is well-defined independently of e_i when $\operatorname{tr}(|T|) < \infty$.

(a) $\operatorname{tr}(S + T) = \operatorname{tr}(S) + \operatorname{tr}(T)$, $\operatorname{tr}(\lambda T) = \lambda \operatorname{tr}(T)$, $\operatorname{tr}(T^*) = \operatorname{tr}(T)^*$.

(b) Trace class operators: $\|T\|_1 := \operatorname{tr}|T| < \infty$, $\|T\|_1 = \|(\sigma_n)\|_{\ell^1}$.

(c) Hilbert-Schmidt operators: $\|T\|_2^2 := \operatorname{tr}(T^*T) < \infty$; complete inner-product $\langle S, T\rangle := \operatorname{tr}(S^*T)$; $\|T\|_2 = \sqrt{\sum_{ij} |\langle e_j, Te_i\rangle|^2} = \|(\sigma_n)\|_{\ell^2}$.

(d) Schatten operators: $\|T\|_p := (\operatorname{tr}|T|^p)^{\frac{1}{p}} = \|(\sigma_n)\|_{\ell^p} < \infty$.

(e) Hölder's inequality: $\|ST\|_r \leqslant \|S\|_p \|T\|_q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

12. *Spectral Theorem*: For T normal and $f \in L^\infty(\sigma(T))$,

$$f(T) := \int_{\sigma(T)} f(\lambda) dP_\lambda \in B(H)$$

meaning $\langle x, f(T)y\rangle = \int_{\sigma_T} f\, d\langle x, P(E)y\rangle$, where $P(E)$ is an orthogonal projection measure, i.e., for any measurable subsets of σ_T, $P(E \cap F) = P(E)P(F), P(E \cup F) = P(E) + P(F)$ for E, F disjoint, $P(E_n) \rightharpoonup P(E)$ for $E_n \to E$, $P(\sigma(T)) = I$. $f(T) = U^{-1}f(\lambda)U$ where $U : H \to H$ is the unitary operator $x \mapsto P_\lambda x$; then

$$(f + g)(T) = f(T) + g(T), \quad (\lambda f)(T) = \lambda f(T), \quad (fg)(T) = f(T)g(T),$$

$$\bar{f}(T) = f(T)^*, \quad f \circ g(T) = f(g(T)), \quad \widehat{f(T)} = f \circ \hat{T}, \quad \|f(T)\| \leqslant \|f\|_{L^\infty(\sigma(T))}$$

Finite Dimensions: Square Matrices

13. The nearest number to a matrix (in the 2-norm) is $\mathrm{tr}(T)/n$.

14. The quasi-nilpotents (radical) are the nilpotents.

15. The matrices with distinct eigenvalues are dense and open in $M_n(\mathbb{C})$ (since $T = D + N$ is close to $D' + N$ where D' has distinct eigenvalues).

16. If $p(x) = \det(T - x)$, then $p(T) = 0$
(since $p(T) = \prod_i p_i(T_i) = \prod_i A_i^{n_i} = 0$, $p_i(x) = (x - \lambda)^n$).

17. *Self adjoint matrices*: If T, with eigenvalues λ_i, is restricted to PTP where P is a projection to a sub-space M of one dimension less than M (for example, by removing the kth row and column), then the new eigenvalues are interlaced
$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n$$

18. *Positive matrices*, $a_{mn} \geqslant 0$. $W(T)$ has its largest extent for a positive real x.

19. $\sqrt[n]{|\det T|} \leqslant \sqrt{n}\max_{i,j}|T_i^j|$; the maximum is achieved by the Hadamard matrices: $HH^* = nI$, $H_0 = [1]$, $H_{k+1} = \begin{pmatrix} H_k & H_k \\ H_k & -H_k \end{pmatrix}$)

8.2 Commutative C^*-algebras

Equivalently, every element is normal.

