Free and Bound Variables

- If **P** is a WFF (well-formed-formula) and **v** is a variable then ∀**v P** and ∃**v P** are WFF too is **v** is bound to **P**.

∀x (Pet(x) → Noisy(x))

All occurences of x are bound within the scope of ∀

∃x (Pet(x) ∧ Noisy(y))

y is not bound within the scope of ∃

- A sentence is a WFF of it has no unbound (free) variables.

Notes on Quantification

- ∀ is often used when we wish to say things like 'every P is a Q'
  - ∀x (P(x) → Q(x))
- ∃ is often used when we wish to say things like 'There is a P which also has the property Q'
  - ∃x (P(x) ∧ Q(x))
- Sometimes the latter sentence is incorrectly translated to ∃x (P(x) → Q(x))
- The meaning is different – compare the truth tables for ∧ and →.

Vacuously True Sentences

- Consider the expression $\forall$**x (Pet(x)** $\rightarrow$ **Noisy(x))** in a world where pets do not exist.
- Clearly nothing can possibly satisfy the first part of the implication – Pet(x) = True, so from the truth table of $\rightarrow$, all the sentence will always be true.

p	**q**	**p** $\rightarrow$ **q**
T	T	T
T	F	F
F	T	T
F	F	T

- We conclude that a universal statement is Vacuously true in a world where the first part does not hold.

Complex Translations (1)

- Every small dog that is at home is happy:

  $\forall$**x (((Small(x)** $\wedge$ **Dog(x)** $\wedge$ **AtHome(x))** $\rightarrow$ **Happy(x))**

- A small dog at home is happy:

  $\exists$**x (Small(x)** $\wedge$ **Dog(x)** $\wedge$ **AtHome(x)** $\wedge$ **Happy(x))**

Complex Translations (2)

- Some boys sits on the left of some girl:

  $\exists$**x** $\exists$**y (Boy(x)** $\wedge$ **Girl(y)** $\wedge$ **LeftOf(x,y))**

- Every boy sits on the left of every girl:

  $\forall$**x** $\forall$**y ((Boy(x)** $\wedge$ **Girl(y))** $\rightarrow$ **LeftOf(x,y))**

Complex Translations (3)

- In the previous example, we had more than one quantifier on the 'outside'. This is called **Prenex** form.
- However, Prenex is not necessarily required. The previous 2 examples could have been expressed as:

  $\exists$x (Boy(x) $\rightarrow$ $\exists$y (Girl(y) $\wedge$ LeftOf(x,y))

  $\forall$x (Boy(x) $\rightarrow$ $\forall$y (Girl(y) $\rightarrow$ LeftOf(x,y))

Complex Translations (4)

- Using multiple quantifiers sometimes makes things slightly tricky. Consider:
  $\forall x \forall y\ ((Boy(x) \wedge Boy(y)) \rightarrow (LeftOf(x,y) \vee RightOf(x,y)))$

- This sentence may be read as "if x and y are boys, then either x is on the left of y or x is on the right of y".
- This makes the wrong assumption that x and y are distinct.

Complex Translations (5)

- To see why, consider:
  $\forall y\ ((Boy(Mark) \wedge Boy(y)) \rightarrow$
  $(LeftOf(Mark,y) \vee RightOf(Mark,y)))$

- But $\forall y$ includes Mark, so we can have:
  $\forall y\ ((Boy(Mark) \wedge Boy(Mark)) \rightarrow$
  $(LeftOf(Mark,Mark) \vee RightOf(Mark,Mark)))$

- It is clear that Mark cannot be left or right of himself.

Complex Translations (6)

- The wrong way was:

  $\forall x \forall y\ ((Boy(x) \wedge Boy(y)) \rightarrow (LeftOf(x,y) \vee RightOf(x,y)))$

- The correct way of making that statement is:

  $\forall x \forall y\ ((Boy(x) \wedge Boy(y) \wedge \mathbf{x \neq y}) \rightarrow (LeftOf(x,y) \vee RightOf(x,y)))$

Complex Translations (7)

- Universal and existential quantifiers may be mixed. Consider “every boy is on the left of a girl”:

  $\mathbf{\forall x\ (Boy(x) \rightarrow \exists y\ (Girl(y) \wedge LeftOf(x,y)))}$

- This can be expressed in Prenex form as:

  $\mathbf{\forall x \exists y\ (Boy(x) \rightarrow (Girl(y) \wedge LeftOf(x,y)))}$

Complex Translations (8)

- When mixing quantifiers, their order is important. Consider the following two sentences:

  ∀x∃y Likes(x,y)
  ∃y∀x Likes(x,y)

- The first states that "everyone (∀**x)** likes someone (∃**y)**".
- The second states that "there is someone (∃**y)** who everyone (∀**x)** likes".