Topological Groups and Rings

joseph.muscat@um.edu.mt
10 March 2016

1 Topological Groups

A **topological monoid** is a topological space with a continuous monoid operation, $(x, y) \mapsto xy$ (in particular $x \mapsto ax$, $x \mapsto xa$),

$$\lim_i (x_i y_i) = \lim_i x_i \lim_i y_i.$$

A **topological group** is a (T_0) topological monoid with continuous inversion $x \mapsto x^{-1}$,

$$\lim_i x_i^{-1} = (\lim_i x_i)^{-1}.$$

(Note: there are groups in which multiplication, but not inversion, is continuous, e.g. ordered groups with Alexandroff topology; other groups have topologies with multiplication that is continuous in each variable, but not jointly.)

The topology depends only on the neighborhood base of 1, since by translation, $\mathcal{N}_x = x\mathcal{N}_1 = \mathcal{N}_1 x$. In fact, $\mathcal{N}_1$ presents a uniform structure with $B_U(x) := xU$ for any $U \in \mathcal{N}_1$ (i.e., $1 \in U$, $U^{-1} \in \mathcal{N}_1$, $\exists V \in \mathcal{N}_1, VV \subseteq U$ by continuity of $*$) (or $B_U(x) := Ux$ or $xU \cap Ux$ or UxU).

Morphisms are the continuous group-morphisms; it is enough to check continuity at 1. Examples of morphisms are $n \mapsto a^n$, $\mathbb{Z} \to G$; conjugation $x \mapsto a^{-1}xa$ is an *inner* automorphism.

Examples:

- $\mathbb{R}^\times \times \mathbb{R}$ with (i) $\binom{x}{y}\binom{a}{b} := \binom{xa}{xb+y/a}$; (ii) $\binom{x}{y}\binom{a}{b} := \binom{xa}{xb+y}$ ($\cong$ Affine($\mathbb{R}$)).
- Sequences of integers with pointwise convergence.
- $\mathbb{Z}$ with the 'evenly spaced topology' of arithmetic sequences $a + b\mathbb{Z}$.
- Any group with topology generated from a family $\mathcal{N}$ of normal subgroups such that $\bigcap \mathcal{N} = \{1\}$ and their cosets; e.g. the discrete topology.
- Free topological groups: The free group A^* on any topological space A, with topology generated by multiplication and inversion.
- The permutations $S(A)$ of a discrete topological space A, with composition and pointwise convergence (induced by A^A). Every topological group can be embedded in some such group.

- The isometries of a metric space, with composition and pointwise convergence. Every topological group can be embedded in $\mathrm{Iso}(X)$ for some Banach space X.
- The automorphisms $\mathrm{Aut}(K)$ of a compact T_2 topological space, with composition and the compact-open topology of $C(K,K)$. Every topological group can be embedded in some such group.

	Finite	**Top. Finitely Generated with (2^{nd}) Countable Base**	2^{nd} **Countable**	**Separable**	
Topological Groups	/////	$\mathbb{Z}_{a+b\mathbb{Z}}$	$\mathbb{Q}^n$, $\mathbb{Z}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$, ℓ^2	$\mathbb{Z}^{\mathbb{R}}$, $\mathbb{R}^{\mathbb{R}}$, $C[0,1]$	$\mathbb{Z}^{2^{\mathbb{R}}}$, ℓ^∞
Locally Compact Groups	/////	$\mathbb{Z}^n$, $\mathbb{R}^n$, $GL(n)$, $H_3(\mathbb{R})$, $\{a,b\}^*$	$\mathbb{Q}_p$, $2^{(\mathbb{N})}$	$2^{\mathbb{R}}\times\mathbb{Z}$, $\mathbb{T}^{\mathbb{R}}\times\mathbb{R}$	Discrete $\mathbb{R}$
Compact Groups	S_n	$SO(n)$	$S_n^{\mathbb{N}}$	$S_n^{\mathbb{R}}$	$S_n^{2^{\mathbb{R}}}$
Abelian Compact Groups	C_n	$\mathbb{T}^n$, $\mathbb{T}^{\mathbb{N}}$	$2^{\mathbb{N}}$	$2^{\mathbb{R}}$, $\mathbb{T}^{\mathbb{R}}$	$2^{2^{\mathbb{R}}}$, $\mathbb{T}^{2^{\mathbb{R}}}$

Subgroups, products $X\times Y$, sums $\sum_i X_i$ (a subgroup of $\prod_i X_i$), functions X^A, its subgroup of continuous functions $C(A,X)$ when A is a topological space, and $X\rtimes_\phi Y$ are also topological groups.

1. Translations $L_a : x\mapsto ax$, $R_a : x\mapsto xa^{-1}$, and inversion $x\mapsto x^{-1}$ are homeomorphisms (hence preserve closed, open, connected, compact, …subsets), and $L_{ab}=L_aL_b$, $R_{ab}=R_aR_b$.

   If U is open in X, and $A\subseteq X$, then $AU(=\bigcup_{a\in A} aU = B_U(A))$, UA, and U^{-1} are also open. $\mathcal{N}_1$ is closed under products and inversion of its sets.

2. (a) $\forall U\in\mathcal{T}_1,\ \exists V\in\mathcal{T}_1,\ xU=Vx$, (similarly) $xyU\supseteq xVyV$.
   (In particular, $VxV\subseteq xU$, $VV\subseteq U$; $V^{-1}x\subseteq xU$, $V^{-1}xV\subseteq xU$).

   (b) Every $U\in\mathcal{T}_1$ contains a "symmetric" open set, i.e., $V^{-1}=V$, e.g. $U\cap U^{-1}$.

   (c) $\bar{A}\bar{B}\subseteq\overline{AB}$, $\bar{A}^{-1}=\overline{A^{-1}}$, $\overline{xA}=x\bar{A}$, $(xA)^\circ=xA^\circ$; $\bar{A}=\bigcap_{U\in\mathcal{T}_1} AU$ ($\Rightarrow\ \bar{U}\subseteq UU$).

(d) If A, B are connected/compact/totally bounded, then so is AB. (But $\mathbb{Z} \times 0 + \{ (n, \frac{1}{n}) : n \in \mathbb{Z} \}$ is not closed in $\mathbb{Q}^2$.)

(e) If K is compact and F closed then KF is closed.

Proof: If $x \notin KF$, then $UK \cap xF^{-1} = \varnothing$ (since T_3), so $U^{-1}x \subseteq (KF)^c$ (since $A \cap BC = \varnothing \Leftrightarrow AC^{-1} \cap B = \varnothing$).

(f) Disjoint compact and closed subsets can be separated using the same open set: $KU \cap CU = \varnothing$.

Proof: For $x \in K$, $\exists V$, $xV \cap C = \varnothing$; let $WWW \subseteq V$, so $xWW \cap CW = \varnothing$ and $K \subseteq x_1W_1 \cup \cdots \cup x_nW_n$; $U := \bigcap_i W_i$; then $KU \cap CU \subseteq (\bigcup_i x_iW_iW_i) \cap CW_i = \varnothing$.

3. (a) A subgroup is either clopen (no boundary), or has no interior.

Proof: If $a, b \in H$ and $aU \subseteq H$ ($U \in \mathcal{N}_1$), then $bU \subseteq ba^{-1}H = H$; its complement of cosets is also open.

For example, $2\mathbb{Z}$ is clopen in $\mathbb{Z}$; $\mathbb{Q}$ is neither closed nor open in $\mathbb{R}$; $\mathbb{R}$ is a boundary set in $\mathbb{C}$.

(b) The closure of a (normal/abelian) subgroup is a (normal/abelian) subgroup.

Proof: If $x_n, y_n \in H$, $x_n \to x$, $y_n \to y$, then $x_ny_n \to xy \in \bar{H}$, $x_n^{-1} \to x^{-1} \in \bar{H}$.

(c) $\bar{H} \smallsetminus H$ is either $\varnothing$ or dense in $\bar{H}$ (since if $y \in \bar{H} \smallsetminus H$ and $x \in H$, then yU contains some $a \in H$; so $xy^{-1}a \in xU \smallsetminus H$).

4. Denote the subgroup generated by A by $[\![A]\!] = \bigcup_{n \in \mathbb{N}} (A \cup A^{-1})^n$. Every set ('topologically') generates a closed subgroup $\overline{[\![A]\!]}$. A 'basis' is a minimal topologically generating set of G.

The subgroup generated by

(a) a non-empty open set is clopen

(b) a compact set is σ-compact

(c) a connected set containing 1 is connected

(d) a finite set is separable, and $\overline{[\![a_1, \ldots, a_n]\!]}$ is second countable.

$[\![A]\!]$ has a (non-compatible) left-invariant metric $d(x, y) := \min\{ n : x \in y(A \cup A^{-1})^n \}$. Note that if $1, a \in A$, then $[\![a^{-1}A]\!] = [\![A]\!]$.

5. (a) The kernel of a morphism is a closed normal subgroup. Closed normal subgroups give quotients that are topological groups.

(b) Stabilizers of some group action, e.g. the centralizer and normalizer of any subset, are closed subgroups.

(c) Discrete subgroups are closed.

Proof: There is $U \in \mathcal{T}_1$ such that $U^{-1}U \cap H = \{ 1 \}$; if $h_i \to x$, then $x^{-1}h_i \in U$ eventually; but $x^{-1}h_i, x^{-1}h_j \in U$ implies $h_i^{-1}h_j \in U^{-1}U \cap H$, so h_i is eventually constant; hence $x \in H$.

6. The topology is homogeneous $T_{3.5}$; so a topological group with an isolated point is discrete (e.g. finite groups); the connected components are either uncountable or points, so the only countably infinite groups are discrete or have the topology of $\mathbb{Q}$. Note that $\mathbb{Z}^{\mathbb{R}}$ is not T_4.

   (Markov) Every $T_{3.5}$ topological space is embedded in some topological group.

7. A filter is *Cauchy* when
$$\forall U \in \mathcal{T}_1,\ \exists A \in \mathcal{F},\ x, y \in A \ \Rightarrow\ x^{-1}y \in U.$$
   If $\mathcal{F}$ is Cauchy and $\mathcal{F} \subseteq \mathcal{G} \to x$ then $\mathcal{F} \to x$.

   Proof: Let $U \in \mathcal{T}_1$, $VV \subseteq U$, then $xV \in \mathcal{G}$ and there is $A \in \mathcal{F}$ with $AA^{-1} \subseteq V$; so $a \in A \cap xV \in \mathcal{G}$, and $A \subseteq aV \subseteq xVV \subseteq xU$, so $xU \in \mathcal{F}$, i.e., $\mathcal{N}_x \subseteq \mathcal{F}$.