Examples:

- $L^\infty(A)$ of bounded measurable functions, with usual product and $f^*(a) = \overline{f(a)}$.
- $C_b(X)$, bounded continuous functions, when X is a locally compact T_2 space; contains the closed ideal $C_0(X)$. For example, $C(K)$ for K compact; e.g. $C(\mathbb{S})$, $\ell^\infty = C_b(\mathbb{N})$, $\mathbb{C}^n = C(n)$.
- The generated subalgebra $Z(A \cup A^*)$; $Z(x)$ for a normal element.

1. $X \equiv C(K)$ via the Gelfand map. The state space consists of the positive Radon measures. The characters are the Dirac functionals $\delta_x(f) = f(x)$.

2. The self-adjoints form a real Banach lattice algebra. They correspond to the real-valued functions.

3. The unitaries correspond to unit-valued functions.

4. Stone-Weierstraß: Any $*$-subalgebra that separates points is dense in X.

8.3 Finite Dimensional Algebras

Equivalently a regular Banach algebra (i.e., every element is regular $\forall a, \exists x,\ axa = a$).

It can be given the non-degenerate bilinear form $\langle x, y\rangle := \mathrm{tr}(x^*y)$ where the elements are considered as matrices.

They are the reflexive C^*-algebras. Proof: If X is infinite dimensional then there an $x \in X$ with $K := \sigma(x) \supseteq A$ countably infinite; so $X \supseteq C^*(x) \cong C(K) \supseteq C(A) \cong c$, which is not reflexive.

The $*$-simple finite-dimensional C^*-algebras are $M_n(\mathbb{C})$ and $M_n(\mathbb{C})^2$ (with $(x, y)^* = (y^*, x^*)$.) Of these the only commutative ones are $n = 1$, i.e., $\mathbb{C}$ and $\mathbb{C}^2$.

8.3.1 Frobenius Algebras

are finite-dimensional algebras with a non-degenerate bilinear form such that $\langle xy, z\rangle = \langle x, yz\rangle$.

Examples: $M_n(\mathbb{F})$ with $\langle x, y\rangle := \mathrm{tr}(xy)$.

8.3.2 Geometric Algebras

A **geometric algebra** is the algebra generated by a real/complex finite-dimensional vector space V such that $v^2 \in \mathbb{R}$ for $v \in V$. Note that $q(v) := v^2$ is thus a quadratic form.

Let $g := [\langle a_i, a_j\rangle] = RDR^*$, with D consisting of p 1s, q -1s and r 0s; the orthogonal columns (in Euclidean sense) of R form an orthogonal basis e_i (wrt the bilinear form); so $e_j e_i = \pm e_i e_j$ or 0.

The algebra has dimension $2^{\dim V}$, generated by the orthogonal basis $e_i \cdots e_j$ ($1 \leqslant i < \cdots < j \leqslant n$, adding 1 separately). As tensors, the elements are graded. The elements of grade r give an $\binom{n}{r}$-dimensional subspace. The highest grade subspace is one-dimensional, called the *pseudo-scalars*, generated by $\omega = e_1 \cdots e_n$.

$$\langle x, y\rangle := \mathrm{Re}(x^*y) = \alpha\beta + \frac{vw + wv}{2} + \cdots$$

Note $vw + wv = (v + w)^2 - v^2 - w^2 \in \mathbb{R}$.

$$\begin{aligned} vw &= \langle v, w\rangle + v \wedge w, & \langle \alpha + v, \alpha + v\rangle &= \alpha^2 + v^2 \\ \langle 1, v\rangle &= 0, & \langle v, w\rangle = 0 &\Leftrightarrow vw = -wv \\ \langle x, yz\rangle &= \langle y^*x, z\rangle = \langle xz^*, y\rangle \\ vv_r &= v \cdot v_r + v \wedge v_r \end{aligned}$$

where $v \cdot v_r := \frac{vv_r - (-1)^r v_r v}{2}$, $v \wedge v_r = \frac{vv_r + (-1)^r v_r v}{2}$ (by induction); more generally

$$v_r v_s = v_r \cdot v_s + \cdots + v_r \wedge v_s$$

where $v_r \cdot v_s$ has grade $|r - s|$, up by two grades, to the highest grade $r + s$.

