UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.(Hons.) Year I
January/February 2007 Assessment Session

MAT1091 Mathematical Methods I: Matrices & O.D.E. **3rd February 2007**

0915 – 1145

Calculators and mathematical booklets are allowed

Answer TWO questions from each section

Section A: (i) Matrices

1. If $A=\begin{pmatrix}3&1&1\\1&3&1\\1&1&3\end{pmatrix}$ and $P=\begin{pmatrix}1&0&1\\-1&1&1\\0&-1&1\end{pmatrix}$ workout AP and P^{-1}.

(20 marks)

Hence or otherwise find the eigenvalues and eigenvectors of A.

(10 marks)

Prove that A is diagonalizable.

Show that for any $\mu \in \mathbb{R}$, $(A-\mu I)$ has the same eigenvectors as A, determine its eigenvalues in terms of μ and determine $(A-3I)^{10}$.

(20 marks)

2. (a) If $v=\begin{pmatrix}1\\1\\0\end{pmatrix}$, determine (i) the 3×3 matrix vv^t, where v^t is the transpose of v;

(ii) $B=vv^t+4I$, where I is the identity matrix;

(iii) the rank of vv^t and the rank of B;

(iv) the common eigenvalue of vv^t and v^tv.

(20 marks)

3. (a) Consider the unconstrained problem of finding the extrema for

$$\left(x^2-4\right)^2+y^2$$

Using partial derivatives, find the critical points and establish which ones are maxima (if any), minima (if any) and saddle points (if any).

(b) (i) Consider the unconstrained problem of minimizing the function

$$3e^x - e^{-x} - 4y^4 + ze^x + 2ze^{-x} + zy^4 - 4z$$

in the three variables x, y, z. Find the critical points.

(ii) Maximize and minimize

$$3e^x - e^{-x} - 4y^4$$

subject to $e^x + 2e^{-x} + y^4 - 4 = 0$

over the variables x, y. State clearly the *points* where the maximum and minimum are attained and the corresponding maximum and minimum *values*.

(c) Show that the differential form

$$\left(y\cos(xy)+e^y\right)dx+\left(x\cos(xy)+xe^y+2y\right)dy = 0$$

is exact and integrate it.

Section B (ii) Ordinary Differential Equations

1. (a) Throughout this question (1a) you may assume that $x > 0$.

   (i) Solve

$$y' = 2 + \frac{y}{x} \qquad (*)$$

   by reducing it to a separable equation.

   (ii) Solve

$$y' = \frac{2x + y + 7}{x + 3}, \qquad y(1) = -1$$

   (iii) Note that Equation (*) is also a 1st order linear ODE. What is the integrating factor?

   (iv) By multiplying throughout by the integrating factor, reduce equation (*) to an exact differential equation. Verify that it is indeed exact.

   (v) Solve

$$y' - \frac{y}{4x} = \frac{1}{2y^3}$$

   (b) Let D denote the operator of differentiation (as usual). Compute

$$\left(D^2 - D - 2\right)\sin\left(e^x\right)$$

   Hence solve

$$y'' - y' - 2y = \cos(3x) - \sin\left(e^x\right)\left(2 + e^{2x}\right)$$

2. (a) Using an Integrating Factor, solve

$$\left(1 + 3x^2\right)y' + 6xy + \left(1 + 3x^2\right)\left(1 + 6x^2\right)y = \left(1 + 3x^2\right)e^{-x^3}$$

   (b) Use the D-operator method throughout this question (2b).

   (i) Solve

$$y'' - 2y' + y = x^3 + 2x$$

   (ii) Solve

$$y'' + 2y' + y = e^{-2x}\left(x^3 + 2x\right) + 2e^{-x}$$

(b) If $A = \begin{pmatrix} 4 & a & 0 \\ 1 & b & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $y = \begin{pmatrix} 6 \\ c \\ 8 \end{pmatrix}$, determine conditions on a, b, and c for the system of equations $Ax = y$ to have (i) a unique solution;
(ii) infinitely many solutions;
(iii) no solutions.

[Hint: Use Gaussian Row Reduction on the augmented matrix $[A : y]$].

(20 marks)

Solve the system of equations for case (ii).

(10 marks)

Det

3. (a) Show that $\det \begin{pmatrix} x & y & z \\ y+z & x+z & x+y \\ 2 & 2 & 2 \end{pmatrix} = 0$

(15 marks)

(b) (i) Work out $\det A$ if $A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 3 \\ 0 & 2 & 0 \end{pmatrix}$.

(5 marks)

(ii) Either use Cramer's Rule, or else use A^{-1}, to solve $Ax = \begin{pmatrix} 7 \\ 11 \\ 4 \end{pmatrix}$.

(15 marks)

det

(iii) If $B = A + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and $C = 2A$,

determine $\det B$ and $\det C$.