8. A function is uniformly continuous when $\forall U \in \mathcal{T}_1, \exists V \in \mathcal{T}_1,\ f(xV) \subseteq f(x)U$, e.g. morphisms; preserve Cauchy filters, total boundedness.

   Proof: By continuity at 1, for any $U \in \mathcal{T}_1$, there is $V \in \mathcal{T}_1$, $\phi V \subseteq U$; hence $\phi(xV) \subseteq \phi(x)U$.

9. The following are closed characteristic subgroups (hence normal):
   (a) The center $Z(G)$
   (b) G_1, the connected component of 1; the other components are its homeomorphic cosets; contains the connected subgroup $[G, G_1]$. For any clopen set U, $UG_1 = U$. The component of $\prod_i G_i$ is $\prod_i G_{i1}$.
   (c) $Q(G) := \bigcap\{\, A : 1 \in A \text{ clopen in } G\,\}$, the quasi-connected component of 1
   (d) $\mathrm{Core}(G) := \bigcap\{\, H \leqslant G : \text{clopen}\,\}$, here called the "core"; any neighborhood of 1 generates a subgroup that covers it. The clopen subgroups form a filter in the lattice of subgroups.
   (e) $\mathrm{NCore}(G) := \bigcap\{\, H \trianglelefteq G : \text{clopen}\,\}$, the "normal core"; any normal neighborhood of 1 eventually covers it; the clopen normal subgroups form a monoid with $\mathrm{NCore}(G)$ as identity.
$$G_1 \subseteq Q(G) \subseteq \mathrm{Core}(G) \subseteq \mathrm{NCore}(G)$$
   In a locally connected group, G_1 is normal clopen, so $\mathrm{NCore}(G) = G_1$.
   (f) The *polycompact radical*, $\mathrm{PRad}(G) := \overline{\bigcup\{\, H \trianglelefteq G : \text{compact}\,\}}$

   Proof: G_1G_1, G_1^{-1}, $\phi^{-1}G_1$ are connected and contain 1. The map $x \mapsto [y, x]$ is continuous, so $[y, G_1]$ are connected and generate the connected subgroup $[G, G_1]$. For $u \in U$ clopen, uG_1 is connected, so $uG_1 \subseteq U$.

For any clopen subset F and $a \in Q$, then Fa^{-1}, F^{-1}, and $\phi^{-1}F$ are clopen and contain 1, so QQ (since $Q \subseteq Fa^{-1}$), Q^{-1}, and ϕQ are subsets of F.

Similarly, for any (normal) clopen subgroup H, the (n)core $C \subseteq \phi^{-1}H$, so $\phi C \subseteq H$; for any neighborhood of 1, $[\![U]\!]$ is clopen, so contains $\mathrm{Core}(G)$. If $H \leqslant K$ and $x \in H$ then $xK \subseteq H$ is open, so H is open.

10. A normal subgroup that is totally disconnected, 0-D, or discrete, commutes with G_1, $Q(G)$, or $\mathrm{Core}(G)$, respectively.

    Proof: $f : x \mapsto [a, x]$ is continuous, and $G \to H$ when $a \in H$; $f^{-1}1$ is the centralizer $c(a)$; fG_1 is connected in H, hence $fG_1 = \{1\}$. For any $b \in H \smallsetminus 1$, there is U_b clopen on 1 not containing b, so $Q \subseteq f^{-1}U_b$ and $Q \subseteq \bigcap_b f^{-1}U_b = f^{-1}1$. If H discrete, $f^{-1}1$ is a clopen subgroup, so $\mathrm{Core}(G) \subseteq c(a)$.

    A discrete normal subgroup is called a *lattice* of G.

11. Series or products: By $\prod_i x_i$ is meant the net of finite products $(\prod_{\substack{i \in A \\ A \text{ finite}}} x_i)$.

    So a product is Cauchy when for any $U \in \mathcal{T}_1$, there is a finite set I such that for any finite set J, $\prod_{i \in J \smallsetminus I} x_i \in U$.

    For sequences, $\prod_{n \in \mathbb{N}} x_n$ is Cauchy when $\prod_n^m x_i \in U$ for n, m large enough.

    (In additive notation, products are written as $\sum$.)

    For abelian groups, $\prod_i (x_i y_i) = (\prod_i x_i)(\prod_i y_i)$ and $\prod_i x_i^{-1} = (\prod_i x_i)^{-1}$.

Topological groups can *act* on topological spaces (where the action $(g, x) \mapsto g \cdot x$ is required to be continuous), and on measure spaces (where the action must preserve measurable (and null) sets). For example, G acts on the cosets of a closed subgroup by $x \cdot yH := xyH$. A function on X is left-*invariant* when $f(x_1, \ldots, x_n) = f(g \cdot x_1, \ldots, g \cdot x_n)$ for any $g \in G$. For example, a measure is left-invariant when $\mu(xE) = \mu(E)$.

A *local group* is a topological space with a partial group structure, i.e., there is a 1, left/right translations by a neighborhood of 1 are defined, as well as inverses, and continuity and associativity hold whenever possible; local groups are *locally isomorphic* when a neighborhood of 1 in X is homeomorphic to a neighborhood of 1 in Y and $\phi(xy) = \phi(x)\phi(y)$, $\phi^{-1}(xy) = \phi^{-1}(x)\phi^{-1}(y)$ hold whenever possible.

Structure of G

1. For H a closed normal subgroup, G/H is a topological group and the morphism $\pi : G \to G/H$ is an open map (since $\pi U = UH$).

   For an open morphism, the map $x \ker \phi \mapsto \phi(x)$ is an isomorphism

   $$G/\ker \phi \cong \operatorname{im} \phi$$

   Also, $(G/K)/(H/K) \cong G/H$, but $HK/K \cong H/H \cap K$ may be false, e.g. for $a \in \mathbb{Q}^{\mathsf{c}}$, $a\mathbb{Z}/(a\mathbb{Z} \cap \mathbb{Z}) \cong \mathbb{Z}$ but $(a\mathbb{Z} + \mathbb{Z})/\mathbb{Z}$ is not discrete.

$\prod_i(G_i/H_i) \cong (\prod_i G_i)/(\prod_i H_i)$ via the open morphism $(x_i) \mapsto (x_i H_i)$.

2. If H is clopen then G/H is discrete.

   If H is compact, then $\pi : G \to G/H$ is a closed (and open) map (F closed in G is compact, so FH is closed).

3. If H and G/H are both connected/compact/separable, then so is G.

   Proof: xH is connected, so if F is clopen, then $xH \subseteq F$ or $xH \subseteq F^{\mathsf{c}}$; so $\pi F = G/H$ as it is clopen. Given an open cover U_α of G, each xH is covered by a finite number of them; of these, πU_α cover G/H.

4. Recall from topology, that G/G_1 is totally disconnected and $G/Q(G)$ is completely disconnected. The clopen subsets of $G/Q(G)$ correspond to those in G.

   $G/\mathrm{NCore}(G)$ has trivial NCore, so it is embedded in a product of discrete groups.

   For G locally connected, G/G_1 is discrete.

   Proof: If H is a normal clopen subgroup, then G/H is discrete; so $x \mapsto (xH)$ for all such H is a 1-1 open morphism.

5. The simple topological groups are either connected or totally disconnected, have no clopen subgroups or are discrete, are abelian or have trivial center, etc.

1.0.1 Normed Groups

Given a left-invariant metric, define the *norm* $\|x\| := d(1,x)$, $G \to \mathbb{R}^+$, so that $d(x,y) = d(1, x^{-1}y) = \|x^{-1}y\|$; then the metric properties become:

$$\begin{array}{c|c} \|xy\| \leqslant \|x\| + \|y\| & d(a,c) \leqslant d(a,b) + d(b,c) \\ \|x^{-1}\| = \|x\| & d(b,a) = d(a,b) \\ \|x\| = 0 \Leftrightarrow x = 1 & d(a,b) = 0 \Leftrightarrow a = b \end{array}$$

Then $\|x\| \geqslant \frac{1}{2}\|xx^{-1}\| = 0$, $\|x_1 \cdots x_n\| \leqslant \|x_1\| + \cdots + \|x_n\|$, $|\|x\| - \|y\|| \leqslant \|x^{-1}y\|$ (so the norm is continuous); $B_r(x) = xB_r$ (where $B_r := B_r(1)$), $B_rB_s \subseteq B_{r+s}$, $B_r^{-1} = B_r$.

The metric is compatible with the topology of G if B_r are open and form a base ("$V \subseteq B_r \subseteq U$"); the boundary of B_r need not be $S_r := \{\, x : \|x\| = r \,\}$.

A metric is left-right invariant $\Leftrightarrow$ $\|yx\| = \|xy\|$ $\Leftrightarrow$ $B_r x = xB_r$ $\Leftrightarrow$ the topology is 'balanced', i.e., there is a base of normal open sets.

Proposition 1

$$\boxed{G \textbf{ is normable} \Leftrightarrow \textbf{it is first countable.}}$$

PROOF: Given any bounded $f : G \to \mathbb{R}$ (can assume $f(1) = 0$), a semi-norm can be created by $|a| := \sup_x |f(xa) - f(x)| \geqslant |f(a)|$. For first countable spaces, there exist compatible norms: Let $V_{1/2^n}$ be a nested sequence of symmetric 1-neighborhoods such that $V_{1/2^{n+1}}^2 \subseteq V_{1/2^n}$; for any dyadic rational $p = 0{\cdot}r_1 \dots r_n$ ($r_i = 0, 1$), let $V_p := V_{r_1/2} \cdots V_{r_n/2^n}$, so that $V_p V_q \subseteq V_{p+q}$; let $f(a) := \inf\{\, t : a \in V_t \,\}$ (extended by 1 in V_1^{c}); hence $f(a) \leqslant t \Leftrightarrow a \in V_t$. $B_t \subseteq V_t \subseteq B_{t+\epsilon}$, since for any $a \in V_t$, $\epsilon > 0$, $x \in G$, there is a dyadic p such that $p \leqslant f(x) < p+\epsilon$; so $xa, x^{-1}a \in V_{p+\epsilon} V_t \subseteq V_{t+p+\epsilon}$, so $|a| = \sup_x |f(xa) - f(x)| \leqslant t + \epsilon$.