1. X^+ is a geometric sub-algebra.
2. $\frac{1}{2}(uvw + wvu) = \langle v, w\rangle u - \langle w, u\rangle v + \langle u, v\rangle w$
3. $u \cdot (v \wedge w) = \langle u, w\rangle v - \langle u, v\rangle w$,
   $u \cdot (v_1 \wedge v_2 \wedge v_3) = \langle u, v_1\rangle v_2 \wedge v_3 - \langle u, v_2\rangle v_1 \wedge v_3 + \langle u, v_3\rangle v_1 \wedge v_2$, etc.
4. Hodge duality: $*x := -\omega x$.
   $*v_r = v_{n-r} = -\omega v_r = -(-1)^{r(n-1)} v_r \omega$, so there is a correspondence between r-vectors and $(n-r)$-vectors.

   $*(xy) = *(x)y$; e.g. $v_r \times w_s := *(v_r \wedge w_s) = *v_r \cdot w_s$, $u \times (v \times w) = -u \cdot (v \wedge w)$, $*(v_r \cdot w_s) = *(v_r) \wedge w_s$.
5. For any morphism T, $y * T(x) = T^*(y) * x$. Eigenvectors can be extended to $Tv_r = \lambda v_r$.
6. Rotation by θ in e_1, e_2 plane: $x \mapsto rxr^*$, where $r = \pm e^{e_2 e_1 \theta/2}$ (called a 'rotor').
   Reflection along direction e is $v \mapsto (eve)^* = -eve$.
   Inversion is $v \mapsto v^{-1} = v/v^2$.

Exterior algebra: $v^2 = 0$ for all $v \in V$. For all u, v, $\langle u, v\rangle = 0$, so $uv = u \wedge v$.

Non-degenerate geometric algebras: $v^2 = 0 \Rightarrow v = 0$. Hence the Clifford algebra is $\mathcal{C}\ell_{p,q}(\mathbb{R})$ or $\mathcal{C}\ell_n(\mathbb{C})$.

There is a conjugation $x \mapsto axa^{*-1}$.

$X = \mathcal{C}\ell_{p,q}(\mathbb{R})$ $Y = \mathcal{C}\ell_{q,p}(\mathbb{R})$	p	$p+1$	$p+2$
q	X	$X^+_{p+1,q} \cong Y$	$\mathcal{C}\ell_{2,0} \otimes Y$
$q+1$	$X^+_{p,q+1} \cong X$	$\mathcal{C}\ell_{1,1} \otimes X$	
$q+2$	$\mathcal{C}\ell_{0,2} \otimes Y$		

Proof: Use the maps $J : e_i \mapsto \begin{cases} e'_i \otimes e''_1 \otimes e''_2 & i \leqslant p \\ 1 \otimes e''_{i-q} & i > q \end{cases}$ for a basis e'_i of $\mathcal{C}\ell_{p,q}(\mathbb{R})$ and e''_i of $\mathcal{C}\ell_{2,0}(\mathbb{R}) = M_2(\mathbb{R})$; or $J : e_i \mapsto \begin{cases} e'_i \otimes e''_1 \otimes e''_2 & i \leqslant p \\ 1 \otimes e''_{i-p} & i > p \end{cases}$; or

$$J : e_i \mapsto \begin{cases} e'_i \otimes e''_1 e''_2 & i \leqslant p \text{ OR } p+1 < i \leqslant p+q+1 \\ 1 \otimes e''_1 & i = p+1 \\ 1 \otimes e''_2 & i = p+q+2 \end{cases}.$$

It follows that $\mathcal{C}\ell_{p+1,q} \cong \mathcal{C}\ell_{q+1,p}$, $\mathcal{C}\ell_{p,q+4} \cong \mathcal{C}\ell_{p+4,q}$, $\mathcal{C}\ell_{p+8,q} \cong M_{16}(\mathcal{C}\ell_{p,q})$; if $p - q = 1 \pmod 4$ then $\mathcal{C}\ell_{p+i,q} \cong \mathcal{C}\ell_{p,q+i}$.

