Topological Spaces

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9 August 2014

A review on filters of subsets and nets is found as an appendix.

1 Convergence Spaces

A filter of subsets captures the idea of a refinement process, or gathering of information; The filter sets may represent imperfect knowledge or a fuzzy 'measurement'. Which set of points remains in the 'limit' of taking smaller sets of the filter? It need not only consist of the points common to all the filter sets (which may be empty), but may include other 'cluster' points.

A **cluster space** is a set X with a **cluster** function mapping proper filters $\mathcal{F}$ of X to subsets of X, such that

$$\mathrm{cl}(\mathcal{F} \cap \mathcal{G}) = \mathrm{cl}(\mathcal{F}) \cup \mathrm{cl}(\mathcal{G}),$$
$$\bigcap \mathcal{F} \subseteq \mathrm{cl}(\mathcal{F})$$

Some notation: $\mathcal{F} \dashrightarrow A$ means $A = \mathrm{cl}(\mathcal{F})$; a *cluster* point of $\mathcal{F}$ is an element $x \in \mathrm{cl}(\mathcal{F})$, here denoted by $\mathcal{F} \dashrightarrow x$. The cluster set of the improper filter 2^X can be taken to be $\varnothing$. The set of cluster points of a subset is $\mathrm{cl}(A) := \mathrm{cl}\mathfrak{F}(A)$.

The morphisms, called **continuous** maps, preserve clustering,

$$\mathcal{F} \dashrightarrow x \;\Rightarrow\; f(\mathcal{F}) \dashrightarrow f(x)$$

i.e., $f\mathrm{cl}(\mathcal{F}) \subseteq \mathrm{cl}f(\mathcal{F})$ and $\mathrm{cl}f^{-1}\mathcal{F} \subseteq f^{-1}\mathrm{cl}(\mathcal{F})$, e.g. constant maps. An isomorphism is called a *homeomorphism*, and the set of morphisms $X \to Y$ is denoted $C(X,Y)$.

Examples: A *trivial* space has $\mathrm{cl}(\mathcal{F}) := X$; a *discrete* space has $\mathrm{cl}(\mathcal{F}) := \bigcap \mathcal{F}$, so that $\mathcal{F} \dashrightarrow x \Leftrightarrow \mathcal{F} \subseteq \mathfrak{F}(x)$. The standard clustering of $\mathbb{N}$ is taken to be discrete. A finite cluster space is determined by $\mathrm{cl}(x)$; *Sierpinski* space is $\mathbf{2} := \{0,1\}$ with $\mathfrak{F}(0) \dashrightarrow \{0\}$, $\mathfrak{F}(1) \dashrightarrow \{0,1\}$; any finite (pre-)ordered space with $\mathrm{cl}(x) = \downarrow x$.

A cluster structure$_1$ on X is *finer* than another$_2$ (on X) when $\mathrm{cl}_1(\mathcal{F}) \subseteq \mathrm{cl}_2(\mathcal{F})$, i.e., $\mathcal{F} \dashrightarrow_1 x \;\Rightarrow\; \mathcal{F} \dashrightarrow_2 x$.

A net *clusters* at x when its filter does,

$$x_i \dashrightarrow x \Leftrightarrow \mathfrak{F}(x_i) \dashrightarrow x$$

Two points x and y are indistinguishable when $\mathcal{F} \dashrightarrow x \Leftrightarrow \mathcal{F} \dashrightarrow y$ for all $\mathcal{F}$. By identifying indistinguishable points, one can take all points to be distinguishable (called the T_0 axiom, henceforth assumed),

$$x \neq y \;\Rightarrow\; \exists \mathcal{F},\ \mathcal{F} \dashrightarrow x \text{ XOR } \mathcal{F} \dashrightarrow y.$$

1. $\mathcal{F} \subseteq \mathcal{G} \;\Rightarrow\; \mathrm{cl}(\mathcal{G}) \subseteq \mathrm{cl}(\mathcal{F})$, i.e., $\mathcal{F} \subseteq \mathcal{G} \dashrightarrow x \;\Rightarrow\; \mathcal{F} \dashrightarrow x$.
   Hence, $\bigcup_i \mathrm{cl}(\mathcal{F}_i) \subseteq \mathrm{cl}(\bigcap_i \mathcal{F}_i)$; in particular, for $\mathfrak{F}(A) \subseteq \mathcal{F} \subseteq \mathcal{M}$ maximal,
   $$\bigcup_{\substack{\mathcal{F}\subseteq\mathcal{M}\\ \mathcal{M}\text{ maximal}}} \mathrm{cl}(\mathcal{M}) \subseteq \mathrm{cl}(\mathcal{F}) \subseteq \bigcap_{A\in\mathcal{F}} \mathrm{cl}(A).$$

2. An equivalent formulation of the axioms is
   $$\begin{aligned}&\mathfrak{F}(x) \dashrightarrow x\\ &\mathcal{F}\cap\mathcal{G} \dashrightarrow x \Leftrightarrow \mathcal{F} \dashrightarrow x \text{ OR } \mathcal{G} \dashrightarrow x\end{aligned}$$
   Proof: If $x \in \bigcap \mathcal{F}$ then $\mathcal{F} \subseteq \mathfrak{F}(x)$, so $x \in \mathrm{cl}\mathfrak{F}(x) \subseteq \mathrm{cl}(\mathcal{F})$.

3. Clustering is a 'pre-closure' operation
   (a) $A \subseteq \mathrm{cl}(A)$; $A \subseteq B \;\Rightarrow\; \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,
   In particular, $\mathrm{cl}(\bigcap_i A_i) \subseteq \bigcap_i \mathrm{cl}(A_i)$, $\bigcup_i \mathrm{cl}(A_i) \subseteq \mathrm{cl}(\bigcup_i A_i)$.
   (b) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
   (c) $\mathrm{cl}(\varnothing) = \varnothing$, $\mathrm{cl}(X) = X$.

   Proof: $A = \bigcap \mathfrak{F}(A) \subseteq \mathrm{cl}\mathfrak{F}(A) = \mathrm{cl}(A)$. $A \subseteq B \;\Rightarrow\; \mathfrak{F}(B) \subseteq \mathfrak{F}(A)$. $\mathfrak{F}(A\cup B) = \mathfrak{F}(A) \cap \mathfrak{F}(B)$. $\mathrm{cl}(\bigcap_i A_i) \subseteq \mathrm{cl}(A_i)$.

4. For any set, one can form the increasing sequence $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{clcl}(A) \subseteq \cdots$, which only terminates when it reaches a **closed** set, i.e., when $\mathrm{cl}(F) = F$, i.e., $\mathfrak{F}(F) \dashrightarrow F$.
   (a) The intersection of closed sets is closed; the finite union of closed sets is closed.
   Proof: $\bigcap_i F_i \subseteq \mathrm{cl}(\bigcap_i F_i) \subseteq \bigcap_i \mathrm{cl}(F_i)$; $\mathrm{cl}(F \cup G) = \mathrm{cl}(F) \cup \mathrm{cl}(G) = F \cup G$.
   (b) There is a smallest closed set containing A, called its *closure* $\bar{A}$. Then,
   $$A \subseteq \bar{A} = \bar{\bar{A}}, \quad \overline{A\cup B} = \bar{A} \cup \bar{B}$$
   Hence $A \subseteq B \;\Rightarrow\; \bar{A} \subseteq \bar{B}$ and $\overline{\bigcap_i A_i} \subseteq \bigcap_i \bar{A}_i$.
   Proof: $\bar{A} := \bigcap\{\, F : \text{closed}, A \subseteq F \,\}$. If $A \cup B \subseteq F$ closed, then $\bar{A}, \bar{B} \subseteq F$ so $\bar{A} \cup \bar{B} \subseteq F$ is the smallest closed set containing $A \cup B$.
   (c) $\varnothing$ and X are closed.

5. The dual concept of clustering is that of "internal" points,

$$\operatorname{int}(A) := \operatorname{cl}(A^{\mathsf{c}})^{\mathsf{c}} = \{\, x : \mathcal{F}(A^{\mathsf{c}}) \not\dashrightarrow x \,\}$$

An internal point of A cannot be "reached" from outside A; we say that A is a *neighborhood* of x when $x \in \operatorname{int}(A)$. It satisfies properties dual to clustering. One can form the decreasing sequence $A \supseteq \operatorname{int}(A) \supseteq \operatorname{int}\operatorname{int}(A) \supseteq \cdots$, which terminates when it reaches an **open** set, $\operatorname{int}(U) = U$.

   (a) A set is open iff its complement is closed. Hence the open sets form a *topology* $\mathcal{T}$: the union, and the finite intersection, of open sets are open.

   (b) The *interior* $A^\circ := (\overline{A^{\mathsf{c}}})^{\mathsf{c}}$ is the largest open set inside A.

$$A^{\circ\circ} = A^\circ \subseteq A, \qquad A \subseteq B \;\Rightarrow\; A^\circ \subseteq B^\circ,$$
$$(A \cap B)^\circ = A^\circ \cap B^\circ, \qquad \bigcup_i A_i^\circ \subseteq (\bigcup_i A_i)^\circ$$

   (c) The *neighborhood filter* of a point is $\mathcal{N}_x := \{\, A : x \in \operatorname{int}(A) \,\}$, which contains $\mathcal{T}_x := \{\, A \in \mathcal{T} : x \in A \,\}$, i.e.,

$$\mathcal{T}_x \subseteq \mathcal{N}_x \subseteq \mathfrak{F}(x)$$

6. $\operatorname{cl}(A) = \{\, x \in X : x \in \operatorname{int}(B) \;\Rightarrow\; A \cap B \neq \varnothing \,\}$
   $\operatorname{int}(A) = \{\, x \in X : x \in \operatorname{cl}(B) \;\Rightarrow\; A \cap B \neq \varnothing \,\}$
   $\mathcal{F} \dashrightarrow x \;\Rightarrow\; \mathcal{F} \vee \mathcal{N}_x$ is proper

   Proof: If $A \cap B = \varnothing$ then $\mathfrak{F}(B^{\mathsf{c}}) \subseteq \mathfrak{F}(A)$ as well as $\mathfrak{F}(B) \subseteq \mathfrak{F}(A^{\mathsf{c}})$. If $x \in \operatorname{int}(B) \;\Rightarrow\; A \cap B \neq \varnothing$, then $A \cap A^{\mathsf{c}} = \varnothing \;\Rightarrow\; \mathfrak{F}(A) \dashrightarrow x$. Hence $A \in \mathcal{F} \dashrightarrow x \in \operatorname{int}(B) \;\Rightarrow\; A \cap B \neq \varnothing$.

7. For any set A, the points of X partition into the interior A°, the *exterior* $\bar{A}^{\mathsf{c}} = A^{\mathsf{c}\circ}$, and the remaining *boundary* ∂A. Then $\bar{A} = A^\circ \cup \partial A$.

$$\partial(A^{\mathsf{c}}) = \partial A, \qquad \partial\partial A \subseteq \partial A,$$
$$\partial(A \cup B) \subseteq \partial A \cup \partial B, \quad \partial(A \cap B) \subseteq \partial A \cup \partial B$$
$$A \text{ is closed } \;\Leftrightarrow\; \partial A \subseteq A,$$
$$A \text{ is open } \;\Leftrightarrow\; \partial A \subseteq A^{\mathsf{c}}.$$

   Proof: $A^\circ \cap \bar{A}^{\mathsf{c}} = (\overline{A^{\mathsf{c}}} \cup \bar{A})^{\mathsf{c}} = (\overline{A^{\mathsf{c}} \cup A})^{\mathsf{c}} = \varnothing$. $\partial\partial A \subseteq \overline{\partial A} \cap \overline{\partial A^{\mathsf{c}}} \subseteq \bar{\bar{A}} \cap \overline{\overline{A^{\mathsf{c}}}} = \partial A$. After removing the interior and exterior of $A \cup B$, what remains is part of $(\partial A \smallsetminus B^\circ) \cup (\partial B \smallsetminus A^\circ)$. $\partial(A \cap B) = \partial(A^{\mathsf{c}} \cup B^{\mathsf{c}}) \subseteq \partial A^{\mathsf{c}} \cup \partial B^{\mathsf{c}}$.

8. For the topology $\mathcal{T}$ of open sets, $A \cap B = \varnothing \;\Leftrightarrow\; A \subseteq \bar{B}^{\mathsf{c}} \;\Leftrightarrow\; B \subseteq \bar{A}^{\mathsf{c}}$. So the exterior is a pseudo-complement (see Ordered Spaces) and $A \mapsto \overline{\bar{A}^{\mathsf{c}}}^{\mathsf{c}} = \bar{A}^\circ$ is a 'closure' map, with the open sets satisfying $A = \bar{A}^\circ$ called *regular open* (equivalently, an open subset that is the exterior of an open set), e.g. the clopen subsets; they form a Boolean lattice, with $A \vee B = \overline{(A \cup B)}^\circ$ and $A' = \bar{A}^{\mathsf{c}}$.

9. For continuous functions f,

$$f\bar{A} \subseteq \overline{fA}, \quad \overline{f^{-1}A} \subseteq f^{-1}\bar{A}, \qquad f^{-1}A^{\circ} \subseteq (f^{-1}A)^{\circ}$$
$$A \text{ closed } \Rightarrow f^{-1}A \text{ closed}, \quad A \text{ open } \Rightarrow f^{-1}A \text{ open}$$
$$\mathcal{T}_{f(x)} \subseteq f(\mathcal{T}_x), \qquad \mathcal{N}_{f(x)} \subseteq f(\mathcal{N}_x)$$

   Proof: If A is closed, $\text{cl}(f^{-1}A) \subseteq f^{-1}\text{cl}(A) = f^{-1}A$. $f^{-1}A \subseteq f^{-1}\bar{A}$ closed, so $\overline{f^{-1}A} \subseteq f^{-1}\bar{A}$. $\bar{A} \subseteq \overline{f^{-1}fA} \subseteq f^{-1}\overline{fA}$, so $f\bar{A} \subseteq \overline{fA}$. $f^{-1}A^{\circ} = f^{-1}\overline{A^{\mathsf{c}}}^{\mathsf{c}} = (f^{-1}\overline{A^{\mathsf{c}}})^{\mathsf{c}} \subseteq \overline{f^{-1}A^{\mathsf{c}}}^{\mathsf{c}} = \overline{(f^{-1}A)^{\mathsf{c}}}^{\mathsf{c}} = (f^{-1}A)^{\circ}$; so if A is open, $f^{-1}A \subseteq (f^{-1}A)^{\circ}$. In fact, these statements are equivalent.

   A function is *open* when $f\mathcal{T}(X) \subseteq \mathcal{T}(Y)$.

10. A subset is *dense* in X when $\bar{A} = X$ ($\Leftrightarrow \forall V \neq \varnothing$ open$, A \cap V \neq \varnothing$).

    A subset is *nowhere dense* in X when A is not dense in any neighborhood, i.e., $\bar{A}^{\circ} = \varnothing$. Subsets, closure, and finite unions are also nowhere dense (since if $C := A \cup B$, then $\bar{C} \cap \bar{A}^{\mathsf{c}} \subseteq \bar{B}$, so $\bar{C}^{\circ} \cap \bar{A}^{\mathsf{c}} \subseteq \bar{B}^{\circ} = \varnothing$ and $\bar{C}^{\circ} \subseteq \bar{A}$, $\bar{C}^{\circ} \subseteq \bar{A}^{\circ} = \varnothing$).

