More on Truth Tables

- Truth tables help when determining the truth or falsity of compound statements. For example:

  $(p \vee q \wedge r) \rightarrow \neg p \vee r$

p	q	r	$q \wedge r$	$p \vee q \wedge r$	$\neg p$	$\neg p \vee r$	$(p \vee q \wedge r) \rightarrow \neg p \vee r$
T	T	T	T	T	F	T	T
T	T	F	F	T	F	F	F
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	F	F	T	T	T
F	F	F	F	F	T	T	T

Logical Equivalence

- Two statements P and Q are **logically equivalent**, if they are true in excactly the same circumstances.
- Logical equivalence is written $\mathbf{P \equiv Q}$
- So...
  - $P \equiv \neg\neg P$
  - $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R \equiv P \wedge Q \wedge R$
  - $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$ **(DeMorgan)**
  - $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$ **(DeMorgan)**

Logical Equivalence Example

- Show that,

  $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Tautologies

- Statements like '*Andrew plays football or he does not* ' are evidently true whatsoever.
- Tautologies are statements that are always true irrespective of the truth values of their atomic statements.
- Again, a truth table may be used to determine whether a statement is a tautology.
- Check if $p \vee \neg p$ is a tautology.
- A tautology is logically equivalent to TRUE.

Contradictions

- A contradiction is the opposite of a tautology.
- Irrespective of the truth values of the atomic sentences, the statement is always false.
- Andrew plays football and he does not play football ($P \wedge \neg P$).
- A contradiction is logically equivalent to FALSE.

Duality

- Let X be a statement involving $\neg$, $\vee$ and $\wedge$.
- Let X* be the statement X with:
  - $\vee$ replaced by $\wedge$
  - $\wedge$ replaced by $\vee$
  - TRUE replaced by FALSE
  - FALSE replaced by TRUE
- X* is the **dual** of X
- Notes:

  $\mathbf{P} \equiv \mathbf{Q}$ then $\mathbf{P^*} \equiv \mathbf{Q^*}$

  $(\mathbf{P} \wedge \mathbf{Q}) \vee \mathbf{R}$ is the dual of $(\mathbf{P} \vee \mathbf{Q}) \wedge \mathbf{R}$

  $(\mathbf{P^*})^* \equiv \mathbf{P}$

Laws of Logic (1)

- Notation:
  - P, Q and R are atomic statements.
  - $\mathbf{T_0}$ is a tautology.
  - $\mathbf{F_0}$ is a contradiction.

- Commutative Law:

  $\mathbf{P \wedge Q \equiv Q \wedge P}$ $\qquad$ $\mathbf{P \vee Q \equiv Q \vee P}$

- Associative Law:

  $\mathbf{(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)}$ $\qquad$ $\mathbf{(P \vee Q) \vee R \equiv P \vee (Q \vee R)}$

Laws of Logic (2)

- Distributive Law:

  $\mathbf{P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)}$

  $\mathbf{P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)}$

- Identity Law:

  $\mathbf{P \wedge T_0 \equiv P}$ $\qquad$ $\mathbf{P \vee F_0 \equiv P}$

- Inverse Law:

  $\mathbf{P \wedge \neg P \equiv F_0}$ $\qquad$ $\mathbf{P \vee \neg P \equiv T_0}$

Laws of Logic (3)

- Double Negation Law:

  $\neg(\neg P) \equiv P$

- Idempotent Law:

  $P \wedge P \equiv P$ $\qquad$ $P \vee P \equiv P$

- DeMorgan's Laws:

  $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\qquad$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

Laws of Logic (4)

- Domination Law:

  $P \vee T_0 \equiv T_0$ $\qquad$ $P \wedge F_0 \equiv F_0$

- Absorption Law:

  $P \wedge (P \vee Q) \equiv P$ $\qquad$ $P \vee (P \wedge Q) \equiv P$

- Negation Law:

  $\neg T_0 \equiv F_0$ $\qquad$ $\neg F_0 \equiv T_0$

Satisfiability (1)

- Intuitively a sentence is satisfiable if it could be true, at least on logical grounds.
- It is irrelevant whether it is physically impossible. For example: RunsFasterThanLight(Andrew) is satisfiable.
- A contradiction can never be satisfied.
- A set of sentences is satisfiable if there is a circumstance under which all sentences are simultaneously true.

Satisfiability (2)

$$\text{Happy(Mark)} \vee \text{Happy(Andrew)}$$
$$\neg\text{Happy(Mark)}$$
$$\neg\text{Happy(Andrew)}$$

- In this example, all sentences are satisfiable.
- Any two are satisfiable.
- But all three can never be true simultaneously (i.e. the set of sentences is not satisfiable).

Logical Truth

- A sentence is logically true regardless of the circumstances.
- For example:

  Home(Andrew) ∨ ¬**Home(Andrew)**

  ¬(**Happy(Andrew)** ∧ ¬**Happy(Andrew)**)
- All tautologies are logically true, but not all logically true sentences are tautologies.

Logical Truth vs. Satifiability vs. Tautologies (1)

- Consider the sentence:

  [Box(a) ∧ **Box(b)]** ∨ ¬ **Box(c)**

A	B	C	A ∧ B	¬C	(A ∧ B) ∨ ¬C
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	T	F	F	T	T

Logical Truth vs. Satifiability vs. Tautologies (2)

- Now, suppose the abbreviation **A** is **Circle(C)**.
  [Circle(c) ∧ Box(b)] ∨ ¬ Box(c)

A	B	C	A ∧ B	¬C	(A ∧ B) ∨ ¬C
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	T	F	F	T	T

Rows are **SPURIOUS** since A and C can never be true together (c can never be a box and circle at the same time)

Logical Truth vs. Satifiability vs. Tautologies (3)

- To determine if a sentence S is satisfiable, logically true or a tautology:

1. Construct the true table for S.
2. S is a tautology IFF it is TRUE in every row.
3. Eliminate any spurious rows.
4. S is satisfiable IFF there is at least one row TRUE.
5. S is logically true IFF all the remaining rows are TRUE.