□

Examples:

- Linearly ordered groups with the interval topology, since $|x'x^{-1}| < \delta$, $|y^{-1}y'| < \delta \;\Rightarrow\; \delta^{-1}(xy)^{-1}\delta^{-1} < x'y' < \delta xy\delta$. Examples: $\mathbb{Z}+$, $\mathbb{Q}+$, $\mathbb{R}^\times\cdot$.
- Subgroups; open images using

  $$\|\pi(x)\| := \inf_{\pi(y)=\pi(x)} \|y\| = \inf_{\pi(y)=1} \|y^{-1}x\| = d(x, \pi^{-1}1)$$

  hence quotients ($\|xH\| = d(x, H)$)
- Countable products of normed groups (since first countability is preserved).
- $\ell^\infty(G)$ with $(a_n)(b_n) := (a_n b_n)$, $\|(a_n)\| := \sup_n \|a_n\|$.

1. A morphism is Lipschitz when $\|f(x)\| \leqslant c\|x\|$.
2. A subset is bounded iff $\forall x \in A,\ \|x\| \leqslant c$, i.e., $A \subseteq B_c$. If A, B bounded then so is AB.
3. $\|\prod_n x_n\| \leqslant \sum_n \|x_n\|$; so in a complete normed group, if $\sum_{n\in\mathbb{N}} \|x_n\|$ converges ('absolutely'), then so does $\prod_{n\in\mathbb{N}} x_n$; moreover sub-'series' (even rearrangements, and inverses) converge as well.

   Proof: $\big\|\prod_{n=M}^N x_n\big\| \leqslant \sum_{n=M}^N \|x_n\|$;

   $\sum_{n=M}^N \|x_{\sigma(n)}^{\pm 1}\| \leqslant \sum_{n=\min\{\,\sigma(M),\dots,\sigma(N)\,\}}^{\max\{\,\sigma(M),\dots,\sigma(N)\,\}} \|x_n\| \to 0$.

   However $\sum_n \frac{(-1)^n}{n}$ converges but $\sum_n \frac{1}{n}$ doesn't.
4. *Root test*: If $r := \lim_{n\to\infty} \|x_n\|^{1/n} < 1$ then $\prod_n x_n$ is Cauchy (e.g. when $\|x_n\| \leqslant \frac{c}{2^n}$); if $r > 1$ then it diverges.

   Ratio test: $r = \lim_{n\to\infty} \frac{\|x_{n+1}\|}{\|x_n\|}$ if it converges.

   Proof: $d(x_1 \cdots x_n, x_1 \cdots x_m) = \|x_{n+1} \cdots x_m\| \leqslant (r+\epsilon)^{m-n}$; else $\|x_n\| \geqslant (r-\epsilon)^n > 1$ for infinitely many n. For n large enough, $(r-\epsilon)^{n-n_0} \leqslant \|x_n\|/\|x_{n_0}\| \leqslant (r+\epsilon)^{n-n_0}$.

5. Separable first countable groups and σ-compact metric groups are second countable (from topology).

6. The norm induced on an open image of a complete group is complete. So quotients are complete if G is complete.

   Proof: Let $\phi(x_n)$ be absolutely convergent in $\operatorname{im}\phi$, so $\sum_n \|\phi(x_n)\|$ converges. For each n, there is a v_n, $\|v_n\| \leqslant \|\phi(x_n)\| + 2^{-n}$, $\phi(x_n) = \phi(v_n)$, so $\sum_n \|v_n\|$ converges as does $\sum_n v_n = v$. Hence $\sum_n \phi(x_n) = \sum_n \phi(v_n) = \phi(\sum_n v_n) \to \phi(v)$.

7. If $\phi : X \to Y$ is a morphism with X is complete, and $\|\phi(x)\| \geqslant c\|x\|$ then ϕ is 1-1 and has a closed image.

8. *Proposition* 2

Open Mapping Theorem

> **An onto morphism from a complete separable normed group to a non-meagre group is open.**

Examples of non-meagre groups are complete metric groups and locally compact groups.

Proof: ϕB_r is not nowhere dense since H is not meagre:

$$H = \phi G = \phi \bar{A} = \phi(AB_r) = \bigcup_n \phi(a_n)\phi B_r.$$

$\overline{\phi B_r}$ form a base for H: there is an interior point $aV \subseteq \overline{\phi B_{r/2}}$, so

$$1 \in W_r := V^{-1}V \subseteq \overline{\phi B_{r/2}\phi B_{r/2}} \subseteq \overline{\phi B_r}.$$

$\forall U \in \mathcal{T}_1(H), \exists \overline{V} \subseteq U, \exists r,\ \phi B_r \subseteq V$ by continuity of ϕ, so $\overline{\phi B_r} \subseteq \overline{V} \subseteq U$.

$W_{r/3} \subseteq \overline{\phi B_{r/3}} \subseteq \phi B_r$: Let $y \in \overline{\phi B_{r/3}}$; then there is an $x_1 \in B_{r/3}$ with $\phi(x_1) \in y\overline{\phi B_{r/9}}$; so $\phi(x_1)^{-1}y \in \overline{\phi B_{r/9}}$; continuing by induction

$$\exists x_n \in B_{r/3^n}, \quad \phi(x_n) \in \phi(x_{n-1})^{-1} \cdots \phi(x_1)^{-1} y\overline{\phi B_{r/3^{n+1}}}$$

Thus $\phi(x_1 \cdots x_n) \in y\overline{\phi B_{r/3^{n+1}}}$, and $\prod_n x_n$ is a Cauchy sequence in G; so $\prod_n x_n \to x \in \overline{B_{r/2}}$ since $\|x_1 \cdots x_n\| \leqslant \|x_1\| + \cdots + \|x_n\| \leqslant r/2$, and

$$y = \lim_{n\to\infty} \phi(x_1 \cdots x_n) = \phi(x) \in \phi\overline{B_{r/2}} \subseteq \phi B_r$$

Finally, $\phi B_r(x)$ is open since $\phi(x)W_{r/3} \subseteq \phi(x)\phi B_r = \phi B_r(x)$, enough to show ϕ is an open mapping.

□

9. *Closed Graph Theorem*: A group-morphism between complete separable normed groups with a graph $\mathrm{Graph}(\phi) = \{ (x, \phi(x)) : x \in G \}$ that is closed in $G \times H$, is continuous.

   Proof: $\mathrm{Graph}(\phi)$ is a complete separable normed group and $\pi_G : \mathrm{Graph} \to G$ is a bijective morphism, hence an isomorphism; thus $\phi = \pi_H \circ \pi_G^{-1}$ is continuous.

10. Every balanced topological group can be embedded in a product of normed groups.

    Proof: For normal $U \in \mathcal{T}_1$, can form a semi-norm $|\cdot|_U$; hence a norm on the metric group G/Z_U where $Z_U = \{ x : |x|_U = 0 \}$; consider $\phi : X \to \prod_U G/Z_U$, $x \mapsto (xZ_U)_{U \in \mathcal{T}_1}$, a 1-1 morphism.

11. The completion of a normed abelian group has a product $[a_n][b_n] := [a_n b_n]$ and a norm $\|[a_n]\| := \lim_{n\to\infty} \|a_n\|$, where $[a_n]$ is an equivalence class of the Cauchy sequence (a_n). Any morphism (being uniformly continuous) can be extended to the completions.

1.1 Locally Compact Groups

Examples

- $\mathbb{C}+$, $\mathbb{C}^\times\cdot$, $\mathbb{H}^\times$.
- The Heisenberg group $\begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$; the Weil-Heisenberg group $\begin{pmatrix} 1 & \mathbb{R} & \mathbb{T} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$.
- Lie groups, i.e., topological groups that are locally isomorphic to $\mathbb{R}^n$.

Closed subgroups, quotients, and finite products are again locally compact groups.

Proposition 3

Haar measure

There is a positive left-invariant Radon measure on Borel sets, which is unique up to a constant.

PROOF: For K compact and $U \in \mathcal{N}_1$, let

$$m_U(K) := \min\{ n \in \mathbb{N} : K \subseteq \bigcup_{i=1}^{n} a_i U, a_i \in K \}$$

then $m_U(\varnothing) = 0$, m_U is increasing, $m_U(K_1 \cup K_2) \leqslant m_U(K_1) + m_U(K_2)$, with equality if $K_1 U \cap K_2 U = \varnothing$; $m_U(xK) = m_U(K)$.

Let K_0 be a compact neighborhood of 1, and

$$\mu_U(K) := m_U(K)/m_U(K_0),$$

then $m_U(K_0) = 1$; $0 \leqslant \mu_U(K) \leqslant m_{K_0}(K)$ for all U since K is covered by $m_{K_0}(K)$ translates of K_0, which in turn are covered by $m_U(K_0)$ translates of U. Hence $\mu_U \in \prod_K [0, m_{K_0}(K)]$, a compact T_2 space; but as $\mathcal{N}_1 \to 1$, the filter $\mu_{\mathcal{N}_1} \to \mu$ uniquely. By continuity of $f \mapsto f(K)$, $\mu(\varnothing) = 0$, $\mu(K_0) = 1$, μ is increasing and finitely sub-additive, translation-invariant, and $K_1 \cap K_2 = \varnothing \Rightarrow \exists U \in \mathcal{T}_1,\ K_1U \cap K_2U = \varnothing \Rightarrow \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. It can thus be extended to a translation invariant Borel measure.

$\square$

Each left-Haar measure has an associated right measure $\mu_r(E) := \mu(E^{-1})$; the left and right-Haar measures need not be the same.

1. Any compact neighborhood K of 1 generates a clopen σ-compact subgroup that is finitely generated modulo K: $[\![K]\!] = K[\![a_1, \ldots, a_n]\!]$. So, topologically, G is the countable sum of σ-compact cosets (including G_1), hence paracompact T_2.

   Proof: Let $L := K \cup K^{-1}$; then $LL \subseteq Ka_1 \cup \cdots \cup Ka_n$ ($a_i \in LL$); $F := [\![a_1, \ldots, a_n]\!]$; then $LLF \subseteq KF \subseteq LF$, so $[\![K]\!] = [\![L]\!] = LF = KF$. Every coset xH has a locally finite refinement of a given open cover.

   Note that picking a nested sequence of open subsets $U_n \subseteq K$, and $V_{n+1}^2 \subseteq U_n \cap V_n$ with $\forall x \in K, x^{-1}V_{n+1}x \subseteq V_n$, gives a compact normal subgroup $H := \bigcap_n V_n$, so $[\![K]\!]/H$ is a σ-compact normed group (hence separable).

2. $\mathrm{Core}(G) = G_1$ (compactly generated).