Hence the first few geometric algebras over $\mathbb{R}$ are (note that $M_n(\mathbb{R}) \otimes \mathbb{F} \cong M_n(\mathbb{F})$, $\mathbb{C} \otimes \mathbb{H} \cong M_2(\mathbb{C})$, $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$)

$p-q-1 \pmod 8$	0	± 1	± 2	± 3	4
$\mathcal{C}\ell_{p,q}(\mathbb{R})$	$\mathbb{R}(m)^2$	$\mathbb{R}(m)$	$\mathbb{C}(m)$	$\mathbb{H}(m)$	$\mathbb{H}(m)^2$

where $\mathbb{F}(n) := M_{2^n}(\mathbb{F})$.

Similarly, $\mathcal{C}\ell_n(\mathbb{C}) \cong \mathbb{C}(n)$ or $\mathbb{C}(n)^2$, $\mathcal{C}\ell_{n+2} \cong M_2(\mathcal{C}\ell_n)$.

Proposition 1

The finite-dimensional real division algebras are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

The only complex finite dimensional division algebra is $\mathbb{C}$.

Proof: Any $x \in X$ satisfies a polynomial $0 = (x-\alpha)\cdots(x^2 - 2\beta x + \gamma)$; hence $x \in \mathbb{R}$ or it satisfies $x^2 - 2\beta x + \gamma = 0$. For $x \notin \mathbb{R}$, x has only two complex eigenvalues λ, $\bar{\lambda}$, so $x^2 \in \mathbb{R} \Leftrightarrow \lambda + \bar{\lambda} = 2\beta = 0 \Leftrightarrow \operatorname{tr}(x) = 0$. Hence X is a geometric algebra.

For a geometric division algebra, $e^2 = 0 \Rightarrow e = 0$, $e^2 = 1 \Rightarrow (e+1)(e-1) = 0 \Rightarrow e \in \mathbb{R}$; if $e_i^2 = -1$, then $(1 - e_1e_2e_3)(1 + e_1e_2e_3) = 0$. So the only possibilities are $\mathcal{C}\ell_0 = \mathbb{R}$, $\mathcal{C}\ell_{0,1} = \mathbb{C}$, $\mathcal{C}\ell_{0,2} = \mathbb{H}$.

□

(There is also the octonion algebra $\mathbb{O}$ which is weakly associative, $x^2y = x(xy), yx^2 = (yx)x$).

8.3.3 Finite-dimensional Complex Lie algebras

Example: The skew-adjoint matrices $u(n)$, satisfying $A^*Q = -QA$, where $Q(x,y)$ is linear in y and anti-linear in x.

Solvable Lie algebras are embedded in the upper-triangular matrices $b(n)$.

Semi-simple Lie algebras are products of simple Lie algebras. These are

Simple Lie algebra	$sl(n)$	$so(2n+1)$	$so(2n)$	$sp(2n)$	g_2	f_4	e_6	e_7	e_8
Corresp. Weyl group	A_{n-1}	B_n	D_n	C_n	G_2	F_4	E_6	E_7	E_8

(They are classified because the Weyl group of reflections along the root vectors form certain Coxeter groups). $so(3) \cong \mathbb{R}^3$ (with cross-product).

8.3.4 Finite-dimensional Jordan algebras

The formally real Jordan algebras (i.e., $\sum_i x_i^2 = 0 \Rightarrow x_i = 0$) are classified - they are the product of the simple ones, i.e.,

1. "Real", the self-adjoint operators on $\mathbb{R}^N$;
2. "Complex", the self-adjoint operators on $\mathbb{C}^N$;
3. "Quaternionic", the self-adjoint operators on $\mathbb{H}^N$;

4. "Octonion", the self-adjoint operators on $\mathbb{O}^3$ (exceptional case);

5. "Spin factor", $\mathbb{R} \times \mathbb{R}^N$ with $(s, \boldsymbol{x}) * (t, \boldsymbol{y}) = (st + \boldsymbol{x} \cdot \boldsymbol{y}, s\boldsymbol{y} + t\boldsymbol{x})$.