    More generally, a subset is *meagre* when it is the countable union of nowhere dense sets; subsets and countable unions are meagre.

11. For a dense subset A, U is regular open iff $U = \overline{(A \cap U)}^{\circ}$.

    Proof: $A \cap U \cap \overline{(A \cap U)}^{\circ} = \varnothing$, so $U \subseteq \overline{A \cap U}$.

Convergence

As a filter is refined, its cluster points are reduced. When x is a cluster point of $\mathcal{F}$ and *all* its refinements, then we say that $\mathcal{F}$ **converges** to x,

$$\mathcal{F} \to x \Leftrightarrow \forall \mathcal{G} \text{ proper } (\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{G} \dashrightarrow x)$$

Convergence is obviously stronger than clustering. For example, the sequence $0, 1, 0, 2, 0, 3, \ldots$ in the discrete space $\mathbb{N}$, with filter generated by the subsets $\{0, n, n+1, \ldots\}_{n \in \mathbb{N}}$, clusters at 0, but does not converge to 0 because of its "tail" at infinity; it contains the refined filter generated by $\{n, n+1, \ldots\}_{n \in \mathbb{N}}$ that does not cluster at 0. Similarly, $\mathfrak{F}(A)$ does not normally converge to any $x \in A$.

Convergence has the following characteristic properties, that can be taken as its axioms

$$\mathfrak{F}(x) \to x,$$
$$\mathcal{G} \supseteq \mathcal{F} \to x \Rightarrow \mathcal{G} \to x$$
$$\mathcal{F} \to x \text{ AND } \mathcal{G} \to x \Rightarrow \mathcal{F} \cap \mathcal{G} \to x$$

(If $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{H}$ proper, then $\mathcal{F} \vee \mathcal{H}$ or $\mathcal{G} \vee \mathcal{H}$ is proper, so $\mathcal{H} \subseteq \mathcal{F} \vee \mathcal{H} \dashrightarrow x$)

A **convergence** space is a set X with a **limit** function on filters such that

$$\lim(\mathcal{F} \cap \mathcal{G}) = \lim(\mathcal{F}) \cap \lim(\mathcal{G}), \quad x \in \lim \mathfrak{F}(x).$$

We write $\mathcal{F} \to x$ when $x \in \lim(\mathcal{F})$, and x is then called a *limit* of $\mathcal{F}$. The improper filter can be taken to converge to all points. The morphisms preserve convergence of filters (or nets)

$$\mathcal{F} \to x \;\Rightarrow\; f(\mathcal{F}) \to f(x).$$

Examples: For a trivial space, $\mathcal{F} \to x$ for all x and $\mathcal{F}$; for a discrete space, the only convergent filters are $\mathfrak{F}(x) \to x$.

A net *converges* to x when its filter converges to x,

$$x_i \to x \Leftrightarrow \mathfrak{F}(x_i) \to x.$$

Constant nets converge, $x_i := x \to x$, (since $\mathfrak{F}(x_i) = \mathfrak{F}(x)$) and subnets of converging nets converge, (since $\mathfrak{F}(x_i) \subseteq \mathfrak{F}(x_{i_j})$).

1. $\mathcal{F} \subseteq \mathcal{G} \;\Rightarrow\; \lim(\mathcal{F}) \subseteq \lim(\mathcal{G})$.

2. A convergence space has a cluster function, where $\mathcal{F} \dashrightarrow x$ is defined by

$$\exists \mathcal{G} \text{ proper}, \mathcal{F} \subseteq \mathcal{G} \to x.$$

   The morphisms are the continuous functions.

   (a) Proper $\mathcal{F} \to x \;\Rightarrow\; \mathcal{F} \dashrightarrow x$ (iff for maximal filters).

   (b) $\mathrm{cl}(\mathcal{F}) = \bigcup_{\mathcal{F} \subseteq \mathcal{M}} \mathrm{cl}(\mathcal{M})$ (so convergence spaces are special cluster ones).

   (c) A sequence (resp. net) clusters at x if it has a sub-sequence that converges to x.

   Proof: If $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{H} \to x$ proper, then $\mathcal{F} \vee \mathcal{H}$ is proper (say), so $\mathcal{F} \subseteq \mathcal{F} \vee \mathcal{H} \to x$; conversely, $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{F} \subseteq \mathcal{H} \to x$.

3. (a) $\mathrm{cl}(A) = \{\, x \in X : \exists \mathcal{F} \to x \text{ proper}, A \in \mathcal{F} \,\} = \bigcup_{A \in \mathcal{F}} \lim \mathcal{F}$
   $\mathrm{cl}(x) = \{\, y \in X : \mathfrak{F}(x) \to y \,\} = \lim \mathfrak{F}(x)$

   (b) $\mathrm{int}(A) = \{\, x \in X : \forall \mathcal{F} \to x \text{ proper}, A \in \mathcal{F} \,\}$.

   (c) $\mathcal{N}_x = \bigcap_{\mathcal{F} \to x} \mathcal{F}$, so $\mathcal{N}_x \dashrightarrow x$.

   Proof: $A \in \mathcal{F} \to x \Leftrightarrow \mathfrak{F}(A) \subseteq \mathcal{F} \to x \Leftrightarrow \mathfrak{F}(A) \dashrightarrow x$. $x \notin \mathrm{cl}(A^{\mathsf{c}}) \Leftrightarrow$ for every proper $\mathcal{F} \to x$, $A^{\mathsf{c}} \notin \mathcal{F}$, hence $A^{\mathsf{c}} \in \mathcal{F} \vee \mathfrak{F}(A^{\mathsf{c}}) \to x$, so $\mathcal{F} \vee \mathfrak{F}(A^{\mathsf{c}})$ is improper, thus $A \in \mathcal{F}$. $A \in \mathcal{N}_x \Leftrightarrow (\mathcal{F} \to x \;\Rightarrow\; A \in \mathcal{F})$.

4. A is open iff $\mathcal{F} \to x \in A \;\Rightarrow\; A \in \mathcal{F}$ iff $x_i \to x \in A \;\Rightarrow\; x_i \to\!\!\!\!\in A$.

   A is closed iff $A \in \mathcal{F} \to x \;\Rightarrow\; x \in A$ iff $A \ni x_i \to x \;\Rightarrow\; x \in A$.

Given functions $\pi_i : Y \to X_i$, then one can make Y an *initial* convergence space by defining

$$\mathcal{F} \to y \Leftrightarrow \forall i, \pi_i(\mathcal{F}) \to \pi_i(y) \text{ in } X_i$$

then π_i are continuous by construction; and $f : X \to Y$ is continuous when $\pi_i \circ f$ are continuous for all i. Dually, given $\pi_i : X_i \to Y$, then can define a *final* convergence space by

$$\mathcal{F} \to y \Leftrightarrow \forall i, (\pi_i(x_i) = y \;\Rightarrow\; \pi_i^{-1}(\mathcal{F}) \to x_i).$$

As special cases of such convergence spaces:

Subspaces:	$Y \subseteq X$	induced by the embedding $Y \to X$, i.e., $\mathcal{F} \to y$ in Y when the generated filter $\uparrow\mathcal{F}$ in X converges to y. Then $\mathrm{cl}_Y(A) = \mathrm{cl}(A) \cap Y$, but $\mathrm{int}_Y(A)$ need not be $\mathrm{int}(A) \cap Y$ (e.g. $\mathbb{Q}, \mathbb{N}, 0$)
Images:	fX	$\mathcal{F} \to y \Leftrightarrow \forall x \in f^{-1}(y),\ f^{-1}(\mathcal{F}) \to x$.
Quotients:	$X/\sim$	induced by $\pi : x \mapsto [x]$, i.e., $\mathcal{F} \to [a]$ iff $x \sim a \;\Rightarrow\; \pi^{-1}(\mathcal{F}) \to x$.
Products:	$\prod_i X_i$	induced by the projections $\pi_{i_0} : (x_i) \mapsto x_{i_0}$; then $\mathcal{F} \to (x_i)_{i\in I} \Leftrightarrow \forall i, \pi_i(\mathcal{F}) \to x_i$; and $(\boldsymbol{x}_j)_{j\in J} \to \boldsymbol{y} \Leftrightarrow \forall i, (x_{ij})_{j\in J} \to x_i$ in X_i (pointwise convergence); $f : X \to \prod_i X_i$ is continuous iff $\pi_i \circ f$ is continuous on X_i. Also, $(A \times B)^\circ = A^\circ \times B^\circ$, $\overline{\prod_i A_i} = \prod_i \bar{A}_i$.
Coproducts:	$\coprod_i X_i$	induced by $\pi_i : X_i \to Y$.
Functions:	$X^Y = C(Y, X)$	$\mathcal{F} \to f$ in X^Y when for any filter in Y, $\mathcal{G} \to y \Rightarrow \epsilon(\mathcal{F} \times \mathcal{G}) \to f(y)$ in X (where $\epsilon(f, y) = f(y)$ is the evaluation map on $X^Y \times Y$). Then, $(X^Y)^Z \cong X^{Y\times Z}$ (i.e., $f(x)(y)$ continuous $\Leftrightarrow$ $f(x, y)$ continuous).

Cauchy Filters

The problem with $\mathcal{F} \to x$ is that in $X \smallsetminus \{x\}$, the filter $\mathcal{F}$ need not converge anymore. What distinguishes filters that truly diverge from those that could converge in a larger space? A set of proper filters that can possibly converge in a larger space are called **Cauchy filters** and satisfy the following axioms:

$$\mathscr{F}(x) \in \text{Cauchy},$$
$$\mathcal{G} \supseteq \mathcal{F} \in \text{Cauchy} \;\Rightarrow\; \mathcal{G} \in \text{Cauchy},$$
$$\mathcal{F}, \mathcal{G} \in \text{Cauchy AND } \mathcal{F} \vee \mathcal{G} \text{ proper} \;\Rightarrow\; \mathcal{F} \cap \mathcal{G} \in \text{Cauchy}$$

The morphisms are those maps that preserve Cauchiness: $\mathcal{F} \in \text{Cauchy} \Rightarrow f(\mathcal{F}) \in \text{Cauchy}$. Subspaces have induced Cauchy filters, $\mathcal{F}$ is Cauchy in Y $\Leftrightarrow$ $\uparrow\mathcal{F}$ is Cauchy in X; for products, $\mathcal{F}$ is Cauchy in $\prod_i X_i$ when $\pi_i\mathcal{F}$ are Cauchy.

Filters are said to be *asymptotic*, $\mathcal{F} \sim \mathcal{G}$ when $\mathcal{F} \cap \mathcal{G}$ is Cauchy. This equivalence relation on Cauchy filters is congruent with respect to subsets and intersections,

$$\mathcal{F} \sim \mathcal{G} \subseteq \mathcal{H} \Rightarrow \mathcal{F} \sim \mathcal{H},$$
$$\mathcal{F} \sim \mathcal{G} \Rightarrow \mathcal{F} \cap \mathcal{H} \sim \mathcal{G} \cap \mathcal{H},$$
$$\mathcal{G} \supseteq \mathcal{F} \in \text{Cauchy} \Rightarrow \mathcal{F} \sim \mathcal{G}.$$

A Cauchy structure induces a convergence structure: $\mathcal{F} \to x$, when $\mathcal{F} \sim \mathfrak{F}(x)$. Hence convergent filters are Cauchy since $\mathcal{F} \sim \mathfrak{F}(x) \sim \mathcal{F}$, so $\mathcal{F} \sim \mathcal{F}$. If $\mathcal{F} \sim \mathcal{G}$ then $\mathcal{F} \to x \Leftrightarrow \mathcal{G} \to x$. A Cauchy space is said to be *complete* when all its Cauchy filters converge. A closed subspace of a complete space is complete.

Cauchy maps preserve completeness and asymptoticity ($\mathcal{F} \sim \mathcal{G} \Rightarrow f(\mathcal{F}) \sim f(\mathcal{G})$), hence are continuous.

	Finite	**(2^{nd}) Countable Base**	**Separable 1^{st} Countable**	**Separable**	
Convergence Spaces $\to$					$C(X,Y)$
Topological Spaces $\mathcal{T}$	Ordered Graphs	$\mathbb{N}_{\leqslant}$	Partic.Pt.$\mathbb{R}$		Excl.Pt.$\mathbb{R}$
T_1 **Spaces** $\{x\}$ closed	/////	cofinite $\mathbb{N}$		cofinite $\mathbb{R}$	cocountable $\mathbb{R}$
Hausdorff Spaces x, y separated	/////	$\mathbb{Q}$ with $\{1/n\}$ closed	Deleted Radius	Strong Ultrafilter	Alexandroff plank
Tychonoff Spaces uniformity	/////	/////	Moore plane	$\mathbb{N}^{\mathbb{R}}$	$\mathbb{N}^{2^{\mathbb{R}}}$
Normal T_1 **Spaces** closed sets separated	/////	/////	Sorgenfrey line	Arens-Fort	Long line
Metric Spaces distance $d(x,y)$	/////	$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{N}^{\mathbb{N}}$, ℓ^2	/////	/////	discrete $\mathbb{R}$, ℓ^∞
Compact T_2 **Spaces**	Discrete Points	$2^{\mathbb{N}}$, $[0,1]^{\mathbb{N}}$	Helly	$2^{\mathbb{R}}$, $\beta\mathbb{N}$, $[0,1]^{\mathbb{R}}$	Fort on $\mathbb{R}$

2 Topological Spaces

A **topological space** is a set X with convergence defined by

$$\mathcal{F} \to x \Leftrightarrow \mathcal{S}_x \subseteq \mathcal{F}$$

where $\mathcal{S}_x = \{ U \in \mathcal{S} : x \in U \}$ and $\mathcal{S}$ is some collection of subsets that cover X. ($\mathcal{S}$ is called a *sub-base* of X, but need not be unique. The larger $\mathcal{S}$ is, the finer is the convergence. These basic sets represent the possible outcomes of fuzzy 'measurements'.)

Examples:

A trivial topological space has $\mathcal{S} = \{ X \}$ (not T_0); a discrete topological space has $\mathcal{S} = \{ \{ x \} : x \in X \}$.

$\mathbb{R}$ with $\mathcal{S}$ consisting of the intervals $]a, b[$.

Any ordered space with the sub-base of (complements of) the closed sets $\downarrow a = \{ x : x \leqslant a \}$. For example, $0 < 1$ gives $\mathbf{2}$; $\mathbb{N}_{\leqslant}$ has open sets $\{ n, n+1, \ldots \}$.

A set (of data) with $\mathcal{S}$ consisting of semi-computable subsets (form a topology).

Every cluster space gives rise to a weaker topological space generated by the open sets $\mathcal{S} := \mathcal{T}$; then $\mathcal{F} \dashrightarrow x \overset{\not\Leftarrow}{\Rightarrow} \mathcal{F} \dashrightarrow_{\mathcal{T}} x$ and $\mathcal{F} \to x \;\Rightarrow\; \mathcal{F} \to_{\mathcal{T}} x$.