   Proof: From 7, G/G_1 is 0-D; for any $x \notin G_1$, there is $K \subseteq G/G_1$ compact open subgroup not containing xG_1, so $G_1 \subseteq \pi^{-1}K \subseteq \{x\}^{\mathsf{c}}$ (a clopen subgroup).

   (Hence groups such that G/G_1 is compact are compactly generated.)

3. The abelian subgroup $\overline{[\![a]\!]}$ is either discrete ($\cong \mathbb{Z}$) or compact.

   Proof: If not discrete, then any neighborhood of 1 contains infinitely many $a^{\pm n}$; so for any compact neighborhood, a^nK ($n > 0$) cover K (since $a^n \in xK^{-1}$ for arbitrarily large n); hence $K \subseteq aK \cup \cdots \cup a^NK$. For any n, pick the first $m \geqslant 0$ with $a^{n+m} \in K$; then $a^{n+m} \in a^kK$; for $i := k - m$, $0 < i \leqslant N$ and $a^n = a^{(n+m-k)+i} \in a^iK$; so a^iK cover $[\![a]\!]$, which is thus totally bounded.

4. Every automorphism has a *modular form*, $\delta(\tau) := \mu(\tau E)/\mu(E) \in \mathbb{R}^+$;

$$\delta\tau\sigma = \delta(\tau)\delta(\sigma)$$

In particular, the *modular* function $\Delta(a)$ is the determinant of the inner automorphism $x \mapsto a^{-1}xa$; then

$$\mu(Ex) = \mu(x^{-1}Ex) = \Delta(x)\mu(E)$$

$$\Delta(xy) = \Delta(x)\Delta(y)$$

Thus Δ is a morphism, so $\Delta(x^{-1}yx) = \Delta(y)$, $[G, G] \subseteq \ker \Delta$.

Proof: Let $\mu(EU) < \mu(E) + \epsilon$ and $\mu(ExU) < \mu(Ex) + \epsilon$; then for $x \in U \cap U^{-1}$, $\mu(Ex) - \epsilon < \mu(E) < \mu(Ex) + \epsilon$; so $|\mu(Ex) - \mu(E)| < \epsilon$ and Δ is continuous at 1.

For a closed normal subgroup, $\Delta_H = \Delta$.

5. *Proposition* 4

Gleason-Yamabe

> **There is a clopen subgroup H containing a compact normal subgroup K, such that H/K is locally Euclidean.**

6. (Cartan-Iwasawa-Malcev) A connected locally compact group has a maximal compact normal subgroup K, all such subgroups are conjugates, and G/K is topologically $\mathbb{R}^n$; the intersection of these maximal compact subgroups is the (poly)compact radical. G is thus a 'projective limit' of Lie groups $\lim_{K\to 1} G_1/K$.

   A general locally compact group contains a compact subgroup such that G/K is topologically $D \times \mathbb{R}^n$ with D a discrete space.

7. Totally disconnected locally compact groups have a base of compact open subgroups; hence are 0-dimensional.

   Any image G/H is again totally disconnected. There is no classification of them, although the characteristically simple ones are nearly so (Caprace).

   Proof: G is 0-D (see Topology), with a base of compact open subsets K. For each $x \in K$, there is $V_x \subseteq K$ such that $V_x x V_x \subseteq K$; so $K \subseteq x_1 V_1 \cup \cdots \cup x_n V_n$; $V := V_1 \cap \cdots \cap V_n$; then $VK \subseteq \bigcup_i V x_i V_i \subseteq K$; hence $[\![V]\!] \subseteq K$ is a clopen subgroup.

8. A locally compact normed group is complete.

   Proof: If (x_n) is Cauchy, then $\|x_m^{-1}x_n\| \to 0$, so $x_n \in x_m K$ for m, n large enough; thus $x_n \to x \in x_m K$.

9. A morphism from a σ-compact to a locally compact group is an embedding. (The image of X is σ-compact, hence locally compact, hence there is a compact subset with non-empty interior.)

1.1.1 σ-Compact Locally Compact Groups

1. *Open mapping theorem*: An onto morphism $\phi : G \to H$ from a σclc group to a non-meagre group is open (so $G/\ker\phi \cong H$).

   Proof: Let $VVV \subseteq U$, $\bar{V}$ compact; then $G = \bigcup_{n\in\mathbb{N}} x_n V$ (since σ-compact); so $\phi G = \bigcup_n \phi(x_n\bar{V}) = \bigcup_n \phi(x_n)\phi\bar{V}$; hence $\phi\bar{V}$ has a non-empty interior W, so $1 \in w^{-1}W \subseteq \phi(v^{-1})\phi\bar{V} \subseteq \phi V\bar{V} \subseteq \phi U$.

   The closed graph theorem follows since the closed Graph of a group morphism is σ-compact locally compact.

2. If H, K are commuting normal subgroups of a σclc group, with $H \cap K = 1$, then $HK \cong H \times K$.

3. Any clopen subgroup has a countable number of cosets.

4. There is a compact normal subgroup whose quotient is a separable normed group.

   Proof: $G = \bigcup_n K_n$ with $K_n \subseteq K_{n+1}$; start with U_0 an open subset of a compact neighborhood. Let $f(a,x) := a^{-1}xa$; by continuity at 1, for any $a \in K_n$, there are V, W open with $f(aV, W) \subseteq U_{n-1}$. Then $K_n \subseteq a_1V_1 \cup \cdots \cup a_kV_k$; let $W_n := W_1 \cap \cdots \cap W_k \in \mathcal{T}_1$, reduced to $U_n \subseteq W_n$ such that $U_n^2, U_n^{-1} \subseteq U_{n-1}$; so $f(K_n, U_n) \subseteq U_{n-1}$. Thus $K := \bigcap_n U_n = \bigcap_n \bar{U}_n$ is a closed subgroup; it is normal since for any $a \in K_N$, $n \geqslant N$,

   $$a^{-1}Ka \subseteq f(K_N, U_n) \subseteq f(K_n, U_n) \subseteq U_{n-1}$$

   Moreover, G/K is first countable (by U_n) and σ-compact.

5. Compactly generated locally compact groups: so are open images; if H, G/H are compactly generated l.c. groups and G is l.c. then G is compactly generated (by $K_HK_{G_1}F$ where F finite set, $K_{G/H} \subseteq \pi K_GF$).

   $G = \bigcup_n K^n$, so K^n has non-empty interior for some n.

An **amenable** group is a locally compact group that has a left-invariant finitely-additive measure on 2^G with $\mu(G) = 1$. Examples include locally compact abelian groups and compact groups. Closed subgroups, finite products, quotients are again amenable.

1.2 Unimodular Groups

are locally compact groups whose left and right Haar measures are the same, thus $\Delta = 1$:

$$\mu(Ex) = \mu(xE) = \mu(E^{-1}) = \mu(E)$$

equivalently, when there is a normal compact neighborhood of 1 (since $\mu(K) = \mu(x^{-1}Kx) = \Delta(x)\mu(K)$).

Examples:

- Discrete groups (with counting measure). Countable locally compact groups are discrete (since homogeneous $T_{3.5}$ spaces are discrete or $\mathbb{Q}$).
- Compact groups (since ΔG is a compact subgroup of $\mathbb{R}$)
- Topologically simple locally compact groups, since $\overline{[G,G]} = G$ or 1 (abelian).
- $GL(\mathbb{R}^n)$, the invertible $n \times n$ matrices, with the induced metric from $\mathbb{R}^{n^2}$, and measure $\mu(T)/|\det T|^n$.

1. The space of compact open normal (con) subgroups form a metric group.

   $f(H,K) := |H/H \cap K|$ satisfies the multiplicative triangle inequality, so $d(H,K) := \ln f(H,K) + \ln f(K,H)$ is a metric on the set of con subgroups.

1.2.1 Locally Compact Abelian Groups

Examples:

- $\mathbb{R}^n$ with translations (the Haar measure is called Lebesgue measure $\mathrm{d}\boldsymbol{x}$, generated from $K_0 = [0,1]^n$)
- $\mathbb{R}^\times$ with scalings (measure $\mathrm{d}x/x$)
- $\mathbb{S}$ with rotations (measure $\mathrm{d}\theta$)
- $\mathbb{Z}^{(\mathbb{N})}$, the finite integer sequences
- Groups topologically generated by one element ('monothetic')

1. They are unimodular.
2. Topologically finitely generated subgroups are the product of a discrete subgroup and a compact one, since $[\![a_1,\ldots,a_n]\!] = [\![a_1]\!] \cdots [\![a_n]\!]$, with each subgroup either discrete or totally bounded.
3. Hence a compactly generated abelian group contains a subgroup $H \cong \mathbb{Z}^n$ such that G/H is compact.

   Proof: $[\![K]\!] = K[\![a_1,\ldots,a_n]\!] = K[\![a_1]\!] \cdots [\![a_n]\!]$; distinguish between $\overline{[\![a_i]\!]}$ that are compact or discrete ($\cong \mathbb{Z}$), hence $[\![K]\!]/H \cong K\overline{[\![a_{m+1}]\!]} \cdots \overline{[\![a_n]\!]}$ compact.
4. $\mathrm{NCore}(G) = \mathrm{Core}(G) = G_1$; so when totally disconnected, can be embedded in a product of discrete abelian groups.
5. *Dual space of Characters*: $G^* := \mathrm{Hom}(G,\mathbb{S})$ is a locally compact abelian group with the compact-open topology generated by the base $\Phi_\epsilon(K) := \{\, \phi \in G^* : \phi K \in e^{2\pi i]-\epsilon,\epsilon[} \,\}$ where K is any compact neighborhood of 1 (uniform convergence on compact sets). (Note: for non-abelian locally compact T_2 groups, G^* need not be a group.)

   'Proof': $\Phi_\epsilon(K^n) \subseteq \Phi_{\epsilon/n}(K)$; $\Phi_\epsilon(K)$ is totally bounded.

(a) G^* separates points of G (i.e., $\forall x \neq 1$, $\exists \phi \in G^*, \phi(x) \neq 1$);

(b) $(G \times H)^* \cong G^* \times H^*$, via $J(\phi, \psi)(x, y) := \phi(x)\psi(y)$ (onto since $\chi(x, y) = \chi(x, 1)\chi(1, y)$, continuous since $J(\Phi_{\epsilon/2}(\pi_G K) \times \Phi_{\epsilon/2}(\pi_H K)) \subseteq \Phi_\epsilon(K) \subseteq (G \times H)^*$);

(c) The *dual* of a morphism $\phi : G \to H$ is $\phi^* : H^* \to G^*$, $\phi^*(\psi) := \psi \circ \phi$;
The *annihilator* and *pre-annihilator* of subsets of G, G^* are the closed subgroups

$$A^\perp := \{\, \phi \in G^* : \phi A = 1 \,\}$$
$$^\perp\Phi := \{\, x \in G : \Phi x = 1 \,\}$$

$A^\perp \cong (G/\overline{[\![A]\!]})^*$ via $\phi \mapsto \phi \circ \pi$.