The first 4 examples all have $x * y = (xy + yx)/2$. Their projections are $\mathbb{R}P^{N-1}$, $\mathbb{C}P^{N-1}$, $\mathbb{H}P^{N-1}$, $\mathbb{O}P^2$.

9 Examples

Finite Dimensional Spaces

1. Euclidean space with inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^{n} \bar{a}_i b_i$. Euclidean theorems apply.

2. Taxicab metric $\|(a, b)\| := |a| + |b|$. Although its topological properties are the same as the Euclidean case, its metric properties are different. There are many shortest paths between two points; the angle between two unit vectors can be taken to be the length of arc on the unit circle; equilateral triangles need not be equiangular, SAS triangles need not be congruent; 'conics' as $d(x, a) = ed(x, b)$, as sum/difference of distances from two points being constant, or as distance from line $d(x, L) = ed(x, a)$; circles may touch at a whole line.

3. Dual numbers: the exterior algebra on $\mathbb{R}$: $a + b\epsilon$ with $\epsilon^2 = 0$. $(a + b\epsilon)^* = a - b\epsilon$. Isomorphic to $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. It is a local ring. For any differentiable function, $f(a + b\epsilon) = f(a) + f'(a)b\epsilon$.

4. $\mathcal{C}\ell_3(\mathbb{R}) = M_2(\mathbb{C})$, can be represented by the Pauli matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (they generate $sl(2)$). Contains the quaternions (as σ_i/i).

5. $\mathbb{H} = \mathcal{C}\ell_{0,2}(\mathbb{R})$, can be represented by $i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$ where $\sigma_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $j, k = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}$ where $\sigma_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Sequence Spaces

6. $\mathbb{R}^{\mathbb{N}}$ with pointwise convergence. Has quasi-norm $\sum_n \frac{1}{2^n} \frac{|a_n|}{1+|a_n|}$. Locally convex, but not locally bounded.

7. ℓ^∞ of bounded sequences with norm $\sup_n |a_n|$, and involution $(a_n)^* := (a_n^*)$, hence a C^*-algebra. Its dual is ba, so not reflexive; not separable. It is injective, i.e., it is complemented in any larger Banach space (via projection $x \mapsto (\pi_i x)$ where π_i are extensions of the coordinate projections). Weak convergence implies pointwise iff weak* convergence.

   c is the closed subspace of convergent sequences (not complemented in ℓ^∞); isomorphic to c_0, the subspace of sequences that converge to 0, a Banach algebra; isomorphic to cs, the space of convergent series with norm

$\|(a_n)\|_{cs} := \sup_n |\sum_{i\geqslant n} a_i|$ ($cs^* \cong bv$). $\|(a_n)+c_0\| = \limsup_n |a_n|$. Its dual is ℓ^1, so not reflexive; Schauder basis e_n, so separable. Not weak complete, e.g. $(1,\ldots,1,0,\ldots)$ is weak Cauchy but does not converge weakly. $e_n \rightharpoonup 0$. It is the only separable injective Banach space. The closed unit ball of c_0 is not weak compact and has no extreme points; the closed unit ball of c has extreme points ± 1. The character space consists of δ_i.

8. ℓ^1, the space of absolutely summable series with norm $\|(a_n)\| := \sum_n |a_n|$, a Banach algebra. Dual space is ℓ^∞, so not reflexive; Schauder basis e_n, so separable. Weak*-convergence iff pointwise convergence and bounded. Weak convergence of sequences iff norm convergence, implies pointwise convergence. The closed unit ball has extreme points $e^{i\theta}e_n$. The characters are $\overline{B}_{\mathbb{C}}$, with $\psi(a_n) = \sum_{n=0}^{\infty} a_n z^n$ 'generating function'.