For points to be distinguishable, $x \neq y \;\Rightarrow\; \mathcal{S}_x \neq \mathcal{S}_y$, i.e.,

$$\exists U \in \mathcal{S},\ x \in U \text{ XOR } y \in U.$$

Topological spaces satisfy stronger properties than convergence spaces:

1. If $\mathcal{F}_i \to x$ for all i then $\bigcap_i \mathcal{F}_i \to x$. Hence if every maximal filter that extends $\mathcal{F}$ converges to x then $\mathcal{F} \to x$. (Thus the convergence of maximal filters determines the convergence space.)

2. $\mathcal{N}_x \to x$ since $\mathcal{S} \subseteq \mathcal{T}$, so $\mathcal{S}_x \subseteq \mathcal{T}_x \subseteq \mathcal{N}_x \subseteq \mathfrak{F}(x)$.

   Proof: If $A \in \mathcal{S}$ and $\mathcal{F} \to x \in A$ then $A \in \mathcal{S}_x \subseteq \mathcal{F}$, so A is open.

3. $\mathcal{N}_x = \mathfrak{F}(\mathcal{S}_x) = \mathfrak{F}(\mathcal{T}_x) = \uparrow \mathcal{T}_x$, the smallest filter converging to x,

   $$\mathcal{F} \to x \Leftrightarrow \mathcal{N}_x \subseteq \mathcal{F} \Leftrightarrow \mathcal{T}_x \subseteq \mathcal{F}$$

   Proof: For $A \in \mathcal{N}_x$, $x \in \operatorname{int}(A)$, so $\mathcal{S}_x \subseteq \mathcal{F} \;\Rightarrow\; A \in \mathcal{F}$, hence $A \in \mathfrak{F}(\mathcal{S}_x)$.

4. $\mathcal{T}$ is the (smallest) topology generated by $\mathcal{S}$.

   Proof: If $x \in U \in \mathcal{T}$, then $U \in \mathcal{N}_x$, so there are $V_i \in \mathcal{S}_x$, $V_x := V_1 \cap \cdots \cap V_n \subseteq U$; $U = \bigcup_{x \in U} \{ x \} \subseteq \bigcup_{x \in U} V_x \subseteq U$, forcing $U = \bigcup_x V_x \in \mathcal{T}(\mathcal{S})$.

5. $\mathcal{F} \dashrightarrow x \Leftrightarrow \mathcal{F} \vee \mathcal{N}_x$ is proper. Hence $\mathcal{N}_x \dashrightarrow y \Leftrightarrow \mathcal{N}_y \dashrightarrow x$;

6. (a) $\operatorname{clcl}(A) = \operatorname{cl}(A)$, so

   $\bar{A} = \operatorname{cl}(A) = \{ x \in X : \forall U \in \mathcal{T}_x,\ A \cap U \neq \varnothing \}$,
   $A^\circ = \operatorname{int}(A) = \{ x \in X : \exists U \in \mathcal{T}_x,\ U \subseteq A \}$.

   For example, $\overline{\{ x \}} = \{ y \in X : \mathfrak{F}(x) \to y \}$. Closure in $Y \subseteq X$ is $\bar{A} \cap Y$.

(b) A point x is an interior, exterior, or boundary point of a set A when $A \in \mathcal{N}_x$, $A^{\mathsf{c}} \in \mathcal{N}_x$, or $A, A^{\mathsf{c}} \notin \mathcal{N}_x$ respectively.

Proof: $x \in \mathrm{int}(A) \Leftrightarrow A \in \mathcal{N}_x \Leftrightarrow A^\circ \in \mathcal{T}_x \Leftrightarrow x \in A^\circ$. Hence $\mathrm{cl}(A) = \mathrm{int}(A^{\mathsf{c}})^{\mathsf{c}} = A^{\mathsf{c}\circ\mathsf{c}} = \bar{A}$.

Note that $\mathcal{N}_x = \{ A \subseteq X : x \in A^\circ \}$, so $\circ$ or closure can be taken as fundamental.

The initial topological space induced by $\pi_i : Y \to X_i$ has the sub-base

$$\mathcal{S}_Y := \{ \pi_i^{-1} U : U \in \mathcal{S}_{X_i} \}.$$

The final topological space induced by $\pi_i : X_i \to Y$ is generated from

$$\mathcal{S}_Y := \{ U : \pi_i^{-1} U \in \mathcal{S}_{X_i} \}.$$

Some examples are

Subspace	$Y \subseteq X$	$\mathcal{S}_Y = \{ U \cap Y : U \in \mathcal{S} \}$,
Quotient	$X/\sim$	$\mathcal{S}_{X/\sim} = \{ U : \pi^{-1} U \in \mathcal{S} \}$,
Products	$X \times Y$	$\mathcal{S}_{X \times Y} = \{ U \times V : U \in \mathcal{S}_X, V \in \mathcal{S}_Y \}$,
Coproducts	$X \sqcup Y$	$\mathcal{S}_{X \sqcup Y} = \mathcal{S}_X \cup \mathcal{S}_Y$.

[The monomorphisms are the 1-1 continuous maps; the epimorphisms are the onto continuous maps; the initial object is $\varnothing$, the terminal object is 1; there are no zero morphisms; category is concrete, complete and co-complete, but not cartesian closed (so not a topos); a 2-morphism is a *homotopy* map.]

1. A net converges when

$$\begin{aligned} x_i \to x &\Leftrightarrow \forall U \in \mathcal{T}_x, x_i \to\!\!\in U, \\ &\Leftrightarrow \forall U \in \mathcal{T}_x, \exists j, \quad i \geqslant j \Rightarrow x_i \in U \end{aligned}$$

2. To write "$f(x) \to y$ as $x \to a$" is another way of stating $f\mathcal{N}_a \to y$, i.e.,

$$\forall V \in \mathcal{T}_y, \exists U \in \mathcal{T}_x, \quad fU \subseteq V.$$

   Thus $f : X \to Y$ is continuous when $\forall a \in X,\ f(x) \to f(a)$ as $x \to a$.

3. The open sets form a complete Heyting algebra ('locale') with $(A \to B) = (A^{\mathsf{c}} \cup B)^\circ$. Its "regular" elements are the clopen subsets, which thus form a Boolean subalgebra.

   Conversely, a complete Heyting algebra $\mathcal{T}$ gives a topological space (called its *spectrum*) $X := \{ x : \mathcal{T} \to \mathbf{2} \text{ frame morphisms} \}$ with topology $U_a := \{ x \in X : x(a) = 1 \}$ for each $a \in \mathcal{T}$; these operations are adjoints not inverses: $X \leqslant \mathrm{Spec}(\mathcal{T}) \Leftrightarrow \mathcal{T} \leqslant \mathrm{Top}(X)$.

4. Every topological space is embedded in some $\mathbf{2}^A$.

   Proof: Let $J := \text{Spec} \circ \text{Top} : X \to \mathbf{2}^{\mathcal{T}}$, $x \mapsto (1_U(x))_{U \in \mathcal{T}}$. Each 1_U is continuous since $1_U^{-1}(1) = U$, so J is continuous. J is open: let $U \in \mathcal{T}$, $y = J(x) \in JU$, so $x \in f_W^{-1}V \subseteq U$ for some open sets $V \subseteq \mathbf{2}$, $W \subseteq X$. Now $\pi_W(y) = f_W(x) \in V$; for any $z \in \pi_W^{-1}V \cap JX$, $z = J(x')$, so $f_W(x') = \pi_W(z) \in V$, so $x' \in f_W^{-1}V \subseteq U$, and $z \in JU$. Finally, J is 1-1 since $f_U(x) = f_U(y)$ for all $U \in \mathcal{T}$ contradicts T_0.

Connected Sets

A space may decompose as a sum of non-trivial (clopen) subspaces $X = A \sqcup B$; otherwise, when 'irreducible' (no open partition), it is said to be **connected**. However, a space may continue to decompose indefinitely without being the (infinite) sum of connected subspaces, e.g. $\mathbb{Q}$, $\mathbb{N}+1$.

1. X is connected iff the only clopen subsets are $X, \varnothing$ $\Leftrightarrow$ every non-trivial subset has a boundary $\Leftrightarrow$ the only continuous functions $f : X \to \{0,1\}$ are constant. (A clopen set cannot intersect a connected set properly.)

   For example, the connected subsets of $\mathbb{R}$ are the intervals (since every proper subset has a boundary point).

2. The connected subsets of a topological space form a connective space. So the union of overlapping connected subsets is connected. Connectedness is preserved by continuous functions (hence quotients). X decomposes into maximal connected subsets called *components*.

   Proof: If $\bigcup_i C_i = A \sqcup B$ then each $C_i \subseteq A$ or $C_i \subseteq B$; so $\bigcap_i C_i \subseteq A \cap B = \varnothing$. If $fX = A \sqcup B$ then $X = f^{-1}A \sqcup f^{-1}B$. The components are $C(x) := \bigcup\{ C : \text{connected}, x \in C \}$.

   Note that the components and their quotient do not determine the topological space, e.g. take $[0,1] \times \{ 1/n : n \in \mathbb{N} \}$ and slide $[0,1] \times \{0\}$ horizontally.

3. The addition of any number of boundary points to a connected set leaves it connected; so components are closed (but not necessarily open).

   Proof: Let $D \subseteq \partial C$; if $C \cup D = A \sqcup B$ then $C \subseteq A$ say; if $D \cap B \neq \varnothing$ then $C \cap B \neq \varnothing$, a contradiction.

4. Some components stick together into (closed) *quasi-components*: In any decomposition $X = A \sqcup B$, a quasi-component lies wholly in A or in B, i.e., $C_1 \subseteq A$ clopen $\Leftrightarrow C_2 \subseteq A$. Thus it equals $Q(C) = \bigcap\{ A : C \subseteq A \text{ clopen} \}$.

   An example of this is $\mathbb{N}$ with two points at infinity a, b; if $a \in A$ open, then A contains an infinite subset of $\mathbb{N}$, so if A is also closed $b \in A$.

5. The number of components is an invariant of the space. When there are a finite number, the components are clopen (so are equal to the quasi-components), and the space is equal to the sum of them. But for an infinite number of components this may be false (e.g. $\mathbb{Q} \not\cong \mathbb{N}$).

6. When $X = A \sqcup \{x\}$, then x is said to be an *isolated* point, i.e., $\{x\}$ is clopen (a non-isolated point is called a *limit point*). The number of isolated points, more precisely the supremum of the cardinality of I where $X = A \sqcup \coprod_{i \in I}\{x_i\}$, is called the *extent* of X. (Thus any subset with more points than the extent has a limit point.) A discrete space consists only of isolated points $\coprod_{x \in X}\{x\}$. At the other extreme, a space without isolated points is called *perfect*.

7. A *totally disconnected* space is one in which the components are single points (must be T_1); a *completely disconnected* space has singleton quasi-components (must be T_2).

   The quotient of a topological space by its components is totally disconnected (any distinct $[x]$, $[y]$ are disconnected by $[x]^c$, $[y]^c$); the clopen subsets of the quotient correspond to those of the space. The quotient by its quasi-components is completely disconnected (distinct $Q(x)$, $Q(y)$ are separated by a clopen set U which must contain whole quasi-components; hence πU is clopen and separates $Q(x), Q(y)$ in X/Q).

Distance and Order Relation

Cluster spaces have a *specialization* order relation in which any filter that clusters or converges to one point is forced to do so at other points that are 'hidden under' it, so to speak:

$$\begin{aligned} x \leqslant y &\Leftrightarrow (\mathcal{F} \dashrightarrow y \;\Rightarrow\; \mathcal{F} \dashrightarrow x) \\ &\Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}, i.e., (\mathcal{F}(y) \dashrightarrow x) \\ &\Leftrightarrow \mathfrak{F}(y) \to x \\ &\Leftrightarrow (\mathcal{F} \to y \;\Rightarrow\; \mathcal{F} \to x) \\ &\Leftrightarrow \mathcal{T}_x \subseteq \mathcal{T}_y \end{aligned}$$

(since $\mathfrak{F}(y) \to x \in A$ open implies $A \in \mathfrak{F}(y)$, so $y \in A$ and $A \in \mathcal{T}_y$.)

1. With respect to this order, continuous functions are monotone, $\overline{\{x\}} = \downarrow\{x\}$, closed sets are lower closed, open sets are upper closed.

   Proof: If $x \in \overline{\{y\}}$ then $f(x) \in f\overline{\{y\}} \subseteq \overline{\{f(y)\}}$. If $x \leqslant y \in \bar{A}$ then $x \in \overline{\{y\}} \subseteq \bar{A}$. If $y \geqslant x \in A^\circ$ then $A^\circ \in \mathcal{T}_x \subseteq \mathcal{T}_y$, so $y \in A^\circ$.

2. $\{x, y\}$ is connected $\Leftrightarrow$ $x \leqslant y$ OR $y \leqslant x$.

3. An ordered space may have several compatible topologies. The coarsest is the one defined previously. The finest is the Alexandrov topology, generated by the open sets $\{x : a < x\}$ (every intersection of open sets is open; the continuous maps are the monotone ones). (These agree for finite sets.)

More generally, take a **distance** function $d : X^2 \to R$, where R is a complete distributive lattice and monoid, with a filter of "positive elements" P, such that

$\forall r \in P, \exists s \in P,\ s+s \leqslant r$; $a+0=a$; $\bigwedge(a+A) = a + \bigwedge A$, and

$$d(x,x) = 0$$
$$d(x,y) \leqslant d(x,z) + d(z,y)$$

P can be taken to be the set $P := \{\, r \in R : 0 \prec r \,\}$, where $s \prec r$ when $\bigwedge A \leqslant s \ \Rightarrow\ \exists a \in A, a \leqslant r$ (implies $s \leqslant r$).

The *ball* about x of radius $r \in P$ is

$$B_r(x) := \{\, y \in X : d(x,y) \prec r \,\}$$

(also $B_r(A) := \{\, y \in X : \exists x \in A, d(x,y) \prec r \,\}$). Because $B_{r\wedge s}(x) = B_r(x) \cap B_s(x)$, these balls generate a topology.

1. Given a topological space $X, \mathcal{T}$, let $R := 2^{\mathcal{T}}$ with reverse order, i.e., $A \leqslant B \Leftrightarrow B \subseteq A$, $0 = \mathcal{T}$, $A \wedge B = A \cup B$; let $A + B := A \cap B$; and let P be the finite subsets of $\mathcal{T}$ (a filter); let

$$d(x,y) := \{\, U \in \mathcal{T} : x \in U \ \Rightarrow\ y \in U \,\}$$

then d is a distance because $d(x,x) = \mathcal{T}$ and if $U \in d(x,z) \cap d(z,y)$, then $x \in U \ \Rightarrow\ z \in U \ \Rightarrow\ y \in U$, so $U \in d(x,y)$.

   (a) Balls are open in $\mathcal{T}$:

$$B_{\{V\}}(x) = \{\, y \in X : d(x,y) \leqslant \{V\} \,\} = \{\, y : x \in V \ \Rightarrow\ y \in V \,\} = \begin{cases} V & x \in V \\ X & x \notin V \end{cases}$$
$$B_{\{V_1,\ldots,V_n\}}(x) = B_{\{V_1\}}(x) \cap \cdots \cap B_{\{V_n\}}(x)$$

   Hence every open set is a a union of balls: $V \subseteq \bigcup_{x \in V} B_{r(x)}(x) \subseteq V$.