(d) If H is a clopen subgroup, then $G^* \to H^*$ is an open map and $H^* \cong G^*/H^\perp$.

(e) $H := \bigcap_{\phi \in G^*} \ker \phi = \overline{[G, G]}$; hence $G^* \cong (G/H)^*$.

6. *Pontryagin duality*: $G^{**} \cong G$ via $x^{**}(\phi) := \phi(x)$.

   Proof: For continuity, take K compact in G^*, L compact in G, $K \subseteq A\Phi_\epsilon(L)$ with $A = \{\, \phi_1, \ldots, \phi_n \,\}$; by continuity, $\phi_i \in \Phi_\epsilon(V_i)$; let $V := L^\circ \cap \bigcap_i V_i$; then for any $\phi \in K$, $x \in V$, $\phi = \phi_i \psi$, so $\phi V \subseteq \phi_i V \psi V \subseteq U_{2\epsilon}$.

7. There is therefore a correspondence between LCA groups and their duals:

G is compact	$\leftrightarrow$	G^* is discrete
normed		σ-compact
connected compact		torsion free
compact totally disconnected		torsion discrete

   (So even compact abelian groups cannot be classified.)

   Proof: If G discrete, then $G^* = \overline{\Phi_\epsilon(1)}$ is compact; if G is compact then $\{\, 1 \,\} = \Phi_\epsilon(G)$ is open. If G is first countable with compact base K_n, then any $\phi \in \Phi_\epsilon(K_n)$ for some n by continuity, so $G^* = \bigcup \overline{\Phi_\epsilon(K_N)}$; if $G = \bigcup_n K_n$, $K_n \subseteq K_{n+1}$, then $\Phi_\epsilon(K_n) \supseteq \Phi_\epsilon(K_{n+1})$ is a countable base.

8. Every locally Euclidean abelian group is of the type $\mathbb{R}^n \times \mathbb{T}^m \times D$ where D is a discrete abelian group.

9. Every compactly generated LCA group is isomorphic to $\mathbb{R}^n \times K$, with K compact. In particular, $G_1 \cong \mathbb{R}^m \times K$ where K is a connected compact abelian group.

   Proof: Any compact neighborhood generates H with $H/\mathbb{Z}^m \cong K$; dually, $H^*/K^* \cong \mathbb{T}^m$. As K^* is discrete, H^* is locally isomorphic to $\mathbb{T}^m$, hence isomorphic to $\mathbb{R}^{n_1} \times \mathbb{T}^{n_2} \times D$ with D discrete. Thus $H^{**} \cong \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times K$.

1.3 Compact Groups

Examples:

- $\mathbb{S}$, the complex numbers of unit length
- $\mathbb{S}^3$, the quaternions of unit length
- Finite groups. Countable compact groups are finite.
- The Cantor set $2^{\mathbb{N}}$ (embedded in $\mathbb{R}$) with addition of binary sequences. More generally, Boolean groups 2^A.

Products, closed subgroups and images are again compact groups.

Slightly more general are the totally bounded groups: images and products are again totally bounded. Every totally bounded group can be completed (embedded in) to a compact group.

1. Clopen subgroups have a finite number of cosets.
2. There is a neighborhood base of normal open sets. So any norm is bi-invariant.

   Proof: Let $U \in \mathcal{T}_1$ and $VVV \subseteq U$; then $G = a_1V \cup \cdots \cup a_nV$; let $W := a_1Va_1^{-1} \cap \cdots \cap a_nVa_n^{-1} \in \mathcal{T}_1$; then for any $x = av \in aV$, $x^{-1}Wx = v^{-1}a^{-1}Wav \subseteq v^{-1}Vv \subseteq VVV \subseteq U$; so $W \subseteq \bigcap_x xUx^{-1} \subseteq U$ a normal open subset.
3. $\mathrm{NCore}(G) = G_1$, since any clopen set on 1 contains a compact open normal subgroup.

   Proof: In G/G_1, any clopen U contains a compact open subgroup H; then $\bigcap_x x^{-1}Hx = \bigcap_{i=1}^n x_i^{-1}Hx_i$ is a con subgroup of H.
4. Every onto morphism between compact groups is open; every bijective morphism is an isomorphism.
5. Connected compact groups: Every neighborhood of 1 contains a con subgroup H such that $G/H \subsetneq GL(\mathbb{R}^n)$. Therefore, by taking a base, G is embedded in a product of $GL(\mathbb{R}^n)$ as an inverse limit.
6. Totally disconnected compact groups (called *profinite*) have bases of con subgroups; hence they can be embedded in a product of (discrete) finite groups via the map $G \to \prod_K G/K$, $x \mapsto (xK)_{K\in\mathcal{N}}$. (From topology, a totally disconnected compact group is either finite or has the Cantor set topology.)
7. Every connected compact abelian group can be embedded in some $\mathbb{T}^A$.

   Every locally connected compact abelian group is of the type $G_1 \times F$ (F finite group).

Every connected compact separable group is monothetic; more generally, every compact separable group such that G/G_1 is monothetic, is monothetic.

Every compact second countable group is topologically generated by at most 2 elements.

1.4 Examples

Finite Groups

$\mathbb{Z}_n$: the characters are $k \mapsto e^{2\pi i k/n}$.

Euclidean Groups

Integers $\mathbb{Z}$: discrete, abelian, linearly ordered; the characters are $n \mapsto e^{int}$ $(0 \leqslant t < 2\pi)$, so $\mathbb{Z}^* \cong \mathbb{S}$.

$\mathbb{Z}^{(\mathbb{N})}$ is second countable, abelian, not locally compact.

Rationals $\mathbb{Q}$: abelian, totally disconnected, linearly ordered, not locally compact. The norms on $\mathbb{Q}$ are either the Euclidean one $|x-y|$ or a p-adic norm for p prime. The p-adic field $\mathbb{Q}_p$ is totally disconnected locally compact containing the compact open subgroup of the p-adic integers.

Reals $\mathbb{R}$: abelian, connected, locally connected, locally compact, linearly ordered. The morphisms $\mathbb{R} \to \mathbb{C}^\times$ are $t \mapsto e^{zt}$ $(z \in \mathbb{C})$; in particular, the characters $\mathbb{R} \to \mathbb{S}$ are $z \in i\mathbb{R}$, i.e., $x \mapsto e^{ixy}$ (since $\ker\phi = 0, y\mathbb{Z}$, or $\mathbb{R}$). The Haar measure on $\mathbb{R}$ is called Lebesgue measure. The closed subgroups are $\mathbb{R}$ and $a\mathbb{Z}$ (discrete).

$\mathbb{R}^\times \cong 2 \times \mathbb{R}$ (via $t \mapsto e^{at}$); abelian, disconnected; its Haar measure is $\mathrm{d}t/|t|$; contains the discrete subgroup $\{\, |x|_p : x \in \mathbb{Q}\,\}$.

Measures come from increasing functions $\mu(a,b] = f(b)-f(a)$ (the number of discontinuities of an increasing function is countable, so $f = f_{\text{cont}} + f_{\text{discrete}}$; μ is continuous since $A_n \to \varnothing$ open sets $\Rightarrow$ $\mu(A_n - K_n) \to 0$ for f right-continuous, so $\bigcap_n K_n = \varnothing$ and $\bigcap_n^N K_n = \varnothing$, so $\mu(A_N) = \mu(A_N - \bigcap_n^N K_n) \leqslant \mu(A_N - K_N) \to 0$; now let $f(x) := \begin{cases} \mu(0,x] & x > 0 \\ -\mu(x,0] & x \leqslant 0 \end{cases}$; then μ is continuous implies f is right-continuous, and $\mu \geqslant 0$ implies f is increasing; so $\mu(a,b] = \mu(0,b]-\mu(0,a]$ or $\mu(a,0]-\mu(b,0]$ or $\mu(0,b]+\mu(a,0]$).

$\mathbb{R}^n$: The only morphisms $\mathbb{R}^n \to \mathbb{C}^\times$ are $\boldsymbol{a} \mapsto e^{\boldsymbol{z}\cdot\boldsymbol{a}}$ (since $\phi(\boldsymbol{a}) = \phi_1(a_1)\cdots\phi_n(a_n)$), so $\mathbb{R}^{n*} \cong \mathbb{R}^n$. Its closed subgroups are isomorphic to $\mathbb{R}^m \times \mathbb{Z}^k$ (the former are the only connected ones, the latter the only totally disconnected ones); every quotient of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^m \times \mathbb{T}^k$.

Complex $\mathbb{C} \cong \mathbb{R}^2$.

$\mathbb{C}^\times \cong \mathbb{S} \times \mathbb{R}$ (via $(\theta,a) \mapsto e^{a+i\theta}$); the morphisms $\mathbb{C}^\times \to \mathbb{R}^\times$ are $z \mapsto |z|^a$.

$\mathbb{H}^\times \cong \mathbb{S}^3 \times \mathbb{R}$.

Torus $\mathbb{S} \cong \mathbb{R}/\mathbb{Z}$: compact, connected, abelian group; the morphisms $\mathbb{S} \to \mathbb{C}^\times$ are the characters $e^{i\theta} \mapsto e^{in\theta}$ $(n \in \mathbb{Z})$ (since $\phi(1) = 1 = e^{2\pi i n}$), so $\mathbb{S}^* \cong \mathbb{Z}$; the only morphism $\mathbb{S} \to \mathbb{R}$ is trivial.

$\mathbb{T}^\mathbb{N}$: 1-top-generated, locally Euclidean; $\mathbb{T}^2$ contains the dense additive subgroup $(1, \alpha)\mathbb{R}$ (modulo 1, $\alpha \in \mathbb{Q}^c$), which is second countable, but not isomorphic to $\mathbb{R}$ (isomorphic only as algebraic groups).

$\mathbb{T}^\mathbb{R}$: 1-top-generated by $(e^{2\pi i r_\alpha})_{\alpha \in \mathbb{R}}$ where r_α are independent irrationals.

Torus solenoid $\mathbb{T}_p$: compact connected abelian.