   $\ell^1(\mathbb{Z})$ has characters S^1 and $\psi(\theta) = \sum_{n\in\mathbb{Z}} a_n z^n$; $\sigma(a_n) = \operatorname{im}\widehat{(a_n)}$; (a_n) has a $*$-inverse iff $\sum_n a_n e^{in\theta} \neq 0$ for all θ. Can be made into a C^*-algebra with $(a_n)^* = (\bar{a}_n)$ and norm $\|x\| = \|L_x\|$, embedded in $B(\ell^2)$.

9. ℓ^p, $p > 1$, with norm $\|(a_n)\| := \sqrt[p]{\sum_n |a_n|^p}$. $I : \ell^p \to \ell^q$ is continuous for $q \leqslant p$; (Pitt) Every operator $\ell^p \to \ell^q$ is compact when $q < p$; hence $\ell^p \not\cong \ell^q$. Dual space is ℓ^{p^*} where $\frac{1}{p} + \frac{1}{p^*} = 1$, so reflexive; uniformly convex; Schauder basis e_n, so separable. Weak convergence iff pointwise convergence and bounded. The set $\{ e_n : n \in \mathbb{N} \}$ is closed (discrete) but $e_n \rightharpoonup 0$; $\{ e_n \} \cup \{ 0 \}$ is weakly compact. $n^{1/p}e_n \not\rightharpoonup 0$ (since unbounded) but 0 is a weak limit point of the sequence ($\forall N, \exists n > N, n^{1/p}e_n \in V_{x,\epsilon}$). The compact operators form the only closed ideal ($p \geqslant 1$).

   ℓ^2 has inner product $\langle (a_n), (b_n) \rangle := \sum_n \bar{a}_n b_n$.

10. ℓ^p, $0 < p < 1$, with quasi-norm $\|(a_n)\| := \sum_n |a_n|^p$. Locally bounded, separable (via e_n), not locally convex. Dual space is isometric to ℓ^∞ via usual $\boldsymbol{x} \mapsto \boldsymbol{x}^*$. The set $\frac{1}{n^{1-p}}e_n$ is totally bounded but its convex hull is unbounded (e.g. $\sum_{n=1}^{N} \frac{1}{n^{1-p}} e_n/N$).

11. James' space: subspace of c_0 with norm

$$\sup_{(n_i)\in\mathcal{O}} \|(a_{n_2} - a_{n_1}, \cdots, a_{n_k} - a_{n_{k-1}}, a_{n_{k+1}}, 0, \ldots)\|_{\ell^2},$$

    where $\mathcal{O}$ is any odd sequence of (increasing) integers. Complete, separable with e_n as a conditional Schauder basis. Not reflexive even though $X \cong X^{**}$.

12. ba, the space of finitely additive signed measures on $\mathbb{N}$, with norm $\|\mu\| := \sup_{E\subseteq\mathbb{N}} \mu(E) - \inf_{E\subseteq\mathbb{N}} \mu(E)$. Not separable. Although the unit ball is weak*-compact it is not sequentially compact, e.g. e_n^* acting on ℓ^∞ has no weak*-convergent subsequence.

Contains the closed subspace bv, of sequences of bounded variation with norm $\|(a_n)\|_{bv} := |a_1| + \sum_n |a_{n+1} - a_n|$; isomorphic to ℓ^1 via $(a_n) \mapsto (a_1, \ldots, a_{n+1} - a_n, \ldots)$. $e_n \not\rightharpoonup 0$.

Function Spaces

13. $L^1[0,1]$, space of functions with norm $\|f\|_1 := \int_0^1 |f|$. Dual space is $L^\infty[0,1]$, so not reflexive; separable by polynomials. Weakly sequentially complete: every weakly Cauchy sequence converges weakly. The closed unit ball has no extreme points.

    $L^1(S^1)$ has character space $\mathbb{Z}$, $\psi_n(a_n) = \int_0^{2\pi} e^{in\theta} f(\theta)\,\mathrm{d}\theta$; the Gelfand map are the Fourier coefficients.