   (b) Every topology is generated by a distance.
   Proof: If V is an open set, then for any $x \in V$, $B_{\{V\}}(x) = V$, so V is open wrt d. Conversely, if V is open wrt d, then for any $x \in V$, there is $A_x \in P$, $B_{A_x}(x) \subseteq V$, so $V \subseteq \bigcup_{x \in V} B_{A_x}(x) \subseteq V$, i.e., $V = \bigcup_{A_x}(x)$ is open in $\mathcal{T}$.

2. Given any $\epsilon \in P$, one can find $\epsilon/2^n$ such that $2(\epsilon/2^{n+1}) \prec (\epsilon/2^n)$, and $2^{n+1}\epsilon := 2(2^n\epsilon)$; furthermore, can extend to positive dyadic fractions $p\epsilon := \sum_n a_n(\epsilon/2^n)$ when $p = \sum_n a_n/2^n$ (binary decomposition of p) (then $p\epsilon + q\epsilon \leqslant (p+q)\epsilon$). This induces a morphism $\phi : R \to \mathbb{R}^+$, $\phi(r) := \bigwedge\{\, p\epsilon : a \leqslant p \,\}$ (if $r \leqslant s$ then $\phi(r) \leqslant \phi(s)$ and $\phi(r+s) = \bigwedge\{\, p \in D : r+s \leqslant p\epsilon \,\} \leqslant \bigwedge\{\, p+q : r \leqslant p\epsilon \text{ AND } s \leqslant q\epsilon \,\} = \phi(r) + \phi(s)$. Hence $\phi_\epsilon \circ d : X^2 \to \mathbb{R}^+$ is a non-symmetric pseudo-metric.

3. $x \leqslant y \Leftrightarrow d(x,y) = 0$. Thus, if $d(x,y) = 0 = d(y,x)$ then $x = y$.

4. $x \in A^\circ \Leftrightarrow \exists r \in P, B_r(x) \subseteq A$.

$x \in \bar{A} \Leftrightarrow \forall r \in P,\ x \in B_r(A)$.

$x \in \partial A \Leftrightarrow \forall r \in P,\ x \in B_r(A) \cap B_r(A^{\mathsf{c}})$.

5. A function $f : X \to Y$ is continuous iff

$$\forall \epsilon \in P_Y, \exists \delta \in P_X,\ d_X(x, y) \prec \delta \ \Rightarrow\ d_Y(f(x), f(y)) \prec \epsilon$$

6. A filter $\mathcal{F}$ is *Cauchy* when $\forall \epsilon \in P, \exists A \in \mathcal{F}, \quad x, y \in A \ \Rightarrow\ d(x, y) \prec \epsilon$.

   Nets are *asymptotic* when $\forall \epsilon \in P, \exists n \in I,\ i, j \geqslant n \ \Rightarrow\ d(x_i, y_j) \prec \epsilon$. A net is Cauchy when every subnet is asymptotic to it, i.e., $d(x_i, x_j) \to 0$.

Compactness

A **compact** space is one in which every proper filter clusters.

1. Equivalently, when every maximal filter converges; or when every open cover has a finite subcover.

   Proof: If V_i are open sets without a finite subcover, then $\varnothing \neq \mathrm{cl}\mathfrak{F}(V_i^{\mathsf{c}})_i \subseteq \bigcap_i V_i^{\mathsf{c}}$, so $\bigcup_i V_i \neq X$. Conversely, if $\mathcal{F}$ is proper, then for $A \in \mathcal{F}$, $\bar{A}^{\mathsf{c}}$ has no finite subcover, $\bar{A}_1^{\mathsf{c}} \cup \cdots \cup \bar{A}_n^{\mathsf{c}} = (A_1 \cap \cdots \cap A_n)^{\mathsf{c}} \neq X$; hence $\mathrm{cl}(\mathcal{F}) = \bigcap_{A \in \mathcal{F}} \bar{A} = (\bigcup_{A \in \mathcal{F}} \bar{A}^{\mathsf{c}})^{\mathsf{c}} \neq \varnothing$.

2. Points, closed subspaces, finite unions, products, and continuous images (quotients) of compact spaces are compact.

   Proof: If $F \in \mathcal{F} \dashrightarrow x$ then $x \in \bar{F} = F$. The finite sub-covers of F and G cover $F \cup G$. Given a proper filter $\mathcal{F}$ on $\prod_i X_i$, then $\pi_i \mathcal{F}$ is proper in X_i, so $\pi_i \mathcal{F} \dashrightarrow x_i$ and $\mathcal{F} \dashrightarrow (x_i)_{i \in I}$. Let $\mathcal{F}$ be a proper filter in fX; then $f^{-1}\mathcal{F}$ is proper in X, so $f^{-1}\mathcal{F} \dashrightarrow x$ and $\mathcal{F} \dashrightarrow f(x)$.

3. Compactness can be split in two weaker conditions:

   Lindelöf: Every open cover has a countable subcover.
   $\Rightarrow$ Every uncountable subset has a limit point.

   Countably compact: Every countable open cover has a finite subcover,
   $\Leftrightarrow$ Every sequence clusters, $\Leftarrow$ Every sequence has a convergent subsequence,
   $\Rightarrow$ *Bolzano-Weierstraß property*: Every infinite subset has a limit point.

   Proof: If X is countably compact and x_n a sequence without a cluster point, then each $x \in X$ is in an open set U_x with only finitely many $x_n \in U_x$; let $U_n := \bigcup\{ U \in \mathcal{T} : x_i \in U \ \Rightarrow\ i \leqslant n \}$, an increasing countable cover of X. Its finite subcover is some U_N, hence $x_{N+1} \in U_N$ a contradiction. Conversely, if U_n is a countable cover, then pick $x_{n+1} \notin U_1 \cup \cdots \cup U_n$; this sequence has a convergent subsequence $x_{n_i} \to a$. If $a \in U_n$ then $x_{n_i} \not\to\in U_n$, a contradiction for large n_i. Let A be an infinite subset; create a sequence of distinct points from it, which has a convergent subsequence; the limit is a limit point of A.

(All these properties are preserved by continuous functions.)

Lindelöf spaces have countable extent (since $X = A \sqcup \coprod_i \{ x_i \}$ has a countable sub-cover).

4. Any topological space can be embedded in a compact space by the addition of one point (∞).

   Proof: Consider $X \cup \{ \infty \}$ where $\mathcal{T}_\infty := \{ A : A^c \text{ is closed compact in } X \}$.

   But not all continuous functions need extend to the one-point compactification: e.g. a sequence $x : \mathbb{N} \to A$ can be extended to $x : \mathbb{N} \cup \{ \infty \} \to A$ only if the sequence converges.

Local and Global Properties

A topological space has both local and global properties: take an open cover of X, each U_i is a topological space in its own right, while the way they connect to each other determine the global properties of X.

A *local morphism* is a map which is a morphism on some neighborhoods of any x, $f(x)$. Local morphisms are in fact the morphisms, but local isomorphisms are more general. A local property is one that is enjoyed by all/none of a class of locally isomorphic topological spaces. A continuous map need not preserve local properties, but open subspaces usually do.

A *fiber bundle* is a topological space E that is locally isomorphic to a product $B \times F$ such that the local projections $\pi : E \to B$ are continuous; equivalently it is two maps $F \to E \to B$; e.g. the 3-sphere is a fiber bundle $S^1 \to S^3 \to S^2$.

A *locally connected* space has neighborhood bases of connected sets; iff there are neighborhoods of connected open sets; the quasi-components are then the components and are open (proof: every x has an open connected neighborhood U which lies in the component $C(x)$, so $C(x)$ is clopen); so cannot be totally disconnected except for discrete spaces.

A *locally compact* space has neighborhood bases of compact sets; preserved by open continuous functions (e.g. quotients); the finite product of locally compact spaces is locally compact (the infinite product is not, unless the spaces are compact).

The *global* properties of an open cover U_i are captured by its *nerve*: the finite subsets of indices j such that $\bigcap_j U_j \neq \varnothing$. For example, compact spaces are 'finite' in the global sense.

Homogeneous Spaces are those for which the automorphism group has one orbit; they 'appear' the same at all points; they are locally isomorphic. A homogeneous space is either discrete (all points are isolated) or perfect T_1. Products are again homogeneous.

3 Size

There are several ways to measure the size of a topological space, apart from its cardinality.

1. The *weight* is the smallest cardinality of a base.
2. The *density* is the smallest cardinality of a dense subset.
3. The *spread* is the largest (sup) cardinality of a discrete subset (hence spread $\geqslant$ extent).

The density and spread are less than the weight: the former because a set of points taken one from each basic open set gives a dense subset; the latter because a discrete set is covered by disjoint open (basic) sets.

If $X \subsetneqq Y$ then the weight and spread of X are at most those of Y.

The number of open disjoint subsets is at most equal to the spread s.

3.1 Separable Spaces

A *separable* space is one that has a countable dense subset, i.e., its density is $\mathbb{N}$; so every point can be approximated to any accuracy by one of these points.

1. Open subspaces, images, and countable sums or products are separable.

   Proof: Given A countable dense; if Y is open, and U open, then there is a point $a \in A \cap U \cap Y$, so $A \cap Y$ is dense in Y. $\overline{fA} \supseteq f\bar{A} = fX$. $\overline{\prod_i A_i} = \prod_i \overline{A_i} = \prod_i X_i$; $(a_n)_{n \in \mathbb{N}}$, with $a_n \in A_n$ for $n \leqslant N$, is dense in $\prod_i A_i$.

2. Every disjoint collection of open sets is countable.

3. The number of regular open subsets is at most $2^{\mathbb{N}}$ (more generally $2^{\mathrm{density}(X)}$) (since U is determined by $A \cap U \in 2^A$ for a dense subset A).

3.2 First Countable Spaces

have countable local weight, i.e., for each x, there are countable sub-bases $\mathcal{S}_x$.

1. Subspaces, open images (quotients) and countable products are first countable.

   Proof: If f is open, then fU_n is a sub-base at $f(x)$. If X_n has countable sub-base $U_{n,m}$, then $\prod_n X_n$ has a sub-base $\pi_n^{-1}U_{n,m}$ $(n, m \in \mathbb{N})$.

2. Every point has a sequence converging to it (take points from U_1, $U_1 \cap U_2$, ... of the countable base.)

   So many convergence properties of filters reduce to properties of sequences, e.g. a set is countably compact iff every sequence has a convergent subsequence; a function is continuous $\Leftrightarrow$ sequentially continuous.

   Proof: If f is not continuous at x, then $\exists U \in \mathcal{T}_x$, $\forall n, V_n \not\subseteq f^{-1}U$, so $\exists a_n \in V_n \setminus f^{-1}U$, so $f(a_n) \notin U$, so $\lim_n a_n = x$ but $\lim_n f(a_n) \neq f(x)$.

3.3 Second Countable Spaces

Have a countable base, i.e., $\mathcal{S}$ is countable.

1. They are separable, first countable, and Lindelöf.

   Proof: Pick one point from each basic open set for a dense subset. Given an open cover U_i, then for each $x \in X$, $x \in V_n \subseteq U_i$, so can choose a countable subcover.

2. Subspaces, open images, and countable products are again second countable.

3. A countable first countable space is second countable, but there are countable spaces that are not first countable (e.g. Arens-Fort space).

4 T_1 Spaces

are $\leqslant$-symmetric T_0 spaces, i.e., $x \leqslant y \Leftrightarrow y \leqslant x (\Leftrightarrow x = y)$, equivalently,

$$\begin{aligned}
&\Leftrightarrow\ d(x,y) = 0 \ \Rightarrow\ x = y, \\
&\Leftrightarrow\ \mathcal{F}(x) \text{ clusters only at } x \\
&\Leftrightarrow\ \text{Points are closed sets,} \\
&\Leftrightarrow\ \mathcal{N}_x \text{ converges only to } x \\
&\Leftrightarrow\ \forall x \neq y,\ \exists \mathcal{F} \to x,\ \mathcal{F} \not\to y \\
&\Leftrightarrow\ \text{Any two points can be separated by open sets} \\
&\Leftrightarrow\ A = \bigcap \{ V \in \mathcal{T} : A \subseteq V \}
\end{aligned}$$

Proof: The first four equivalences are trivial. If $x \neq y$, then $\mathcal{F}(x) \not\to y$. If $\mathcal{F} \not\to y$ then $\mathcal{T}_y \not\subseteq \mathcal{F}$ so $\exists U \in \mathcal{T}_y, U \notin \mathcal{T}_x \subseteq \mathcal{F}$. $A = \left(\bigcup_{x \notin A} \{ x \}\right)^{\mathsf{c}} = \bigcap_{x \notin A} \{ x \}^{\mathsf{c}} \supseteq \bigcap_{A \subseteq V \in \mathcal{T}} V \supseteq A$. $y \notin \{ x \} = \bigcap_{x \in V \in \mathcal{T}} V \Rightarrow \exists V \in \mathcal{T}, x \in V, y \notin V$. If $x < y$ then $x \in U, y \in U^{\mathsf{c}}$, so $\bar{y} \subseteq U^{\mathsf{c}}$, hence $x \not\leqslant y$, a contradiction.

Example: Cofinite topology on X (generated by the cofinite sets, i.e., sets whose complements are finite).

1. Subspaces, images, products, and the one-point compactification are also T_1.

   Proof: For products, let $(x_i) \neq (y_i)$, so $x_j \neq y_j$ for some j. Then x_j, y_j are separated by open sets U_j, V_j, so $\pi_j^{-1} U_j$ and $\pi_j^{-1} V_j$ separate $(x_i), (y_i)$.

2. For first countable T_1 spaces, limit points are limits of convergent sequences. Hence BW-compact sets are countably compact.

3. Finite T_1 spaces are discrete.

4.1 T_2 – Hausdorff Spaces

Proper filters have at most unique limits

$$\begin{aligned} &\mathcal{F} \to x \text{ AND } \mathcal{F} \to y \ \Rightarrow \ x = y \\ \Leftrightarrow &\mathcal{F} \to x \text{ AND } \mathcal{F} \dashrightarrow y \ \Rightarrow \ x = y \\ \Leftrightarrow &\mathcal{N}_x \text{ clusters only at } x \\ \Leftrightarrow &\text{Any two points can be separated by disjoint open sets} \end{aligned}$$

Proof: The last statement, $\exists A \in \mathcal{T}_x, B \in \mathcal{T}_y,\ A \cap B = \varnothing$, is equivalent to $\mathcal{N}_x \vee \mathcal{N}_y$ being improper. $\mathcal{N}_x \vee \mathcal{N}_y$ converges to both x and y, so it can only be proper when $x = y$. If $\mathcal{N}_x \dashrightarrow y$ then $\mathcal{N}_x \vee \mathcal{N}_y$ is proper. $\mathcal{F} \to x \ \Rightarrow \ \mathcal{N}_x \subseteq \mathcal{F} \ \Rightarrow \ \mathrm{cl}(\mathcal{F}) \subseteq \mathrm{cl}(\mathcal{N}_x) = \{ x \}$.

In particular, convergent nets/sequences have unique limits, denoted $\lim_{n\to\infty} x_n$. T_2 spaces are T_1.

1. Subspaces and products are also T_2 (but not necessarily images or the one-point compactification).

2. Every topological space can be reduced to a T_2 space (take quotient by the relation $x \sim y \Leftrightarrow \forall f : X \to A(T_2)$ onto, continuous, $f(x) = f(y)$).