Unit Quaternions $\mathbb{S}^3$: compact, connected group; its center is the only normal subgroup $\{\pm 1\}$; the cosets of the subgroup $\mathbb{S}$ form the *Hopf* fibration.

Cantor group $2^\mathbb{N}$: compact, totally disconnected, abelian.

Matrix Groups

Heisenberg group $\mathbb{R}^{2n+1}$ with $\begin{pmatrix} \boldsymbol{a} \\ b \\ \boldsymbol{c} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ y \\ \boldsymbol{z} \end{pmatrix} := \begin{pmatrix} \boldsymbol{a} + \boldsymbol{x} \\ b + y + \boldsymbol{a} \cdot \boldsymbol{z} \\ \boldsymbol{c} + \boldsymbol{z} \end{pmatrix}$: connected, locally Euclidean; center is $(\mathbf{0}, x, \mathbf{0})$; contains the discrete Heisenberg group $H_3(\mathbb{Z})$, which is 2-generated. There are various versions depending on the interpretation of the dot product.

Affine($\mathbb{R}$): $(a, v)(b, w) := (ab, v + aw)$, where $a, b \neq 0$; disconnected, locally Euclidean but not unimodular; trivial center; contains the normal subgroup $(1, \mathbb{R})$.

p-adic Groups

The Prüfer p^∞-group $\mathbb{Z}[1/p]/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$: countable, discrete, abelian subgroup of $\mathbb{Q}/\mathbb{Z}$; contains a tower of subgroups $p^{-n}\mathbb{Z}/\mathbb{Z}$; its dual is $\mathbb{Z}(p)$, and conversely.

2 Topological Rings and Modules

A **topological ring** is a ring with a topology such that addition and multiplication are continuous. (Note that $x \mapsto -1x$ is then continuous.)

A **topological module** is a topological ring acting continuously on a topological abelian group,

$$\lim_n a_n x_n = (\lim_n a_n)(\lim_n x_n).$$

A *topological algebra* is a topological ring with a chosen subfield in its center. The morphisms are the maps that preserve $+, \cdot, \to$.

Examples:

- Linearly ordered rings with the interval topology. (They are either discrete or order-dense; if $0 < b < a < 1$ then $\epsilon := \min(b, a - b)$ satisfies $2\epsilon < a$; similarly $4\eta < a$, so $(1 + \eta)^2 < 1 + a$.)
- $\mathbb{Q}$ with topology generated by the filter base of sets $m\mathbb{Z}$ ($m \neq 0$).
- Rings with an ideal I, and the natural I-adic topology generated by the filter base consisting of I^n ($n \in \mathbb{N}$). If $\bigcap_n I^n = 0$ then the topology is Tychonov.
- Products of modules and rings X^A are again topological (with pointwise convergence).
- Matrices $M_n(R)$ are again topological (with topology induced from R^{n^2}).
- The endomorphisms of an abelian topological group, with the topology generated by $U_K := \{ \phi : \phi K = 0 \}$ for finite subsets K.

The (Cauchy) completion is still a topological ring with

$$[x_n] + [y_n] := [x_n + y_n], \quad [x_n][y_n] := [x_n y_n].$$

An invertible morphism between complete modules is automatically an isomorphism. For example, $x \mapsto a^{-1}xa$ is a ring automorphism.

1. Recall from topological groups that for any open set U about 0
   $B_U(x) = x + U = U + x$; $-U \in \mathcal{N}$
   $\exists V \in \mathcal{N}, \quad V + V \subseteq U$
   $\exists V \subseteq U, \ -V = V$
   For any $A \subseteq X$, $A + U$ is open.
   In addition,
   $\exists V \in \mathcal{T}_0, \quad VV \subseteq U$ (continuity of $\cdot$)
   $\exists V \in \mathcal{T}_0, \quad xV \subseteq U$ (continuity at $(x, 0)$)
   $x \mapsto ax$ (or xa) is a homeomorphism if a is invertible.
   A, B compact/connected $\Rightarrow$ $A + B, AB$ compact/connected.

2. The closure of a sub-module, sub-ring, or ideal is again the same. So a set A generates the closed sub-module $\overline{[\![A]\!]}$ and closed ideal $\overline{\langle A\rangle}$.

   Centralizers $Z(A)$, such as the center, are closed subrings.

3. $\sum_{n,m} a_n x_m = (\sum_n a_n)(\sum_m x_m)$; in particular $\sum_n a x_n = a \sum_n x_n$.

4. R connected $\Rightarrow$ X connected (since $X = \bigcup_{x\in X} Rx$). For example, any topological ring that contains a connected subring is connected.

5. A subset B of a topological module is *bounded* when

   $$\forall U \in \mathcal{T}_0(X), \exists V \in \mathcal{T}_0(R), \quad VB \subseteq U.$$

   A subset B of a topological ring is *bounded* when

   $$\forall U \in \mathcal{T}_0, \exists V \in \mathcal{T}_0, \quad VB \cup BV \subseteq U.$$

   Subsets, finite unions, closures, $M+N$, BM, are again bounded. For example, compact subsets and Cauchy sequences. Morphisms preserve bounded sets. X is bounded if R is discrete.

   Proof: $\overline{B} = \bigcap_{U\in\mathcal{T}_0}(B+U)$ so $V\overline{B} \subseteq VB + VV \subseteq W + W \subseteq U$. $(V_1 \cap V_2)(M+N) \subseteq V_1 M + V_2 N \subseteq W + W \subseteq U$. For each $x \in K$, $V_x x \subseteq U$; $x+V_x$ cover K; take $V := \bigcap_{i=1}^n V_{x_i}$. For any W, $n \geqslant N \Rightarrow (x_n - x_N) \in W$; for each x_i, $V_i x_i \subseteq W$; $V := \bigcap_{i=1}^N V_i \cap W$; then $Vx_n \subseteq V_N x_N + VW \subseteq W + WW \subseteq U$. If $VA \subseteq T^{-1}U$ then $VTA \subseteq U$.

6. An idempotent on X, $P^2 = P$, called a *projection*, has a closed image $\operatorname{im} P = \ker(1-P)$.

7. (a) If $\{\, x : \bigcup_i T_i x \text{ bounded}\,\}$ is not meagre then T_i are equicontinuous.

   (b) If T_n are equicontinuous morphisms and $T_n x \to Tx$ for all x, then T is a morphism.

   Proof: Let $A := \{\, x : \forall i, T_i x \in \bar{W}\,\} = \bigcap_i T_i^{-1}\bar{W}$ closed; then $\{\, x : \bigcup_i T_i x \text{ bdd}\,\} \subseteq \bigcup_n nA$, so nA contains an open set, i.e., $x + V \subseteq A$; thus $T_i V \subseteq -T_i x + T_i A \subseteq \bar{W} + \bar{W} \subseteq U$. $\forall U, \exists W, V, \forall n,\ T_n V \subseteq W \subseteq \bar{W} \subseteq U$; hence $TV \subseteq U$.

8. If $a_n \to 0$ and x_n bounded then $a_n x_n \to 0$. If (a_n), (x_n) are Cauchy, then so is $(a_n x_n)$.

   Proof: $a_n x_n - a_m x_m = a_n(x_n - x_m) + (a_n - a_m)x_m \in B_1 V_1 + V_2 B_2 \subseteq W + W \subseteq U$.

9. If M is a clopen sub-module and $B \subseteq X$ is bounded, then $[M : B] := \{\, a \in R : aB \subseteq M\,\}$ is a clopen left ideal of R.

   Proof: If $a \in [M : B]$ and $VB \subseteq M$, then $(x+V)B \subseteq xB + VB \subseteq M$.

10. A (left) *topological divisor of zero* satisfies $ax_i \to 0$ for some $x_i \not\to 0$. Not invertible.

    A *topological nilpotent* satisfies $a^n \to 0$.

    An ideal is *topologically nilpotent* when $I^n \to 0$, i.e., $I^n \subseteq U$ for any $U \in \mathcal{T}_0$.

    An element is a (left) *topological quasi-regular* when for every $U \in \mathcal{T}_0$ there is a b such that $(1-a)b \in 1 + U$, i.e., $1 \in \overline{(1-a)R}$.

    A *topological quasi-nilpotent* is such that xa is a topological quasi-regular for all x.

11. The kernel of a morphism is a closed submodule/ideal. Closed submodules and closed ideals give quotients.

    Isomorphism theorems: $(R/I)/(J/I) \cong R/J$. $(\prod_i R_i)/(\prod_i I_i) \cong \prod_i R_i/I_i$.

12. The following are closed sub-modules/ideals:

    (a) The connected component of 0.

    (b) The *module core* $\bigcap\{\, M : \text{clopen sub-module}\,\}$. The clopen submodules form a filter $\mathcal{I}$ in the lattice of sub-modules.
    The *ring core* is $\bigcap\{\, I : \text{clopen ideal}\,\}$.

    Proof: aC_0 is connected and contains 0, so $aC_0 \subseteq C_0$.

13. Hence simple modules are either connected or totally disconnected. For example, topological division rings.

14. If R is complete with Core $\to 0$, and $a^n \to 0$, then a is quasi-regular,

    $$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

    Proof: $\sum_n a^n$ converges to some s since for any $U \in \mathcal{T}_0$, $\exists V \subseteq U$ clopen subgroup, and $n \geqslant N \;\Rightarrow\; a^n \in V$, so $\sum_n a^n$ is Cauchy; $s = 1 + as$.

2.1 Normed Modules/Rings

A **normed** module is one that has a translation-invariant metric (1st countable), acted upon by a normed ring, such that

$$\begin{aligned} \|x+y\| \leqslant \|x\| + \|y\|, &\qquad \|-x\| = \|x\|, \\ \|x\| \geqslant 0, &\qquad \|x\| = 0 \Leftrightarrow x = 0, \\ \|ax\| \leqslant \|a\|\|x\|. & \end{aligned}$$

Examples:

- $\mathbb{R}^n$ with product multiplication and $\|(a_i)\| = \max_i |a_i|$.