    $L^1(\mathbb{R})$ has character space $\mathbb{R}$, $\psi_\xi(f) = \int e^{ix\xi} f(x)\,\mathrm{d}x$; the Gelfand map is the Fourier transform.

    $L^1(\mathbb{R}^+)$ has character space $\mathbb{R}^+ \times i\mathbb{R}$, $\psi_z(f) = \int_0^\infty e^{-zx} f(x)\,\mathrm{d}x$; the Gelfand map is the Laplace transform.

14. $L^p[0,1]$, $1 < p$, with norm $\|f\|_p := \sqrt[p]{\int_0^1 |f|^p}$. Dual space is L^{p^*} where $\frac{1}{p} + \frac{1}{p^*} = 1$, so reflexive; uniformly convex since

    $$2(\|f\|^{p^*} + \|g\|^{p^*})^{p-1} \leqslant \|f+g\|^p + \|f-g\|^p \leqslant 2(\|f\|^p + \|g\|^p), \ (p \leqslant 2)$$

    (reversed inequalities for $p \geqslant 2$); separable. $I : L^p[0,1] \to L^q[0,1]$ is continuous for $q \leqslant p$, with meagre image (unit ball has no interior in L^q). The closed unit ball has its boundary as extreme points.

    $L^2[0,1]$ has inner product $\langle f, g\rangle := \int_0^1 \bar{f} g$; isomorphic to ℓ^2. The Hilbert-Schmidt operators are the integral operators with kernel in $L^2[0,1]^2$.

15. $L^p[0,1]$, $0 < p < 1$. Locally bounded, but there are no non-trivial open convex subsets; hence trivial dual space (no morphisms into a locally convex space); the only weakly closed subspaces are 0 and X. No Schauder basis.

16. $L^\infty[0,1]$, space of bounded (ae) functions with norm $\|f\|_\infty := \sup_{x \text{ a.e.}} |f(x)|$. Isomorphic to ℓ^∞; not separable. The closed unit ball has extreme points $|f| = 1$ a.e..

17. $L^0[0,1]$, the space of measurable functions with $f_n \to 0$ when $\forall \epsilon > 0$, $\mu\{\, x : |f_n(x)| \geqslant \epsilon \,\} \to 0$ as $n \to \infty$.

18. $C(\Omega)$, the space of continuous functions with complete quasi-norm: if (f_n) is Cauchy, then (f_n) is Cauchy in each $C(K_i)$, so $f_n \to f$ in K_i; take f as patch of all these f's; then $|f_n - f| = \sum_i \frac{1}{2^i} \frac{|f_n - f|_i}{1 + |f_n - f|_i} < \frac{1}{m}$, i.e., $f_n \to f$ in $C(\Omega)$.

$C(K)$ is separable iff K is metrizable (similarly $C_0(X)$). Dual space consists of regular Borel measures of bounded variation (not separable: uncountable δ_t). Weak-convergence iff pointwise and bounded. The closed unit ball has extreme points δ_x, $x \in K$.

$C[0,1]$ with involution $f^*(t) = \overline{f(t)}$, a C^*-algebra; has character space $[0,1]$, δ_t; its Gelfand map is the identity, $\sigma(f) = \operatorname{im} f$. The closed ideals correspond to closed subsets of $[0,1]$ as $\mathcal{I}_A = \{ f : fA = 0 \}$. $\sigma(f) = \operatorname{im}(f)$.

$C(\mathbb{R}^N)$. Locally convex but not locally bounded; not separable (contains ℓ^∞). The closed unit ball has extreme points ± 1 (or $|f| = 1$ if over $\mathbb{C}$).

Matrix Algebras

19. $B(\ell^2)$, not separable (contains ℓ^∞).

20. $B(c_0)$. Each eigenvalue belongs to a closed disk about T_{ii} of radius $\sum_{j \neq i} |T_{ji}|$.