3. Compact subsets are closed; disjoint compact sets can be separated by disjoint open sets.

   Proof: For any $x \in K$, $y \notin K$, there are disjoint open sets $x \in U_x$ and $y \in V_x$; hence $K \subseteq U_y := \bigcup_{i=1}^n U_{x_i}$ disjoint from $y \in V_y := \bigcap_{i=1}^n V_{x_i}$. For a disjoint compact subset with elements y, $V := \bigcup_{j=1}^m V_{y_j}$ and $U := \bigcap_{j=1}^m U_{y_j}$ separate the two.

4. If $f : Y \to X$, $g : Z \to X$ are continuous, then the set $\{ (y, z) \in Y \times Z : f(y) = g(z) \}$ is closed; for example, the graph and kernel of f, and the equalizer of $f, g : Y \to X$ in Y. So if f, g agree on a dense subspace of Y, then they agree on all of Y: Continuous functions are determined on dense subsets. The epimorphisms to T_2 spaces are the dense continuous functions.

   Proof: If $(y, z) \notin A$, then $f(y) \neq g(z)$, so are separated by open sets U, V; then $(y, z) \in f^{-1}U \times g^{-1}V \subseteq A^{\mathsf{c}}$.

5. In a T_2 space with a dense subset A, every point x is determined by $\mathcal{T}_x$, and every $U \in \mathcal{T}_x$ is determined by the points of A in it; thus $|X| \leqslant 2^{2^{d(X)}}$; in particular, separable T_2 spaces have cardinality at most $2^{\mathbb{R}}$.

   If the space is separable first countable, every point is determined by a sequence taken from A, so its cardinality is at most $\mathbb{N}^{\mathbb{N}} \equiv \mathbb{R}$.

$T_{2.5}$ Spaces

are stronger versions of Hausdorff spaces in which any two points can be separated by disjoint closed neighborhoods.

T_3 **Spaces (Regular)**

are (T_0) spaces in which any point and a disjoint closed set can be separated by disjoint open sets; equivalently, there is a base of closed neighborhoods, $\forall U \in \mathcal{T}_x, \exists V \in \mathcal{T}_x,\ V \subseteq \bar{V} \subseteq U$; thus a base of regular open subsets.

1. T_3 spaces are $T_{2.5}$. (If $x \not\leqslant y$, then $x \in V \subseteq \bar{V} \subseteq U$, $y \in U^{\mathsf{c}} \subseteq \bar{V}^{\mathsf{c}}$, so T_2, hence $T_{2.5}$).

2. Subspaces and products are T_3.

3. Every compact subset can be separated from a disjoint closed set (separate points of K from F, then take finite sub-cover).

4. If X is dense in $\tilde{X}$ and Y is T_3 then any continuous function $f : X \to Y$ extends uniquely to a continuous function $\tilde{f} : \tilde{X} \to Y$, where $\tilde{f}(\tilde{x}) := \lim_{x\to\tilde{x}} f(x)$.

   Proof: Well-defined because Y is T_2; continuous because for any closed neighborhood F of $\tilde{f}(\tilde{x})$, there is an open set V in X such that $fV \subseteq F$; so $\tilde{f}V \subseteq \overline{f(V \cap X)} \subseteq F$.

5. Countable T_3 spaces are Lindelöf (hence $T_{3.5}$, see later), so connected T_3 spaces are uncountable, except for singletons or $\varnothing$.

6. There are only two countably infinite homogeneous T_3 spaces — $\mathbb{N}$ and $\mathbb{Q}$.

7. For a countably compact T_3 space, meagre subsets have empty interior. Hence X is not meagre.

   Proof: Let $\bigcup_n A_n$ be closed nowhere dense sets with interior $U \neq \varnothing$. Starting with $V_1 := U$ and continuing iteratively, $\varnothing \neq V_n \not\subseteq A_n$, so $\exists x_n \in V_n \smallsetminus A_n$; hence $x_n \in V_{n+1} \subseteq \overline{V_{n+1}} \subseteq V_n \smallsetminus A_n$. Decreasing closed sets in a countably compact space have non-empty intersection, $x \in \bigcap_{n\geqslant 2} \overline{V_n} \subseteq U \smallsetminus \bigcup_n A_n$ a contradiction.

A space is *zero dimensional 0-D* when every point has a neighborhood base of clopen sets (i.e., without a boundary). 0-D spaces are (obviously) completely disconnected T_3 spaces. Subspaces are again 0-D. Can be embedded in 2^C where C are the clopen sets, via $x \mapsto \{ i : x \in U_i \}$. For example, the discrete spaces and $\mathbb{Q}$.

More generally, a space is called *n-dimensional* when every neighborhood has a set with boundary of dimension $n - 1$.

5 $T_{3.5}$ Tychonoff Spaces - Uniform Spaces

are (T_0) spaces such that an associated distance is symmetric,

$$d(y, x) = d(x, y).$$

The induced pseudo-metric $g := \phi \circ d : X^2 \to \mathbb{R}^+$ is continuous, where $\phi : R \to \mathbb{R}^+$. (Proof: for any $\delta > 0$, let $p < \delta$ be a dyadic fraction; then $d(y,z) \prec p\epsilon$ implies $|g(x,y) - g(x,z)| \leqslant g(y,z) = \phi \circ d(y,z) \leqslant p < \delta$ since $g(x,y) \leqslant g(x,z) + g(z,y)$).

The distance generates a **uniformity** on X, namely the filter $\mathcal{U}$ generated from the relations $U_r := \{ (x,y) \in X^2 : d(x,y) \prec r \}$ $(r \in P)$. The reflexive, symmetric, transitive and the $s + s \leqslant r$ properties induce

Each $r \in \mathcal{U}$ is reflexive,	since $d(x,x) = 0$
$r \in \mathcal{U} \;\Rightarrow\; r^{-1} \in \mathcal{U}$,	$d(y,x) = d(x,y)$
$\forall r \in \mathcal{U}, \exists s \in \mathcal{U}, s \circ s \subseteq r$	$d(x,y) \leqslant d(x,z) + d(z,y) \prec 2s \leqslant r$

Moreover, assuming no indistinguishable points, it can be assumed that $\bigcap \mathcal{U} = \{ (x,x) : x \in X \}$. Conversely, any uniform space gives rise to a symmetric distance. There is a unique finest uniformity that is compatible with the topology (take union of all compatible uniformities).

A filter is *Cauchy* when $\forall U \in \mathcal{U},\ \exists A \in \mathcal{F},\ A \times A \subseteq U$, i.e.,

$$\forall r \succ 0,\ \exists A \in \mathcal{F},\ \forall x,y \in A, \quad d(x,y) \leqslant r$$

Proof: $\mathfrak{F}(x)$ is Cauchy since for any $r \succ 0$, take $A := B_{r/2}(x)$. Given $r \succ 0$, $A \in \mathcal{F}$, $B \in \mathcal{G}$, $d(A,A) \leqslant r$, $d(B,B) \leqslant r$, then for any $x,y \in A \cup B \in \mathcal{F} \cap \mathcal{G}$ and $z \in A \cap B$, so $d(x,y) \leqslant d(x,z) + d(z,y) \leqslant 2r$.

Every uniform space can be uniquely completed (by taking the set of minimal Cauchy filters; then X is embedded in it via $x \mapsto \mathfrak{F}(x)$).

A function $f : X \to Y$ is *uniformly continuous* when $f\mathcal{U}_X$ refines $\mathcal{U}_Y$, i.e.,

$$\forall \epsilon \succ_Y 0, \exists \delta \succ_X 0, \quad d(x,y) \prec \delta \;\Rightarrow\; d(f(x), f(y)) \prec \epsilon$$

Hence they are Cauchy maps (and preserve asymptotic nets), so continuous. Any uniformly continuous function extends uniquely to a uniformly continuous function on the completion spaces, via $\tilde{f}(x) := \lim_n f(x_n)$.

Examples: $\mathbb{R}^A$ with pointwise topology.

1. Any point and a disjoint closed set can be separated by a real-valued function. (Hence T_3).

   Proof: Given any $x \in X$ and $\epsilon \in P$ such that $B_\epsilon(x) \subseteq F^c$, let $f(y) := 1 \wedge g(x,y)$, where g is the pseudo-distance defined above; then $f(x) = 0$ and for $y \notin \overline{B_\epsilon(x)}$, $d(x,y) \not\prec \epsilon$, so $g(x,y) \geqslant 1$.

   Incidentally, this shows that if X is connected then $X \to [0,1]$ is onto, so X is uncountable; unless the only closed sets are trivial, $X = \{ x \}$ or $\varnothing$.

2. Tychonoff spaces are those spaces that can be (densely) embedded in a compact-T_2 space (e.g. via the map $J(x) := (f(x))_{f \in C} \in [0,1]^{C(X,[0,1])}$).

   Proof: J is continuous because each component is, $x \mapsto f(x)$. J is 1-1 because $C := C(X,[0,1])$ distinguishes points. It is an open map because

C distinguishes points from closed sets: let $x \in U$ open in X, then $\exists f \in C$, $f(x) = 0$, $fU^{\mathsf{c}} = 1$; let $V := \pi_f^{-1}[0,1[$ open in $[0,1]^C$. Then $(f(y))_{f\in A} = J(y) \in V \Rightarrow f(y) < 1 \Rightarrow y \in U$, so $J(x) \in V \cap J(X) \subseteq J(U)$.

There is a unique 'Stone-Cech' compactification or completion, βX, in which every continuous function $f : X \to Y$ (Y compact T_2) extends to $\tilde{f}$ on βX. Each $x \in \beta X$ corresponds to a maximal ideal of $C_b(X)$, $\mathcal{I}_x = \ker \delta_x = \{ f \in C_b(X) : \tilde{f}(x) = 0 \}$.

3. Subspaces and products are again Tychonoff (but not images or quotients).

   Proof: The subspace distance remains symmetric. $\prod_i X_i \subsetneq \prod_i [0,1]^{A_i} = [0,1]^{\sum_i A_i}$.

4. A set is *totally bounded* when for any $r \in P$, there is a finite set F such that $A \subseteq B_r(F)$; preserved by uniformly continuous functions. Subspaces, finite unions, and products of totally bounded are totally bounded.

   (a) Every filter in a totally bounded subset has a Cauchy refinement; i.e., maximal filters are Cauchy.

   (b) Any sequence in a totally bounded subset has a Cauchy subsequence (since choose $r_n \to 0$; some set in $B_{r_n}(F_n)$ must contain an infinite number of terms).

5. A Tychonoff space has a *proximity* relation $A\,\delta\,B$ defined by $\forall r \in P, B_r(A) \cap B_r(B) \neq \varnothing$. Then

$$A\,\delta\,B \Leftrightarrow B\,\delta\,A, \qquad x\,\delta\,y \Leftrightarrow x = y, \qquad A \not\delta \varnothing$$
$$A \cap B \neq \varnothing \Rightarrow A\,\delta\,B, \quad A\,\delta\,B \cup C \Leftrightarrow (A\,\delta\,B) \text{ OR } (A\,\delta\,C)$$
$$\text{If for all } E, A\,\delta\,E \text{ OR } B\,\delta\,E^{\mathsf{c}} \text{ then } A\,\delta\,B$$

5.0.1 Locally Compact T_2 Spaces

When each neighborhood has a base of compact sets; equivalently, every point has a compact neighborhood (since K is T_3 so for $x \in K^\circ$, $U \in \mathcal{T}_x$, $U \cap K^\circ$ contains a closed (compact) neighborhood).

Closed subspaces, and open images, are again locally compact T_2.

1. The one-point compactifications are T_2, hence are Tychonoff. They are the open subspaces of compact T_2 spaces. Example: $\mathbb{R}$ compactifies to $\mathbb{S}$.

2. They inherit several properties from compact T_2 spaces:

   (a) 0-D $\Leftrightarrow$ Totally disconnected.

   (b) Meagre subsets have empty interior.

   Proof: Let $\bar{U}$ be a compact neighborhood of a (so completely disconnected); for any other $x \in \bar{U}$, there is a clopen subset $a \in V_x \subseteq \{ x \}^{\mathsf{c}}$; $\bar{U} \smallsetminus U$ is compact, hence is covered by a finite number of these; thus $V_1 \cap \cdots \cap V_n \subseteq U \cap \bar{U}^{\mathsf{c}}$; $K := V_1 \cap \cdots \cap V_n \cap \bar{U} \subseteq U$ is a clopen subset on a.

3. Can define the categorical Y^X when X is locally compact T_2, as $C(X,Y)$ with the compact-open topology i.e., generated by the functions f such that there are $K \subseteq X$ compact and $U \subseteq Y$ open with $fK \subseteq U$. If Y is T_0, T_1, T_2 or T_3, then so is $C(X,Y)$.

5.1 T_4 Spaces (Normal)

Any two disjoint closed sets or points can be separated by disjoint open sets. Equivalently,

$F \subseteq U$ (closed, open) $\Rightarrow$ $F \subseteq V \subseteq \bar{V} \subseteq U$; hence can fill out with a continuum of sets - first dyadic, then $V_t := \bigcup_{i/2^n \leqslant t} V_{i,n}$ such that $s < t \;\Rightarrow\; V_s \subseteq \bar{V}_s \subseteq V_t$;

$\Leftrightarrow$ Any two disjoint closed sets or points can be separated by continuous functions (Urysohn: take $f(x) := \inf\{\, t : x \in V_t \,\}$);

$\Leftrightarrow$ Continuous functions on a closed subspace $f : F \to \mathbb{R}$ have a continuous extension to X and X is T_1 (Tietze).

1. Closed subspaces are T_4.

2. Every locally finite open cover has a partition of unity $\sum_i f_i(x) = 1$ subordinate to it. (Locally finite means that each point has a neighborhood that intersects only finitely many sets in the cover.)

   Proof: Given locally finite U_i; it has a locally finite refinement V_i such that $\overline{V_i} \subseteq U_i$; for each i, V_i can be separated from U_i by a continuous function $g_i : X \to [0,1]$ with $g_i U_i^{\mathsf{c}} = 0$. Let $g := \sum_i g_i$ (well-defined by local finiteness). Since each x is covered by some V_i, $g > 0$; hence $f_i := g_i/g$ is a locally finite partition of unity with supp $f_i \subseteq U_i$.

5.1.1 T_5 Spaces (Hereditarily normal)

when any two separated sets ($A \cap \bar{B} = \varnothing = B \cap \bar{A}$) or points can be separated by disjoint open sets. Equivalently, every subspace is T_4.

Important examples of T_5 spaces are:

Linearly ordered (bounded) spaces with topology generated by the closed sets $\uparrow x = [x,1]$ and $\downarrow x = [0,x]$.

1. Limits are order-morphisms, $x_n \leqslant y_n \;\Rightarrow\; \lim_n x_n \leqslant \lim_n y_n$.

2. Linear Orders are T_5.

   Proof: Given $A \cap \bar{B} = \varnothing = \bar{A} \cap B$, let A_α be the convex components of A in $\bar{B}^{\mathsf{c}}$, and B_β of B in $\bar{A}^{\mathsf{c}}$. If the upper end of A_α is not open, then it has a maximum a; $a \notin \bar{B}$, so $\exists x, A_\alpha < x < \overline{B_\beta}$ for all $A_\alpha < B_\beta$. Similarly for the lower end and for B_β. So $A_\alpha \subseteq U_\alpha$, $B_\beta \subseteq V_\beta$ open, $U_\alpha \cap V_\beta = \varnothing$; $U := \bigcup_\alpha U_\alpha$, $V := \bigcup_\beta V_\beta$; $U \cap V = \varnothing$.

