UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.(Hons.) Year I
January/February 2006 Assessment Session

MAT1091 Mathematical Methods I: Matrices & O.D.E. **28th January 2006**

0915 – 1145

Calculators and mathematical booklets are allowed

Answer TWO questions from EACH section

Section A

1. Let $R = \begin{bmatrix} 5 & 1 & a \\ 1 & 5 & b \\ a & b & 6 \end{bmatrix}$ and $S = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

   If $a \neq 0$ and $b \neq 0$, determine the condition on the values of a and b for the system of equations $Rx = \begin{pmatrix} 5 \\ 1 \\ c \end{pmatrix}$ not to have a unique solution. For what values of c will the equations have infinitely many solutions?

   If $a = 0$ and $b = 0$, show that R and S share the same eigenvalues. Determine whether R and S are diagonalizable.

2. Let $A = \begin{pmatrix} 0 & 4 & 5 \\ 1 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix}$.

   Determine the elementary matrix E_1 if $E_1 A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & 5 \\ 2 & 2 & 5 \end{pmatrix}$.

   Calculate $\det E_1$ and k if $\det E_1 A = k \det A$.

   If $B = \begin{pmatrix} 0 & 4 & 5 \\ 0 & 4 & 5 \\ 2 & 2 & 5 \end{pmatrix}$, what is $E_1 B$?

By considering $\det E_1B$, or otherwise, show that $\det B = 0$.

If E_2 and E_3 are also elementary matrices such that $E_3E_2E_1\,A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & 5 \\ 0 & 0 & p \end{bmatrix}$,

where $p \neq 0$, determine possible matrices for E_2 and E_3.
Hence or otherwise, find $\det A$ and $\det 2A$.

Solve the system of equations $Ax = \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix}$.

3. (a) $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix}$

Work out QQ^t and **hence** determine Q^{-1}.

Show that $\det Q = -1$.

(b) If $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, calculate AB.

Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ are eigenvectors of A.

Determine the eigenvalues of A and the matrix P such that $P^{-1}AP$ is a diagonal matrix D.

Find an expression for A^7 in terms of D and P.
[It is not necessary to calculate P^{-1}].

Section B

4. Solve the following differential equations:

   (a) $(2x-2y-8)dx+(x-3y-6)dy=0$

   (b) $\dfrac{dy}{dx}-5y=-\dfrac{5}{2}xy^3$

   (c) $\dfrac{d^2y}{dx^2}+16y=\sin 4x$ (Use the D-operator method)

5. Find the general solutions of the following differential equations. Use the D-operator method:

   (a) $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=14e^{2x}-3$

   (b) $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=xe^{-2x}$

6.(a) A surfboard company has developed the yearly profit equation:

$$P(x,y)=-2x^2+4xy-3y^2+4x-2y+77$$

where x is the number (in thousands) of standard surfboards produced per year, y is the number (in thousands) of competition surfboards produced per year, and P is profit (in thousands of dollars) per year. How many of each type of surfboard should be produced per year to realize a maximum profit? What is the maximum profit?

(b) Consider the equation:

$$\left(\frac{\sin 2x}{y}+x\right)dx+\left(y-\frac{\sin^2 x}{y^2}\right)dy=0$$

i) Show that this equation is a total differential equation.
ii) Find the general integral.

ac/MAT1091/jan06