Proof Systems

- The technique we used so far for establishing the validity of arguments was based on the evaluation of their corresponding truth tables.
- The a problem starts appearing when the number of variables grows large – the truth table starts becoming too complex to work out.
- We will consider the use of **Formal deductive Systems** to establish the validity of a sentence or argument.

Formal Deductive Systems

- In formal systems we use an algebraic technique.
- A Formal Deductive System (FDS) requires:
  - A formal language.
  - A set of axioms. These are basic formulae from which theorems are derived. Axioms don't need to be proven. They are fundamental stated facts.

    (e.g. $b \times 1 = b$)
  - Inference rules. These are rules that allow us to derive new sentences from existing ones.

More on Inference Rules

- Using Inference Rules we can derive new rules from existing ones.
- A common rule is called **Modus Ponens**:
  - If we know a fact **P**, and also know **P→Q**, then we know that **Q** is provable too.
- Another one is **Modus Tollens**:
  - If we know **Q** is false, and **P→Q**, then we know that **P** must be false as well.
  - **If there is fire here, then there is oxygen here.**
    There is no oxygen here.
    Therefore, there is no fire here.

Proofs vs. Theorems

- A proof is a sequence of formulae such that:
  - It is an axiom.
  - It is derivable from earlier formulae in the sequence using inference rules.
- A proof whose last element is the formulae **F**, is called the **Proof of F**.
- A formula **F** is a theorem in a system if there is a **Proof of F** in the system, denoted by σ^F.
- σ is the system.

Deduction

- A proof is derived from axioms.
- A deduction allows the possibility of a number of assumptions.
- So, a deduction is a sequence of formulae such that:
  - It is an axiom.
  - **A number of assumptions/hypothesis.**
  - It is derivable from earlier formulae in the sequence using inference rules.

System L (1)

- System L is a FDS.
- It requires the following objects:
  - An alphabet.
  - A set of WFFs.
  - A set of axioms (possibly empty).
  - A set of inference rules.
- The **alphabet** consists of: the set of propositional variables, punctuation symbols, and logical operators.

System L (2)

- If **P** is a **WFF**, it must conform to the following rules (Note that WFFs are written in uppercase):
  - **P** is a propositional variable, or
  - **P** is of the form **¬Q**, where **Q** is a **WFF**, or
  - **P** is of the form **(Q→R)** where **Q** and **R** are WFFs.
- In System L, we have only one rule – Modus Ponens.
- Axiom Schems:
  - **A1)** **(U→(V→U)**
  - **A2)** **((U→(V→W))→((U→V)→(U→W))**
  - **A3)** **(¬U→¬V)→(V→U)**

Deductions in L

- Given the hypothesis:

(1) (p→(q→r))→(~~p→~(q→r))

(2) p→(q→r)

(3) q→r

- Show that: **¬p**

Deduction:

(a) **~~p→~(q→r)**

by MP on (1) (2)

(b) **(~~p→~(q→r))→((q→r)→~p)**

by A3 ~p/U, (q→r)/V

(c) **(q→r)→~p**

by MP on (a) (b)

(d) **~p**

by MP on (3) (c)