3. A subset is topologically dense $\Leftrightarrow$ order-dense outside gaps.

4. The order is compact $\Leftrightarrow$ order-complete.

   Proof: If X is incomplete, then there is a non-empty subset A without a supremum; so the open sets $[0, a[$ $(a \in A)$ and $]u, 1]$ $(u \in \mathrm{UB}(A))$ cover X, but no finite subcover exists. Conversely, if X is complete and U_i an open cover, then the set $A := \{ x \in X : [0, x[\text{ is covered by a finite number of } U_i \}$ is non-empty $(0 \in A)$, so has a supremum y, which is covered $]a, b[\subseteq U$. This contains some $x \in A$, so $[0, b[$ is covered by the finite number of I_i and U; thus $y, b \in A$. Hence the supremum is 1, i.e., there is a finite subcover.

5. The order is connected $\Leftrightarrow$ order-complete and dense (i.e., no 'cuts' or gaps).

   Proof: Any cut disconnects X: if a non-empty subset A has no supremum, then $U :=$ Upperbounds(A) is both open and closed. If $x < y$ is a gap, then $[0, y[$ and $]x, 1]$ disconnect X. Conversely if X is complete but disconnected by disjoint clopen sets A, B with $a \in A$, $b \in B$, $a < b$; then let $C := A \cap \downarrow b$ and $c := \bigvee C$; if $c \in A$, then it is exterior to B, so $c \in]x, y[\subseteq A$ making $c < y$ a gap; else if it is in B, then $c \in]x, y[\subseteq B$, so $x < c$ is a gap.

6. Cauchy-complete $\Leftrightarrow$ paracompact T_2.

7. The density is the spread s or s^+.

5.1.2 T_6 Spaces (Perfectly Normal)

when every two disjoint closed sets can be perfectly separated, i.e., there is a continuous function $f : X \to [0, 1]$, $f^{-1}0 = A$, $f^{-1}1 = B$; equivalently when T_4 and every closed set is the countable intersection of open sets.

5.2 Paracompact T_2 Spaces

Open covers have open locally-finite refinements (i.e., the refinement sets are covered by the cover sets, e.g. subcovers). Their (Lebesgue) dimension is n such that every open cover has an $(n + 1)$-locally finite refinement.

Example: *CW-complexes*

1. Closed subspaces and finite sums are paracompact T_2, but not necessarily products or images.

2. For a locally finite cover, $\overline{\bigcup_i V_i} = \bigcup_i \bar{V}_i$.

   Proof: Any x has a neighborhood that intersects only $V_1, \ldots, V_n$ say; so $x \in \overline{\bigcup_i V_i} = \overline{\bigcup_{i>n} V_i} \cup \bar{V}_1 \cup \cdots \cup \bar{V}_n$, hence $x \in \bigcup_{i \leqslant n} \bar{V}_i \subseteq \bigcup_i \bar{V}_i$.

3. Paracompact T_2 are normal.

   Proof: Given F, G closed sets, hence paracompact. For each $x \in F$, $y \in G$, there are open sets $U_{x,y}$ such that $x \in U_{x,y}$, $y \in \overline{U_{x,y}}^{\mathsf{c}}$ since T_2. For fixed

x, $U_{x,y}$ have a locally finite refinement V_i on G; $V_x := \bigcup_i V_i \supseteq G$. So $x \in \bigcap_i \bar{V}_i^{\mathsf{c}} = (\bigcup_i \bar{V}_i)^{\mathsf{c}} = \overline{V_x}^{\mathsf{c}}$. $\bar{V}_x^{\mathsf{c}}$ have a locally finite refinement W_j on F; $W := \bigcup_j W_j \supseteq F$. Then $G \subseteq \bigcap_x \bar{V}_x^{\mathsf{c}} = (\bigcup_x \bar{V}_x)^{\mathsf{c}} \subseteq (\bigcup_j \bar{W}_j)^{\mathsf{c}} = \bar{W}^{\mathsf{c}}$.

Important examples of paracompact T_2 spaces are:

5.2.1 Metric Spaces

have a distance or metric $d : X^2 \to \mathbb{R}^+$.

A morphism may be required to be Lipschitz: $d(f(x), f(y)) \leqslant c\,d(x, y)$; hence uniformly continuous.

1. Metric spaces are first countable paracompact T_6.

   Proof: $B_{1/n}(x)$ form a countable base at x. Disjoint closed subsets are perfectly separated by $d(x, A)/(d(x, A) + d(x, B))$. Given an open cover U_α, for each α, pick a largest ball $B_\alpha := B_{1/2^n}(x_\alpha) \subseteq U_\alpha \cap U_\beta$ for some $\beta < \alpha$ ($B_0 \subseteq U_0$); let $V_\alpha := U_\alpha \smallsetminus \bigcup_{\beta<\alpha} \overline{\frac{1}{2}B_\alpha}$. Then V_α is a locally point-finite refinement of U_α. For each x, let $m_x := \frac{1}{2}\sup\{\, r : \exists \alpha,\ B_r(x) \subseteq V_\alpha \,\}$, $W_\alpha := \bigcup\{\, B_{m_x/2}(x) : V_\alpha$ is first to contain $B_{m_x}(x) \,\}$ is the required locally finite cover.

2. A space is metrizable $\Leftrightarrow$ paracompact T_2 locally metrizable (Smirnov)
   $\Leftrightarrow$ T_6 with a σ-discrete basis (Bing).

3. There are no coproducts. The image by a 1-1 function is metrizable by $d(f(x), f(y)) := d(x, y)$. The pre-image by a countable number of functions (that separate points) is metrizable by $d(x, y) := \sum_n a_n d_n(f_n(x), f_n(y))$, where $\sum_n a_n < \infty$. In particular the countable product of metric spaces is again a metric space: take $f_n := \pi_n$ so $d((x_n), (y_n)) := \sum_n a_n d_n(x_n, y_n)$.

4. The completion of a metric space is a metric space.

   Proof: Take $\tilde{X}$ to be the space of Cauchy sequences, with $d((x_n), (y_n)) := \lim_{n\to\infty} d(x_n, y_n)$, identifying asymptotic sequences (indistinguishable by d).

5. A set is *bounded* when for $x, y \in A$, $d(x, y) \leqslant c$. Bounded sets are preserved by Lipschitz maps.

6. Every complete metric space is either σ-compact or contains $\mathbb{N}^{\mathbb{N}}$ (Hurewicz).

7. For complete metric spaces, a meagre subset has empty interior.

   Proof: Let $\bigcup_n A_n$ be closed nowhere dense sets with interior $U \neq \varnothing$. Let, iteratively, $B_{r_{n+1}}(x_{n+1}) \subseteq B_{r_n}(x_n) \smallsetminus A_n$ (with $B_{r_1}(x_1) \subseteq U \smallsetminus A_1$); x_n is a Cauchy sequence which converges to $x \in U \smallsetminus \bigcup_n A_n$.

8. For Lipschitz maps, let $c_f := \sup_{x\neq y} \frac{d(f(x), f(y))}{d(x,y)}$. Then $c_{f\circ g} \leqslant c_f c_g$.

9. (Banach) A Lipschitz map with $c < 1$ has a unique fixed point $f(x) = x$, that can be reached via $x_{n+1} := f(x_n)$.

10. The set of distances on X is closed under addition and multiplication by positive real numbers. Any function $f : X^2 \to \mathbb{R}^+$ can be turned into a metric by taking the shortest paths: replace f by $g(x, y) := f(x, y) + f(y, x)$, let $d(x, y) := \inf\{ \sum_{i=1}^{n-1} g(x_i, x_{i+1}) : x_1 = x, x_n = y \}$, identify pairs $d(x, y) = 0$.

11. The image of a metric space need not be a metric space (with the same induced metric). For example, the identity map on $\mathbb{R}$ from Euclidean to cofinite topology; or consider the subsets $A_r := \{ (x, y) \in \mathbb{R}^2 : xy = r \}$, then $X/A = \{ A_r : r \in \mathbb{R} \}$ is homeomorphic to $\mathbb{R}$, but not using the metric of $\mathbb{R}^2$ (because any two A_r are arbitrarily close to each other).

12. A metric space is *self-similar* by f when $d(f(x), f(y)) = \lambda d(x, y)$ $(\lambda \neq 1)$; i.e., $fB_r(x) = B_{\lambda r}(f(x))$. If f is invertible, cannot be bounded. For example, fractals.

13. An *ultrametric* satisfies the stronger triangle inequality $d(x, y) \leqslant \max(d(x, z), d(z, y))$ (e.g. discrete metric). Every triangle is isosceles; so any two balls are either disjoint or concentric; so $B_r(x)$ partition X for any fixed r; hence balls (and their "surface") are clopen; hence 0-D. Conversely, a p-'metric' satisfies $d(x, y) \leqslant \sqrt[p]{d(x, z)^p + d(z, y)^p}$ $(0 < p)$.

5.3 Lindelöf T_3 Spaces

They are paracompact T_2.

Proof: Given an open cover, each x is in some U_x, so there are open sets $x \in V_x \subseteq \overline{V_x} \subseteq U_x$. Hence there is a countable sub-cover V_n; let $W_n := U_n \setminus \overline{V_1 \cup \cdots \cup V_{n-1}}$. For any $x \in X$, let n be the minimum such that $x \in \overline{V_n}$; then $x \in W_n$ (and $x \in W_i \Rightarrow i \leqslant n$), so these form a locally finite sub-cover.

Examples:

1. Separable paracompact T_2 spaces.

   Proof: Given countable dense set A and open cover U_i; for $x \in U_x$, there is an open set $x \in W_x \subseteq \bar{W}_x \subseteq U_x$; the open cover W_x has a locally finite refinement V; hence choose the countable V_x for $x \in A$; since V is locally finite, $\bigcup_i U_i \supseteq \bigcup_n \bar{V}_n = \overline{\bigcup_n V_n} \supseteq \bar{A} = X$.

2. Second countable T_3 spaces $\Leftrightarrow$ separable metric spaces $\Leftrightarrow$ Lindelöf metric spaces; they are embedded in $[0, 1]^{\mathbb{N}}$ (Urysohn metrization theorem).

   Proof: A metric space with dense subset x_n, has countable base $B_{1/m}(x_n)$. For each n, the open cover $B_{1/n}(x)$ of a Lindelöf metric space has a countable subcover. Second countable T_3 are Lindelöf paracompact, hence there is a locally finite partition of unity f_n subordinate to the base U_n.

Let $J : x \mapsto (f_n(x))_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ (a metric space); J is obviously 1-1 and continuous; J^{-1} is continuous: if $x \in U$ is open in X, there is $x \in U_n \subseteq \overline{U_n} \subseteq U$, so $\sum_{i\leqslant n} f_i(x) = 1$; but $y \notin U \Rightarrow f_i(y) = 0$, so $d(J(x), J(y)) = \sum_i 2^{-i}|f_i(x) - f_i(y)| \geqslant \frac{1}{2^n}$.

Complete separable metric spaces are covered by (i.e., are a quotient of) $\mathbb{N}^{\mathbb{N}}$; compact metric spaces embed as closed subsets of $[0,1]^{\mathbb{N}}$; 0-D separable metric spaces are embedded in $2^{\mathbb{N}}$.

Examples: Totally bounded metric spaces (and subspaces such as locally compact metric spaces).

3. σ-compact T_3 spaces (i.e., admit a countable cover of compact sets); e.g. second countable locally compact T_2 (metric) spaces; connected locally compact paracompact T_2 spaces (since take a locally finite cover of compact neighborhoods U_i; starting with U_0, take a finite $U_1, \ldots, U_n$ that intersect it, then finite others that intersect these, etc. ; then $\overline{\bigcup_n U_n} = \bigcup_n \bar{U}_n = \bigcup_n U_n = X$).

(Arhangel'skii) When first countable, $|X| \leqslant 2^{\mathbb{N}}$.

5.4 Compact T_2 Spaces

Equivalently, totally bounded complete Tychonoff spaces.

Proof: $B_r(x)$, $x \in K$, has a finite sub-cover. If $\mathcal{F} \not\to x$, then $\exists A \in \mathcal{T}, x \in A, A \notin \mathcal{F}$, so a finite sub-cover, yet $\exists B_i, y \in B_i \Rightarrow B_i \subseteq B_{A_i}(y)$. If V_j cover K totally bounded then $K \subseteq B_r(F)$, so some $B_r(x_1)$ needs an infinite number of V_j for cover; continue with $r \to 0$ to get a Cauchy sequence $x_n \to x$; but $B_r(x_n) \subseteq B_s(x) \subseteq V$ is a finite cover.

1. Quasi-components are components. (Hence totally disconnected compact spaces are 0-D.)

   Proof: Suppose $Q = A \cup B$, $A \subseteq U$, $B \subseteq V$, $A \cap B = \varnothing$; let $C := (U \cup V)^{\mathsf{c}}$ compact, and let C_i be the clopen sets containing Q; then C_i^{c} cover C, so the clopen set $F^{\mathsf{c}} := C_1^{\mathsf{c}} \cup \cdots \cup C_n^{\mathsf{c}}$ covers C, and $Q \subseteq F$; but $U \cap F$, $V \cap F$ are clopen ($\overline{U \cap F} \subseteq \bar{U} \cap F \cap (U \cup V) = U \cap F$), so A, B are separated by a clopen set, $B = \varnothing$ and Q is connected.

2. Any 1-1 continuous map from a compact T_2 space to a T_2 space is an embedding (since it preserves closed/compact sets).

   Hence the topology of a compact T_2 space is rigid (since if X' is weaker, then the identity map $X \to X'$ is an embedding; if finer, use $X' \to X$).

3. Any continuous map between compact T_2 spaces is uniformly continuous.

   Proof: For any $V \subseteq Y$, there is a W such that $W \circ W \subseteq V$; for each y there is $B_{W_y}(y) \subseteq B_{W^{-1}}(y)$; as f is continuous, $f^{-1}B_{W_x}(f(x))$ is an open cover of X, so it has a refinement U; i.e., for any $x \in X$, there is an $a \in X$, $B_U(x) \subseteq f^{-1}B_{W_a}(f(a))$, so $fB_U(x) \subseteq B_{W_a}(f(a)) \subseteq B_{W^{-1}\circ W}(f(x)) \subseteq B_V(f(x))$.

4. $C(X) \cong C(Y)$ (isometric) $\Leftrightarrow X \cong Y$.

5. A *Boolean* space (or Stone space) is a 0-D compact T_2 space. The topological space associated with a Boolean algebra is a Boolean space, and the (clopen) topology of a Boolean space is a Boolean algebra.

6. Stone-Cech compactification of a set: given any set A, let $\beta A := \{\, \mathcal{F} : \text{maximal filter} \,\}$ with $\mathcal{S}$ being $U_a = \{\, \mathcal{F} : a \in \mathcal{F} \,\}$ for any $a \subseteq A$. βA is a Boolean space.

7. For Y $T_{3.5}$ and X compact, $C(X, Y)$ is the space of uniform convergence.

8. Every compact metric space is the continuous image of the Cantor set.

   For example, there is a space-filling path $C \to [0,1]$, $\boldsymbol{x} \mapsto \sum_n \frac{x_n}{2^{n+1}}$, extended to $C \cong C^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$, extended to $[0,1] \to [0,1]^{\mathbb{N}}$ by Tietze's theorem.