- $\mathbb{R}^2$ with $\binom{a}{b}\binom{x}{y} := \binom{ax}{(ay+bx)/2}$, or $\frac{1}{2}\binom{ax\pm by}{ay+bx}$, or $\binom{ax}{(ay+bx+by)/3}$.
  $\mathbb{R}^3$ with $\begin{pmatrix}a\\b\\c\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} := \begin{pmatrix}ax\\by\\(az+cy)/2\end{pmatrix}$
- $\ell^\infty(R)$ with addition, multiplication and $\|(a_n)\| := \sup_n \|a_n\|$; if R is complete, so is $\ell^\infty(R)$; not separable unless $R = 0$. Contains the closed subring of convergent sequences, and its closed maximal ideal c_0 of those that converge to 0.
- The set of bounded functions $A \to \mathbb{R}$ with the supremum norm; its subring of continuous functions $C_b(A)$, when A is a topological space, and its closed ideal $C_0(A)$ when A is locally compact T_2.
- $\ell^1(R)$ with addition, convolution and $\|(a_n)\| := \sum_{n=0}^\infty \|a_n\|$. It is complete if R is.
- $\ell^1(G, R)$ for any (finite) group, with $\boldsymbol{e}_g * \boldsymbol{e}_h := \boldsymbol{e}_{gh}$. For example, $\ell^1(\mathbb{Z})$.
- Products are again normed, $\|(x, y)\| := \max(\|x\|, \|y\|)$.

The completion of a normed module is again normed, $\|[x_n]\| := \lim_{n\to\infty} \|x_n\|$, so can assume complete. Any morphism between modules extends uniquely to the completions (including the completion of the ring).

1. $1 \leqslant \|1\|$ and $\|n\| \leqslant n\|1\|$; $\|a^n\| \leqslant \|a\|^n$.
2. $B_r B_s \subseteq B_{rs}$ so a set is metrically bounded iff bounded; $aB_r \subseteq B_{\|a\|r}$.
   $A \subset X$ is *balanced* when $B_1[0]A \subseteq A$,i.e., $\|a\| \leqslant 1, x \in A \Rightarrow ax \in A$, e.g. B_r.
3. For any a, $\|a^n\|^{1/n} \to \rho(a) \leqslant \|a\|$ (from above).
   (a) $\rho(1) = 1$, $\rho(ab) = \rho(ba)$, $\rho(a^{-1}xa) = \rho(x)$.
   (b) $\rho(a^n) = \rho(a)^n$, $\rho(a) = \|a\| \Leftrightarrow \|a^n\| = \|a\|^n$.
   (c) $\rho(a) < 1 \Rightarrow a^n \to 0$ (topological nilpotent)
   $\rho(a) > 1 \Rightarrow a^n \to \infty$.
   (d) If $ab = ba$ then $\rho(ab) \leqslant \rho(a)\rho(b)$, so $\rho(a^{-1})^{-1} \leqslant \rho(a)$.

   Proof: Let $\rho(a) := \inf_n \|a^n\|^{1/n}$. The division $n = q_n n_0 + r_n$ satisfies $r_n/n \to 0$, $q_n/n \to \frac{1}{n_0}$, so $\rho(a) \leqslant \|a^n\|^{1/n} \leqslant \|a^{n_0}\|^{q/n}\|a\|^{r/n} \to \|a^{n_0}\|^{1/n_0} < \rho(a) + \epsilon$.
4. In a complete normed ring, the power series $\sum_{n=0}^\infty a_n x^n$ converges absolutely if $\rho(x) < 1/\limsup_n \|a_n\|^{1/n}$ (by the root test).
5. $(1-a)^{-1} = 1+a+a^2+\cdots$ for $\rho(a) < 1$, since $(1-a)\sum_{n=0}^N a^n = 1-a^{N+1} \to 1$.

6. $x - axb = c$ has the solution $x = \sum_{n=0}^{\infty} a^n c b^n$ if $\rho(a)\rho(b) < 1$.

7. The invertible elements form an open topological group, since $x \mapsto x^{-1}$ is continuous and the distance between an invertible a and a non-invertible b is at least $\|a^{-1}\|^{-1}$.

   Proof: $\|x^{-1} - y^{-1}\| \leqslant \|x^{-1}\|\|y - x\|\|y^{-1}\|$

   $$(a+h)^{-1} = (1 + a^{-1}h)^{-1}a^{-1} = a^{-1} - a^{-1}ha^{-1} + \cdots$$

   when $\|h\| \leqslant \|a^{-1}\|^{-1}$.

   Example: Invertible Matrices $\mathrm{GL}(R^n) := \{ A \in M_n(R) : \det A \text{ invertible} \}$.

8. The closure of a proper left-ideal is proper. So maximal ideals are closed.

2.1.1 Valued Rings

An **absolute value** is a norm with $|ax| = |a||x|$.
Then $|1| = 1$ (unless $R = 0$); $|-x| = |x|$ is redundant; $\left|x^{-1}\right| = 1/|x|$.

Examples

- *Euclidean value*: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ with $|x|_\infty := \sqrt{x^* x}$.
- Discrete rings with $|x|_0 := \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$ 1.
- $\mathbb{Q}(\sqrt{2})$ with $|a + b\sqrt{2}| := \mathrm{abs}(a - b\sqrt{2})$.
- $R[x]$ with $|p| := 2^{\deg(p)}$.
- $R[[x]]$ with $\left|a_n x^n + a_{n+1}x^{n+1} + \cdots\right| := \frac{1}{2^n}$.

1. There are no proper topological divisors of zero since $|ax_n| \to 0 \;\Rightarrow\; a = 0$. Hence a commutative valued ring extends to its field of fractions.

   The topological nilpotents are B_1.

   The *unit group* is $\{ x : |x| = 1 \}$.

2. The *radius of convergence* of a power series $\sum_n a_n x^n$ is $R = 1/\limsup_n \|a_n\|^{1/n}$, i.e., it diverges if $\rho(x) > R$.

3. If there are $|ab| = 1$ with $|a| > 1$, then absolute values are equivalent $\Leftrightarrow \|x\| = |x|^\alpha$ $(\exists \alpha > 0)$.

   Proof: If non-discrete then $\|x\| < 1 \Leftrightarrow x^n \to 0 \Leftrightarrow |x| < 1$; similarly $\|x\| > 1 \Leftrightarrow |x| > 1$. $\|x\| = \|a\|^{s(x)} \Leftrightarrow |x| = |a|^{s(x)}$ since $\|x\| \lesseqgtr \|a\|^{m/n} \Leftrightarrow \|x^n b^m\| \lesseqgtr \|ab\|^m = 1 \Leftrightarrow |x^n b^m| \lesseqgtr 1 \Leftrightarrow |x| \lesseqgtr |a|^{m/n}$. But $\|a\| = |a|^\alpha$, so $\|x\| = \|a\|^{s(x)} = |a|^{s(x)\alpha} = |x|^\alpha$.

4. $1 \leqslant \sup_{|x|\leqslant 1} |1+x| \leqslant \sup_{x,y\neq 0} \frac{|x+y|}{|x| \vee |y|} \leqslant 2.$

   $\sup_{|x|\leqslant 1} |1+x| = 1$ iff $|n| \leqslant 1$ for all $n \in \mathbb{N}$ (non-Archimedean).

   Proof: If $|x| \leqslant 1$ then $|1+x|^n = |(1+x)^n| = |1+nx+\cdots+x^n| \leqslant (n+1)$, so $|1+x| \leqslant (1+n)^{1/n} \to 1$. (Similarly, for commutative rings or division rings, an absolute value is an ultrametric $\Leftrightarrow$ non-Archimedean $\Leftrightarrow$ $\mathbb{N}$ is bounded.)

Proposition 5

Ostrowski

The absolute values on $\mathbb{Z}$ are the discrete, the Euclidean, and the p-adic values (for each prime p).

They extend to $\mathbb{Q}$ by $|a/b| = |a| / |b|$.

PROOF: One type is discrete. Another is the Euclidean value $|n|$. Let $1 < \|a\|(\leqslant |a|)$. For any $|x| > 1$, $|a|^n = r_0 + r_1 |x| + \cdots + r_m |x|^m \leqslant (m+1) |x|^{m+1}$ (base-$|x|$ expansion) with $0 \leqslant r_i < |x|$. Then $\|a\|^n \leqslant (m+1)\|x\|^{m+1}$; but $\|a\|^n$ grows faster if $\|x\| \leqslant 1$; so $\|x\| > 1$. In fact, $\|a\| \leqslant (m+1)^{\frac{1}{n}} \|x\|^{(m+1)/n} \to \|x\|^{\log_{|x|}|a|}$, so $\frac{\log \|a\|}{\log |a|} \leqslant \frac{\log \|x\|}{\log |x|}$. Hence $\frac{\log \|x\|}{\log |x|} = \alpha < 1$ by symmetry of x and a, i.e., $\|x\| = |x|^{\alpha}$.

Otherwise for all n, $\|n\| \leqslant 1$ and there is $\|a\| < 1$. There must be a prime factor of a with $\|p\| < 1$. If q is another prime with $\|q\| < 1$ then $\exists a_n, b_n \in \mathbb{Z}$, $a_n p^n + b_n q^n = 1$, so $1 = \|1\| \leqslant \|p\|^n + \|q\|^n \to 0$ as $n \to \infty$, a contradiction. So p is unique with $\|p\| < 1$, the rest satisfy $\|q\| = 1$. Hence $\|n\| = \|p^k q^r \ldots\| = \|p\|^k = |n|_p^{\alpha}$ where $\|p\| = p^{-\alpha}$ for some $\alpha > 0$.

□

2.1.2 Non-Archimedean Valued Division Rings

Examples:

- $\mathbb{Q}$ with the p-adic value.
- Any division ring with finite characteristic, i.e., $n1 = 0$ (since $\mathbb{N}$ is then bounded).
- *p-adic value*: PID with $|p|_p := \frac{1}{2}$ for some fixed prime p, and $|q|_p := 1$ for all other primes, extended to its field of fractions by $|x/y| := |x| / |y|$. In particular for $\mathbb{Z}$ and $\mathbb{Q}[x]$. It is an ultrametric $|x+y| \leqslant |x| \vee |y|$.
- $F(x)$ with $|\sum_n a_n x^n| := \max_n |a_n|$.

1. Recall that an ultrametric space is 0-D, hence totally disconnected (balls are clopen).

2. (a) If $|a| > |x_n|$ then $|a + \sum_n x_n| = |a|$.
   Proof: $|x| < |a| \leqslant |a + x| \vee |x| = |a + x|$, so $|a| = |a + x_1| = |a + x_1 + x_2| = \cdots$.

   (b) $\sum_n a_n$ is Cauchy $\Leftrightarrow$ $a_n \to 0$ (since $|a_n + \cdots + a_m| \leqslant |a_n| \vee \cdots \vee |a_m| \to 0$.