   Space-filling paths: Connected locally connected second countable compact T_2 spaces are the images of $[0,1]$ (Hahn-Mazurkiewicz).

9. A countable compact T_2 space is a second countable metric space. More generally, the weight is at most $|X|$.

   Proof: For $m, n \in X$ and base $\mathcal{U}$, $m \in U_n \subseteq n^{\mathsf{c}}$, so X is first countable; hence second countable T_3 and metrizable.

6 Measurable Spaces

A *generalized measure* is a pseudo-metric on subsets, $d : 2^X \to [0, \infty]$, which is invariant under symmetric differences $d(A \triangle C, B \triangle C) = d(A, B)$ (i.e., adding or removing the same points from A, B does not change their distance); hence $d(A, B) = d(A \triangle B, \varnothing) =: \mu(A \triangle B)$. Equivalently, it is a mapping $\mu : 2^X \to [0, \infty]$ such that

$$\mu(A \triangle B) \leqslant \mu(A) + \mu(B), \qquad \mu(\varnothing) = 0.$$

It follows that

1. $\mu(A \sqcup B) \leqslant \mu(A) + \mu(B)$, $|\mu(A) - \mu(B)| \leqslant \mu(A \triangle B)$

2. μ is continuous on sets of finite measure, $A_n \to A \Rightarrow \mu(A_n) \to \mu(A)$.

   Proof: $\mu(A) = \mu(A \triangle B \triangle B) \leqslant \mu(B) + \mu(A \triangle B)$. $|\mu(A_n) - \mu(A)| \leqslant \mu(A \triangle A_n) \to 0$

3. If $A_n \to A$ then $A_n \triangle B \to A \triangle B$.

Sets are measure-indistinguishable when $\mu(A \triangle B) = d(A, B) = 0$, denoted $A = B$ a.e. (equal *almost everywhere*). Sets with $\mu(A) = 0$ are called *null*, when $A = \varnothing$ a.e..

Example: The Boolean algebra generated from a given partition A_i (atoms), and weights $\mu(A_i) := w_i$.

A **measure** is the special case where

$$\mu(E_1 \sqcup E_2 \sqcup \cdots) = \mu(E_1) + \mu(E_2) + \cdots$$

It follows that

1. $E \subseteq F \;\Rightarrow\; \mu(E) \leqslant \mu(F)$
2. $\mu(E \triangle F) = \mu(E \smallsetminus F) + \mu(F \smallsetminus E)$
3. $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$
4. $\mu(\bigcup_n E_n) \leqslant \sum_n \mu(E_n)$
5. $E_n \subseteq E_{n+1} \;\Rightarrow\; \mu(\bigcup_n E_n) = \sup_n \mu(E_n)$
6. $E_n \supseteq E_{n+1} \;\Rightarrow\; \mu(\bigcap_n E_n) = \inf_n \mu(E_n)$ OR $\inf_n \mu(E_n) = \infty$
7. $E_n \to E$ pointwise $\;\Rightarrow\;$ E is measurable AND $\mu(E_n) \to \mu(E)$ if $E_n \subseteq F$, $\mu(F) < \infty$.
8. The countable sum of measures is another measure.

Proof: Pointwise convergence of sets means $\forall x \in E, \exists N, n \geqslant N \;\Rightarrow\; x \in E_n$ AND $\forall x \notin E, \exists N, n \geqslant N \;\Rightarrow\; x \notin E_n$. Let $A_N := \{\, x : n \geqslant N \;\Rightarrow\; x \in E_n \,\} = \bigcap_{n \geqslant N} E_n$, $B_N := \{\, x : n \geqslant N \;\Rightarrow\; x \notin E_n \,\} = \bigcap_{n \geqslant N} E_n^{\mathsf{c}}$, both increasing. Then $E = \bigcup_N A_N$ is measurable and $\mu(E) = \sup_N \mu(A_N)$.

Not all subsets need be measurable; the measurable sets form a σ-subalgebra $\sigma[\mu] \subseteq 2^X$, closed under countable unions and complements.

A measurable space can be 'completed' by adding the sets that are indistinguishable from measurable sets with $\mu(F) := \mu(E)$ when $F = E$ a.e., i.e., by adding the null sets; (note there may be many more indistinguishable subsets than measurable ones e.g. the Borel sets in $\mathbb{R}$ are $2^{\mathbb{N}}$ in number, but the number of null sets, such as the subsets of the Cantor set, is $2^{\mathbb{R}}$).

Examples: The discrete (counting) measure where $\delta_A(E) := \#(E \cap A)$ for any subset $E \subseteq X$. The 'Lebesgue' σ-algebra of a topological space $\sigma[\mathcal{T}]$: the completion of the Borel σ-algebra (generated by the open sets). Products of measure spaces have a measure $\xi(E \times F) = \mu(E)\nu(F)$.

Any generalized measure $m^* : 2^X \to [0, \infty]$ with the properties

$$m^*(\varnothing) = 0, \quad A \subseteq B \;\Rightarrow\; m^*(A) \leqslant m^*(B),$$
$$m^*(A \cup B) \leqslant m^*(A) + m^*(B), \quad A_n \to A \;\Rightarrow\; m^*(A_n) \to m^*(A)$$

gives rise to a complete measure: Let a set E be measurable when for any subset A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^{\mathsf{c}}),$$

and let $\mu(E) := m^*(E)$. It follows that if E, F are measurable then so are E^c and $E \cup F$, since

$$\begin{aligned}
&m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c) \\
&= m^*(A \cap (E \cup F)) + m^*(A \cap E^c \cap F^c) \\
&= m^*(A \cap (E \cup F) \cap E) + m^*(A \cap (E \cup F) \cap E^c) + m^*(A \cap E^c \cap F^c) \\
&= m^*(A \cap E) + m^*(A \cap F \cap E^c) + m^*(A \cap E^c \cap F^c) \\
&= m^*(A \cap E) + m^*(A \cap E^c) = m^*(A).
\end{aligned}$$

Indistinguishable sets ($m^*(E \triangle F) = 0$) are measurable

$$\begin{aligned}
m^*(A) &\leqslant m^*(A \cap (E \cup F)) + m^*(A \cap (E^c \cap F^c)) \\
&\leqslant m^*(A \cap E) + m^*(A \cap (F \smallsetminus E)) + m^*(A \cap E^c) \\
&\leqslant m^*(A) + m^*(E \triangle F) = m^*(A).
\end{aligned}$$

Proposition 1

Radon

A finitely additive measure on compact subsets extends to a regular Borel measure.

PROOF: For U open, let $m(U) := \sup\{\, m(K) : K \subseteq U,\ \text{compact}\,\}$ (well-defined and increasing); $m(U_1 \cup U_2) \leqslant m(U_1) + m(U_2)$ since $K \subseteq U_1 \cup U_2 \ \Rightarrow\ K = K_1 \cup K_2$, $K_i \subseteq U_i$.

$$m(\bigcup_n U_n) \leqslant \sum_n m(U_n)$$

since $K \subseteq \bigcup_n U_n \ \Rightarrow\ K \subseteq \bigcup_{n=1}^N U_n \ \Rightarrow\ m(K) \leqslant \sum_{n=1}^N m(U_n)$.

For any subset A, let $m(A) := \{\, m(U) : A \subseteq U,\ \text{open}\,\}$ (well-defined and increasing); $m(\bigcup_n A_n) \leqslant \sum_n m(A_n)$ since pick $A_n \subseteq U_n$ open, $m(U_n) \leqslant m(A_n) + \epsilon/2^n$; then $m(\bigcup_n A_n) \leqslant m(\bigcup_n U_n) \leqslant \sum_n m(U_n) \leqslant \sum_n m(A_n) + \epsilon$.

Thus m is an outer measure that can be extended to a measure μ. This is a Borel measure since for any open U and any subset A, pick

$$\begin{gathered}
V \supseteq A \text{ open}, m(V) \leqslant m(A) + \epsilon \\
K_1 \subseteq U \cap V \text{ compact}, m(K_1) \geqslant m(U \cap V) - \epsilon \\
K_2 \subseteq V \smallsetminus K_1 \text{ compact}, m(K_2) \geqslant m(V \smallsetminus K_1) - \epsilon,
\end{gathered}$$

$$\begin{aligned}
m(A \cap U) &\leqslant m(V \cap U) \leqslant m(K_1) + \epsilon \\
m(A \cap U^c) &\leqslant m(V \cap K_1^c) \leqslant m(K_2) + \epsilon \\
\therefore m(A \cap U) + m(A \cap U^c) &\leqslant m(K_1 \cup K_2) + 2\epsilon \\
&\leqslant m(V) + 2\epsilon \leqslant m(A) + 3\epsilon
\end{aligned}$$

Thus U is measurable.

$\square$

1. Contains the measure space generated by the finite-measure sets $\mathcal{M}_f$ (with $\mu(E) = \lim_n \mu(E_n)$ for $E_n \to E$; this is well-defined since $E_n \to E$ and $F_n \to F \;\Rightarrow\; E_n \cup F_n \to E \cup F$ and $E_n \setminus F_n \to E \setminus F$; so
$$\begin{aligned}\mu(E \cup F) = \lim_n \mu(E_n \cup F_n) &= \lim_n \mu(E_n) + \mu(F_n) - \mu(E_n \cap F_n)\\ &\leqslant \mu(E) + \mu(F)\end{aligned}$$
$$\begin{aligned}A_n \to \varnothing \;\Rightarrow\; &E_{nm} \to A_n \to \varnothing\\ \Rightarrow\; &E_{nm} \to \varnothing \;\Rightarrow\; \mu(E_{nm}) \to 0 \;\Rightarrow\; \mu(A_n) \to 0\end{aligned}$$
so
$$\mu(A) = \inf\{\, \mu(E) : A \subseteq E \in \mathcal{M} \,\} = \sup\{\, \mu(E) : E \subseteq A, E \in \mathcal{M} \,\}$$
Proof:
$$\begin{aligned}\mu(A) < \infty \Leftrightarrow A \in \mathcal{M}_f &\Leftrightarrow E_n \to A, E_n \in \mathcal{M}\\ &\Leftrightarrow E_n \nearrow A \text{ AND } \tilde{E}_n \searrow A \;\Rightarrow\; \mu(E_n) \nearrow \mu(A) \text{ AND } \mu(\tilde{E}_n) \searrow \mu(A),\end{aligned}$$
and $\mu(A) = \infty \;\Rightarrow\; \forall E \supseteq A, \mu(E) = \infty$ and $\mu(E_n) \nearrow \infty$,

2. Can be extended to the locally measurable sets i.e., $\forall E$ measurable, $A \cap E$ is measurable, with measure
$$\mu(A) = \sup_{E \subseteq A, E \in \mathcal{M}} \mu(E) = \inf_{A \subseteq V, V open} \mu(V)$$
In this sense the measure is finite only when the set is measurable (since $\mu(A \cap E_n) \to \mu(A) < \infty \;\Rightarrow\; A \cap E_n \to A \;\Rightarrow\; A \cap E = A a.e \;\Rightarrow\; A \in \mathcal{M}$)

   Proof: The locally measurable sets are closed under complements since for A locally measurable and E measurable $A^{\mathsf{c}} \cap E = E \setminus (A \cap E) \in \mathcal{M}$ so A^{c} is locally measurable; if A_n are locally measurable, then $\bigcup_n A_n \cap E = \bigcup_n (A_n \cap E)$ is measurable; $\mu(A \sqcup B) = \sup_E \mu(A \sqcup B \cap E) = \sup_E \mu(A \cap E) + \mu(B \cap E) = \mu(A) + \mu(B)$; $A_n \to \varnothing \;\Rightarrow\; A_n \cap E \to \varnothing$, so if $\forall n, \mu(A_n) = \infty$ then $\exists E_n, \mu(A_n \cap E_n) \geqslant n$, so $N \leqslant \mu(A_n \cap \bigcup_n E_n) \to 0$ a contradiction, so for large N, $\mu(A_N) < \infty$, so $\mu(A_N) \to 0$.

3. Note that when X is covered by a countable number of measurable sets, then the locally measurable sets are the measurable sets.

4. If a measurable space is generated by a countable number of sets A_n of finite measure that together cover X, and E is of finite measure, then $\forall \epsilon > 0, \exists B$ measurable, $d(A, B) < \epsilon$.

5. In a second countable locally compact T_2 space, the compact sets generate the Borel sets.

   Proof: V open $\Rightarrow$ $K_n \nearrow V$ so $\mu(E) = \sup\{\,\mu(K) : K \subseteq E\,\} = \inf\{\,\mu(U) : E \subseteq U\,\}$ since $E \subseteq \bigcap_n K_n \;\Rightarrow\; \mu(K_n \smallsetminus E) = \inf\{\,\mu(U) : K_n \smallsetminus E \subseteq U\,\}$, so $\mu(K_n \cap E) = \mu(K_n) - \mu(K_n \smallsetminus E) = \sup(\mu(K_n) - \mu(U)) \leqslant \sup \mu(K_n \smallsetminus U) = \sup \mu(K) = \sup\{\,\mu(K) : K \subseteq E\,\}$.

6. Subspaces: $\mu(E|A) = \mu(E \cap A)$ (for probability, normalize to $\mu(E|A) = \mu(E \cap A)/\mu(A)$).

   Products: $X \times Y$ the measurable sets are generated from $E \times F$, with $\mu(E \times F) = \mu(E)\mu(F)$.

   E and F are said to be independent when $\mu(E \cap F) = \mu(E)\mu(F)$ (so $\mu(E|F) = \mu(E)$). For a partition into subsets F, $\mu(E) = \sum_F \mu(E|F)\mu(F)$.

7. *Hausdorff measure* for metric spaces: $\mu_\alpha(E) := \sup\{\,\sum_i r_i^\alpha : E \subseteq \bigcup_i B_{r_i}(x_i), r_i < \delta/2\,\}$.

8. Measurable Functions: $\forall V$ measurable, $f^{-1}V$ is measurable; similarly locally measurable functions; (f is called Baire measurable when for V open, $f^{-1}V$ is Borel); e.g. the constants, 1_A for A locally measurable, continuous functions are Borel measurable; we write $f = g a.e.$ when $f(x) = g(x)$ a.e.x; the composition of measurable functions is measurable; (f, g) is measurable $\Leftrightarrow$ f, g are measurable.

   For a Borel measure space, f is measurable when $f^{-1}V$ is measurable for V Borel; continuous functions are therefore measurable.

The *information* of a set E wrt X is $\log_2 \frac{\mu(X)}{\mu(E)}$ (measured in 'bits'); the *entropy* of a partition of X is $H = -\sum_i \mu(E_i) \log_2 \mu(E_i)$. The entropy is maximum when the partition sets have equal measure; it is additive and continuous. $H(E, F) = H(E|F) + H(F)$; $H(f(E)) \leqslant H(E)$ with equality when f is invertible (since $H(f(E)|E) = 0$, so $H(E) = H(E, f(E)) = H(f(E)) + H(E|f(E))$); If E, F are independent then $H(E|F) = H(E)$; $H(E \times F) \leqslant H(E) + H(F)$.