3. F contains the *valuation ring* $B_1[0] := \{ x : |x| \leqslant 1 \}$, which is a local ring with its unique (clopen) maximal ideal B_1. Its ideals contain whole balls $\{ x : |x| \leqslant r \}$. When commutative, $B_1[0]/B_1$ is called the *residue class field* of F.

   Proof: If $x \in I$ and $|y| \leqslant |x|$, then $\left|x^{-1}y\right| \leqslant 1$ so $y = xv \in I$. In particular, $I \subseteq B_1$ else $1 \in I$.

4. The valuation ring is Noetherian $\Leftrightarrow$ ideals are principal $\Leftrightarrow$ values of elements are discrete.

   Proof: If R is Noetherian, then $I = \langle a_1, \ldots, a_n \rangle$ with a_1 of maximum value; then $a_i \in \langle a_1 \rangle$. If values are α^n, then an ascending chain of ideals would have a maximum value α^m.

5. The only valuations of a PID (and its field of fractions) which are bounded by 1 are the discrete and the p-adic ones.

   Proof: The values on $\mathbb{N}$ are bounded by 1, so $R \cap B_1$ is an ideal in R, hence $\langle p \rangle$ with p prime; as $|x| < 1 \Leftrightarrow |x|_p < 1$, the two values are equivalent.

6. The non-Archimedean valuations of $F[t]$ which become discrete on F are the discrete, $2^{\deg(q)}$, and the p-adic ones (for some irreducible polynomial p). For example, $\mathbb{F}_{q^n}[t]$.

   Proof: If neither discrete nor p-adic, then $|t| = \alpha > 1$, so $|at^n| = |a|\,|t|^n = \alpha^n$ and $|a_n t^n + \cdots| = \alpha^n$.

7. An algebraic field extension of F has a non-Archimedean value that extends that of F.

2.2 Locally Compact Rings

Closed sub-rings, quotients and finite products are again so.

1. The connected component of 0 is the intersection of the clopen subrings.

   Proof: Consider G/C_0; there is a base of compact open subgroups K; as K is bounded, $\exists U \subseteq K$, $UK \subseteq K$, so $U \cdots U \subseteq UUK \subseteq UK \subseteq K$, so the subring $[\![U]\!] \subseteq K$ is clopen hence compact.

2. (Kaplansky) The Jacobson radical is closed.

3. Recall from topological groups that locally compact normed rings are complete.

4. Connected locally compact rings are of the type $R \times P$ where R is a finite-dimensional algebra over $\mathbb{R}$ and P is a compact abelian group.

2.2.1 Locally Compact Division Rings

also require that inversion $x \mapsto x^{-1}$ is continuous, so it is a topological group for multiplication. A locally compact field is called a *local field*.

1. There is a compatible valuation $|a| = \delta(a) = \frac{\mu(aE)}{\mu(E)}$ from the automorphism $x \mapsto ax$.

2. $B_r[a] := \{ x : |x - a| \leqslant r \}$ is compact.

   Proof: $\forall r, \exists s,\ B_s[0] \subseteq K \subseteq B_r[0]$, pick $x_0 \in B_s[0]$, $y_0 \in B_r[0]$; then the homeomorphism $x \mapsto y_0 x_0^{-1} x$ takes the compact set $B_s[0]$ to $B_r[0]$.

3. The only connected locally compact division rings are 0, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

   Proof: If $0 \neq 1$ then $\mathbb{Q} \subseteq R$, so it has one of three types of values. The discrete and p-adic values give totally disconnected completions of $\mathbb{Q}$, hence $\mathbb{R}$ is embedded in R. As a locally compact vector space on $\mathbb{R}$ it is finite dimensional. From Frobenius' theorem, it is one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (see Topological Vector Spaces).

4. (van Dantzig, Hasse, Jacobson) The disconnected ones are discrete or (ultrametric) division ring extensions of $\mathbb{Q}_p$ or $\mathbb{F}_{p^n}((x))$.

5. $GL(F^n)$ is a locally compact group if F is a locally compact field.

6. The roots of unity of a locally compact non-Archimedean field form a finite group.

2.3 Compact Rings

1. Compact rings are totally disconnected, since the only connected compact ring is 0.

2. Topological quasi-nilpotents are quasi-nilpotents.

   Proof: $(1 - x)R$ is compact and $1 \in \overline{(1 - x)R}$, so $1 = (1 - x)(1 - y)$.

3. The Jacobson radical is topologically nilpotent. They are a J-adic ring, i.e., there is a base of clopen ideals J, so $R \subsetneq \lim_{\leftarrow} R/J$ (with R/J discrete rings).

   Proof: The open compact subgroups K form a base; $\exists U \subseteq K$, $UR \subseteq K$, so $\exists V \subseteq U, VR \subseteq U$ and $RVR \subseteq K$, so $\langle V \rangle$ is an open ideal in K.

4. Compact semi-simple rings are products of finite simple rings.

2.4 Examples

1. $\mathbb{Q}/\mathbb{Z} \cong \sum_p \mathbb{Z}(p^\infty)$: totally disconnected, abelian, not locally compact; every element has finite order, the torsion subgroup of $\mathbb{S}$.

2. *p-adic Numbers* $\mathbb{Q}_p$: the Cauchy-completion of $\mathbb{Q}$ with the p-value $|m/n| := 1/p^k$ where $m/n = p^k r/s$, an ultrametric; uncountable; topologically $2^{\mathbb{N}} \smallsetminus 0$, so perfect, locally compact T_2, totally disconnected; has characteristic 0; it is a (non-ordered) field; ($\cong \mathbb{Z}(p)[\frac{1}{p}]$); has many distinct algebraic closures. For any $x \in \mathbb{Q} \smallsetminus 0$, write $x = \frac{m}{n} p^r$, so can find $na = m \pmod p$; take $x_1 := x - ap^r$ and continue to get a sequence of a's with $x = \sum_{n=r}^{\infty} a_n p^n$. $\mathbb{Q}_p$ is its own dual as a topological group. Equivalently, it is the field of quotients of the inverse limit of $\mathbb{Z}/\langle p^n \rangle$.

   For $x \in \mathbb{Q}$, $x \neq 0$, then $\prod_p |x|_p = 1/|x|$ (over primes), i.e., $\prod_{val} |x|_v = 1$ (where $v = 1, p, \infty$ are the discrete, p-adic, and Euclidean valuations.)

3. *p-adic Integers* $\mathbb{Z}(p)$: the Cauchy-completion of $\mathbb{Z}$ with the p-adic topology generated by the ideal $[\![p]\!]$; topologically $2^{\mathbb{N}}$; it is the closed unit ball of $\mathbb{Q}_p$, a compact subring of $\mathbb{Q}_p$, which contains $\mathbb{Z}$, not locally connected. $p\mathbb{Z}(p)$ is a maximal ideal in $\mathbb{Z}(p)$, and $\mathbb{Z}(p)/p\mathbb{Z}(p) \cong \mathbb{Z}_p$. $\mathbb{Z}(p)$ contains a copy of $\mathbb{Z}$ as $(m + p^n)_{n \in \mathbb{N}}$, so it is a compactification of it.

4. $\mathbb{Z}[1/p]$ with the p-adic metric: locally compact, not locally connected; $\mathbb{Z}[\frac{1}{p}] \subseteq \mathbb{R} \times \mathbb{Q}_p$.

5. *Real Numbers* $\mathbb{R}$: completion of $\mathbb{Q}$.

   $$\min(a, b, \ldots) \leqslant (\frac{\alpha}{a} + \frac{\beta}{b} + \ldots)^{-1} \leqslant a^\alpha b^\beta \ldots \leqslant \alpha a + \beta b + \ldots \leqslant (\alpha a^p + \beta b^p + \ldots)^{1/p} \leqslant \max(a, b, \ldots)$$

   (the middle means are called harmonic, geometric, arithmetic and p-root mean square means) (proofs: Young - geometric $\leqslant$ arithmetic since $x^\alpha - 1 \leqslant \alpha(x-1)$ so take $x = a/b$, equivalently written as $(ab)^r/r \leqslant a^p/p + b^q/q$ where $1/r = 1/p + 1/q$; includes Cauchy's inequality $ab \leqslant (a^2 + b^2)/2$)

   Every real number has a decimal expansion (essentially unique); every real number has a continued fraction expansion $a_0 + 1/(a_1 + \ldots)$ with positive a_i, this expansion terminates for rational numbers and recurs for quadratic surds.

6. *Complex Numbers* $\mathbb{C} := \mathbb{R} \oplus i\mathbb{R}$ with $i^2 = -1$, conjugate $(a + ib)^* = a - ib$ (i.e., $j^* = -j$), value $|z|^2 = z^* z = zz^*$.

   Contains the topological sub-field of definable numbers (those complex numbers which are characterized by some statement i.e., $y = x \Leftrightarrow \Phi(y)$), its sub-field of computable numbers (which can be generated by some algorithm), which contains the algebraic numbers; they are all countable and algebraically closed.

7. *Quaternions* $\mathbb{H} := \mathbb{C} \oplus j\mathbb{C}$ with $x+yj = (a+bi)+(c+di)j = a+bi+cj+dk$ where $k = ij$, $(x+yj)^* = x^* - jy^*$ (i.e., $j^* = -j$), $yj = -jy$ ($ji = -k$); $j^2 = -1$; it has a value $|z|^2 = z^*z = zz^*$, $(zw)^* = w^*z^*$; a division ring with center $\mathbb{R}$.

8. *Octonions* $\mathbb{O} := \mathbb{H} \oplus k\mathbb{H}$ with $(a+bk)(c+dk) = (ac - d^*b) + (da + bc^*)k$ ($a(bk) = (ba)k, (ak)b = (ab^*)k, (ak)(bk) = -b^*a$) and $(a+kb)^* = a^* - kb$; $(ijk)^* = -ijk$; not associative but alternative (i.e., associative on any two elements), has inverses.

9. Sedonians: $\mathbb{S} := \mathbb{O} \oplus e\mathbb{O}$ (using $a(eb) = e(a^*b), (ea)b = e(ba), (ea)(eb) = -ba^*, e^* = -e$); it is power-associative ($x^m x^n = x^{m+n}$) only; has zero-divisors.

References:

A.V. Arkhangel'skii, Mikhail Tkachenko, *Topological Groups and Related Structures.*

T.K. Subrahmonian Moothathu *Topological Groups*

J. Gleason *Haar Measures*

C. Barwick *Exercises on LCA Groups: An Invitation to Harmonic Analysis*

K.H. Hoffmann *Introduction to Topological Groups*

P. Clarke Lecture Notes on Valuation Theory