Radon measures also satisfy (i) finiteness: compact subsets have finite measure, and (ii) regularity: for measurable E, open U,

$$\mu(U) = \sup\{\,\mu(K) : K \text{ compact}, K \subseteq U\,\}, \qquad \mu(E) = \inf\{\,\mu(U) : E \subseteq U \text{ open}\,\}$$

7 Examples

Finite spaces have a unique base of open sets $U_x := \bigcap\{\,U \in \mathcal{T} : x \in U\,\}$. The space is determined by its specialization order, with $\mathcal{F} \to x \Leftrightarrow {\downarrow} x \in \mathcal{F}$; they are in 1-1 correspondence with finite ordered spaces.

The number of finite (T_0) topologies (up to isomorphism) are:

Points	0	1	2	3	4	5	6	7	8	9	10
Topologies	1	1	2	5	16	63	318	2045	16999	183231	2567284

For example, two-point topologies are either discrete or Sierpinski space **2**.

Discrete spaces Σ_A are the finest possible:) the only convergent filters are $\mathcal{F}(x)$, all functions from Σ_A are continuous. It is a metric space $d(x,y) = 1-\delta_{xy}$; every subset is clopen, so 0-D, locally compact and locally connected (in fact, Σ_A are the only locally connected totally disconnected spaces). The separable discrete spaces are $\Sigma_{\mathbb{N}} = \mathbb{N}$ and Σ_n; only the latter are compact.

The one-point compactification of Σ_A is called a *Fort* space – a compact T_2 space that is not first countable except for $A \subseteq \mathbb{N}$ when it is second countable.

The 'extension' $X \# Y$ of two topological spaces is $X \sqcup Y$ with open sets being $X \cup V$ and $U \in \mathcal{T}(X)$.

Excluded point topology: $\Sigma_A \# \{0\}$ (every open set excludes 0, except X); the closed sets contain 0, except $\varnothing$; every proper filter converges to 0. So compact, connected, and not T_1. First countable and locally connected but separable (and second countable) iff $A \subseteq \mathbb{N}$.

Particular point topology: $\{0\} \# \Sigma_A$ (every open set contains 0, except $\varnothing$); the closed sets do not contain 0, except X. So connected, locally connected and not T_1; separable since $\overline{\{0\}} = X$, but Lindelöf iff $A \subseteq \mathbb{N}$.

Evenly spaced topology on $\mathbb{Z}$: with the basic open sets $a + b\mathbb{Z}$ (arithmetic sequences); these are clopen, so X is 0-D, second countable; has a metric $|n| = \frac{1}{2^k}$ where k is the largest power of the prime decomposition of n; every open set, except $\varnothing$, is infinite; not compact or complete (e.g. the sequence of primes does not converge) and not locally compact. ($\bigcup_{p \text{ prime}} p\mathbb{Z} = \{\pm 1\}^{\mathsf{c}}$ not closed, so the primes cannot be finite.)

p-adic $\mathbb{Z}$: basic open sets $a + p^k\mathbb{Z}$; metric $|m| = |p^k \cdots| := \frac{1}{2^k}$, clopen balls, so 0-D; countable; not locally compact.

Euclidean spaces

$\mathbb{Q}$: topology generated by open intervals $]p,q[$, hence second countable; has an incomplete metric $|p-q|$ (e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is divergent Cauchy); it is the unique countable perfect T_3 space; $]a,b[$ with a,b irrational form a clopen base, so it is 0-D. Bounded sets are totally bounded. The compact subsets can be complicated. The one-point compactification of $\mathbb{Q}$ is connected, T_1 (but not T_2 or first countable at ∞); but removing ∞ gives a totally disconnected space.

$\mathbb{R}$ is the completion of $\mathbb{Q}$. The intervals are the only separable connected linearly ordered spaces; $[0,1]$ is the unique compact metric space that is connected and locally connected with two non-cut points. $\mathbb{Q}^{\mathsf{c}}$ is 0-D but not meagre. The one-point compactification of $\mathbb{R}^N$ is $\mathbb{S}^N$; $\mathbb{S}^2$ every automorphism has a fixed point. Every 1-1 continuous function $f : U \to \mathbb{R}^N$ where $U \subseteq \mathbb{R}^N$ is open, is an embedding, and fU is open.

$\mathbb{R}/\mathbb{N}$, the infinite rose with stalk; connected, and locally connected; σ-compact but not locally compact or first countable at 0; separable by $\mathbb{Q}$.

Hawaiian earring, the one-point compactification of $\mathbb{R} \smallsetminus \mathbb{Z}$; connected but not locally connected, has no universal cover.

Long Line consists of $|\mathbb{R}|$ intervals $[0,1[$ joined together with the order topology ($\omega_1 \times [0,1[$ in each direction); it is first countable T_5 but not separable, Lindelöf, or paracompact.

$[0,\omega_1[$ is the linear order of countable ordinals; its extent is $|\mathbb{N}|$ but its spread is $|\mathbb{R}|$.

Baire Spaces are infinite products of Euclidean spaces (with pointwise convergence). They are all T_2.

$\mathbb{N}^{\mathbb{N}}$ (*Baire space*) has a complete metric $d(x_n,y_n)=\inf\{\,\frac{1}{n}:i\leqslant n \Rightarrow x_i=y_i\,\}$; embedded in $\mathbb{R}$ as the irrational numbers (via continued fractions). The set of sequences $N_{\boldsymbol{x}}$ that agree with a finite sequence $\boldsymbol{x}$ form a countable clopen base; so 0-D and second countable; homogeneous, so perfect. $\bar{A}$ can be obtained by forming the tree of finite initial sequences of A, and then taking all sequences that are generated by this tree. A closed set is compact when the tree it generates has finite branching at nodes; hence not σ- or locally compact.

$2^{\mathbb{N}}$ (*Cantor space*): (or any countable product of finite discrete spaces such as $3\times5\times2\times8\times\cdots$); a homogeneous, compact subspace of Baire space; can be embedded in $\mathbb{R}$ as a nowhere dense set via ternary expansion; separable by the 'finite' $0-1$ sequences (endpoints). It is the only perfect, totally disconnected, compact metric space. Any perfect complete metric space contains a copy of the Cantor set (in the construction of the usual Cantor set, replace the intervals by nested closed balls of radii $1/3^n$).

Proof: Let X be perfect, totally disconnected, compact metric space; then $x=Q(x)$ quasi-component. Hence the clopen sets of diameter 1 cover X; of the finite sub-covers, there are covers of clopen sets of diameter $1/2^n$; hence there is a homeomorphism $2^{\mathbb{N}}\to X$.

$\mathbb{N}^{\mathbb{R}}$ basic open sets are clopen, so 0-D; is $T_{3.5}$ but not T_4; separable but not first countable; contains $\mathbb{N}^{\mathbb{N}}$, so not σ-compact or locally compact. $2^{\mathbb{R}}$ is compact but not sequentially compact: the sequence $f_n(x):=n^{th}$ digit in binary expansion of x, has no convergent subsequence.

$\mathbb{R}^{\mathbb{N}}$ is second countable (separable using finite rational sequences); connected; not locally compact (e.g. $(\delta_{in})_n$ has no convergent subsequence at 0; then scale and translate).

$\mathbb{R}^{\mathbb{R}}$ is separable (using rational step functions) but not first countable.

$[0,1]^{[0,1]}$ is a connected compact T_2 space; not first countable. Contains the closed separable first countable subspace, called *Helly* space: $\{\,f\in[0,1]^{[0,1]}:$ increasing$\,\}$.

$C[0,1]$ with uniform topology is a complete metric space; separable (by rational polynomials); not locally compact or σ-compact (every neighborhood has a sequence which does not converge); connected and locally connected (balls are convex). $\mathbb{R}^A$ with uniform topology is separable iff second countable iff A finite.

Exotic Lines

Sorgenfrey line - generated by $[a,b[$; e.g. $-\frac{1}{n}\not\to 0$ but $\frac{1}{n}\to 0$; $[a,b[$ is clopen, so finer than $\mathbb{R}$; hence 0-D; T_6; the only compact subsets are countable (hence not σ-compact or locally compact); Lindelöf, separable (by $\mathbb{Q}$), first countable (by $[a,r[$, $r\in\mathbb{Q}$), but not second countable or metrizable.

Michael's line - generated by open intervals and irrational singletons; hence finer than $\mathbb{R}$; $\mathbb{Q}$ is closed and $[a,b]$ with $a,b\in\mathbb{Q}^{\mathsf{c}}$ are clopen; first countable and

0-D, but not separable (irrationals are uncountable); paracompact T_2, but not metrizable, Lindelöf or locally compact.

Moore plane - $\mathbb{R}\times\mathbb{R}^+$ with neighborhoods $B_\epsilon(x,y)$ ($\epsilon \leqslant y$) and $B_y(x,y)\cup\{(x,y)\}$. Sequences can only converge to $\mathbb{R}\times\{0\}$ from above. Is $T_{3.5}$ but not T_4; separable (by $\mathbb{Q}^2$) but not second countable (contains the discrete subspace $\mathbb{R}$); connected.

Deleted Radius: $\mathbb{R}^2$ with open sets generated from disks with right horizontal radius deleted.

See π-Base for many more examples.

Review on filters and nets in 2^X

A *filter* $\mathcal{F}$ is a collection of subsets of X that is upper-closed and lower-directed; a *net* $(x_i)_{i\in I}$ is a map $I\to X$ from a directed set (i.e., any two elements have an upper-bound); a *sequence* is the special case of a mapping from $\mathbb{N}$ (see Ordered Sets):

1. $\forall A,B\in\mathcal{F},\ A\cap B\in\mathcal{F}$, and $A\subseteq C \ \Rightarrow\ C\in\mathcal{F}$.

2. The intersection of filters is a filter, so a filter can be generated from any collection of sets via $\mathfrak{F}(\mathcal{S}) := \uparrow\{A_1\cap\cdots\cap A_n : A_i\in\mathcal{S}, n\in\mathbb{N}\}$, e.g. $\mathfrak{F}(A) := \{B\subseteq X : A\subseteq B\}$, $\mathfrak{F}(\varnothing) = 2^X$ (the improper filter).

   A filter on a finite set must be of this form, $\mathcal{F}=\mathfrak{F}(A)$ where $A=\bigcap\mathcal{F}$. But infinite sets may have 'free' filters with $\bigcap\mathcal{F}=\varnothing$, e.g. $\mathcal{F}=\uparrow\{n,n+1,\dots\}$ in $\mathbb{N}$, or $\mathcal{F}=\uparrow\{\,]0,1/n[\,: n\in\mathbb{N}\}$ in $\mathbb{Q}$.

3. $A\subseteq B\ \Leftrightarrow\ \mathfrak{F}(B)\subseteq\mathfrak{F}(A)$.

4. A proper filter can be thought of informally as generated by some nested collection of subsets: well-order $\mathcal{F}$ and take $A_1\supseteq A_1\cap A_2\supseteq A_1\cap A_2\cap A_\omega\supseteq\cdots$.

5. The set of filters form a complete distributive lattice

$$\begin{aligned}
&\mathcal{F}\wedge\mathcal{G}=\mathcal{F}\cap\mathcal{G},\\
&\mathcal{F}\vee\mathcal{G}=\mathfrak{F}(\mathcal{F}\cup\mathcal{G})=\uparrow\{A\cap B : A\in\mathcal{F}, B\in\mathcal{G}\},\\
&\textstyle\bigvee_i\mathcal{F}_i=\mathfrak{F}(\bigcup_i\mathcal{F}_i)
\end{aligned}$$

   For example, if $\mathcal{F}\cap\mathcal{G}\subseteq\mathcal{H}$ proper, then $\mathcal{F}\vee\mathcal{H}$, or $\mathcal{G}\vee\mathcal{H}$, is proper (since $\mathcal{H}=\mathcal{H}\vee(\mathcal{F}\cap\mathcal{G})=(\mathcal{H}\vee\mathcal{F})\cap(\mathcal{H}\vee\mathcal{G})$). $\mathfrak{F}(A)\cap\mathfrak{F}(B)=\mathfrak{F}(A\cup B)$, $\mathfrak{F}(A)\vee\mathfrak{F}(B)=\mathfrak{F}(A\cap B)$.

6. Every proper filter $\mathcal{F}$ can be extended (refined) to a maximal filter $\mathcal{M}$: its complement is an ideal, $A\cup B\in\mathcal{M}\ \Leftrightarrow\ A\in\mathcal{M}$ OR $B\in\mathcal{M}$, in particular $\forall A\subseteq X, A\in\mathcal{M}$ OR $A^{\mathsf{c}}\in\mathcal{M}$.

7. $\mathcal{F} = \bigcap\{\, \mathcal{M} : \text{maximal}, \mathcal{F} \subseteq \mathcal{M} \,\} = \bigvee\{\, \mathfrak{F}(A) : A \in \mathcal{F} \,\}$. If $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{M}$ maximal, then $\mathcal{F} \subseteq \mathcal{M}$ OR $\mathcal{G} \subseteq \mathcal{M}$.

8. Given a filter $\mathcal{F}$ on X and a function $f : X \to Y$, $f(\mathcal{F})$ is defined to be that filter generated by $f\mathcal{F} = \{\, fA : A \in \mathcal{F} \,\}$, namely $\{\, B \subseteq Y : f^{-1}B \in \mathcal{F} \,\}$. Similarly, $f^{-1}(\mathcal{F})$ is the filter generated by $\{\, f^{-1}A : A \in \mathcal{F} \,\}$; it is proper iff for all $A \in \mathcal{F}$, $A \cap \operatorname{im} f \neq \varnothing$.

   $$f(\mathfrak{F}(A)) = \mathfrak{F}(fA), \quad \mathfrak{F}(f^{-1}A) = f^{-1}\mathfrak{F}(A), \quad \mathcal{F} \subseteq ff^{-1}\mathcal{F}, \quad f^{-1}f\mathcal{F} \subseteq \mathcal{F}.$$

9. Maximal filters are mapped to maximal filters when f is onto (since $f\mathcal{M} \subset \mathcal{F} \;\Rightarrow\; \mathcal{M} \subseteq f^{-1}\mathcal{F}$).

10. A net $(x_i)_{i \in I}$ is said to be *eventually* in A, $x_i \twoheadrightarrow\!\in A$ when

    $$\exists j \in I,\ i \geqslant j \;\Rightarrow\; x_i \in A.$$

11. A net is the point version of a filter: every net generates a filter

    $$\mathfrak{F}(x_i) := \{\, A \subseteq X : x_i \twoheadrightarrow\!\in A \,\} = \uparrow\{\, \{\, x_i : i \geqslant j \,\} : j \in I \,\}$$

    conversely, pick a point from each set in a filter, $x_i \in A_i$, to form a net when ordered by $i \geqslant j \Leftrightarrow A_i \subseteq A_j$. For example, the filter corresponding to a constant net $x_i = x$ is $\mathfrak{F}(x)$.

12. A *subnet* is a composition $J \to I \to X$ such that $J \to I$ is increasing and $\forall a \in I, \exists j \in J, i_j \geqslant a$.

13. A *sequence* $\mathbb{N} \to X$ is an example of a net, and a *subsequence* $\mathbb{N} \to \mathbb{N} \to X$ (with $\mathbb{N} \to \mathbb{N}$ strictly increasing) an example of a subnet. But not every subnet of a sequence, $J \to \mathbb{N} \to X$ is a subsequence, e.g. $(a_1, a_1, a_4, a_3, \ldots)$.