Chapter 8

INTERSECTING FAMILIES OF SETS AND PERMUTATIONS: A SURVEY

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Abstract

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if any two sets in $\mathcal{A}$ have at least t common elements. A central problem in extremal set theory is to determine the size or structure of a largest t-intersecting sub-family of a given family $\mathcal{F}$. We give a survey of known results, conjectures and open problems for various important families $\mathcal{F}$, namely, power sets, levels of power sets, hereditary families, families of signed sets, families of labeled sets, and families of permutations. We also provide some extensions and consequences of known results.

1 Introduction

Unless otherwise stated, we shall use small letters such as x to denote elements of a set or non-negative integers or functions, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose elements are sets themselves). It is to be assumed that arbitrary sets and families are *finite*. We call a set A an *r-element set*, or simply an *r-set*, if its size $|A|$ is r (i.e. if it contains exactly r elements). A family is said to be *uniform* if all its sets are of the same size.

The set $\{1,2,\dots\}$ of positive integers is denoted by $\mathbb{N}$. For $m,n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ is denoted by $[m,n]$, and if $m = 1$ then we also write $[n]$. For a set X, the *power set* $\{A \colon A \subseteq X\}$ of X is denoted by 2^X, and the uniform sub-family $\{Y \subseteq X : |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$.

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For a family $\mathcal{F}$ of sets, we denote the union of all sets in $\mathcal{F}$ by $U(\mathcal{F})$ and we denote the size of a largest set in $\mathcal{F}$ by $\alpha(\mathcal{F})$. For an integer $r \geq 0$, we denote the uniform sub-family $\{F \in \mathcal{F} : |F| = r\}$ of $\mathcal{F}$ by $\mathcal{F}^{(r)}$ (note that $\mathcal{F}^{(r)} = \binom{X}{r}$ if $\mathcal{F} = 2^X$), and we call $\mathcal{F}^{(r)}$ the *r'th level of* $\mathcal{F}$. For a set S, we denote $\{F \in \mathcal{F} : S \subseteq F\}$ by $\mathcal{F}(S)$. We may abbreviate $\mathcal{F}(\{x\})$ to $\mathcal{F}(x)$. If $x \in U(\mathcal{F})$ then we call $\mathcal{F}(x)$ a *star of* $\mathcal{F}$. More generally, if T is a t-element subset of a set in $\mathcal{F}$, then we call $\mathcal{F}(T)$ a *t-star of* $\mathcal{F}$.

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A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. More generally, $\mathcal{A}$ is said to be *t-intersecting* if $|A \cap B| \geq t$ for any $A, B \in \mathcal{A}$. So an intersecting family is a 1-intersecting family. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if $|\bigcap_{A \in \mathcal{A}} A| \geq t$ (i.e. there are at least t elements common to all the sets in $\mathcal{A}$); otherwise, $\mathcal{A}$ is said to be *non-trivial*. So a t-star of a family $\mathcal{F}$ is a trivial t-intersecting sub-family of $\mathcal{F}$ that is not contained in any other. If there exists a t-set T such that $\mathcal{F}(T)$ is a largest t-intersecting sub-family of $\mathcal{F}$ (i.e. no t-intersecting sub-family of $\mathcal{F}$ has more sets than $\mathcal{F}(T)$), then we say that $\mathcal{F}$ has the *t-star property at T*, or we simply say that $\mathcal{F}$ has the *t-star property*. If either $\mathcal{F}$ has no t-intersecting sub-families (which is the case if and only if $\alpha(\mathcal{F}) < t$) or all the largest t-intersecting sub-families of $\mathcal{F}$ are t-stars, then we say that $\mathcal{F}$ has the *strict* t-star property. We may abbreviate '1-star property' to 'star property'.

Extremal set theory is the study of how small or how large a system of sets can be under certain conditions. In this paper we are concerned with the following natural and central problem in this field.

Problem: *Given a family $\mathcal{F}$ and an integer $t \geq 1$, determine the size or structure of a largest t-intersecting sub-family of $\mathcal{F}$.*

We provide a survey of results that answer this question for families that are of particular importance, and we also point out open problems and conjectures. The survey papers [25] and [32] cover a few of the results we mention here and also go into many variations of the above problem; however, much progress has been made since their publication. Here we cover many of the important results that have been established to date, restricting ourselves to the problem stated above.

The most obvious families to consider are the power set $2^{[n]}$ and the uniform sub-family $\binom{[n]}{r}$, and in fact the problem for these families has been solved completely. However, there are other important families on which much progress has been made, and there are others that are still subject to much investigation. The families defined below are perhaps the ones that have received most attention and that we will be concerned with.

Hereditary families: A family $\mathcal{H}$ is said to be a *hereditary family* (also called an *ideal* or a *downset*) if all the subsets of any set in $\mathcal{H}$ are in $\mathcal{H}$. Clearly a family is hereditary if and only if it is a union of power sets. A *base* of $\mathcal{H}$ is a set in $\mathcal{H}$ that is not a subset of any other set in $\mathcal{H}$. So a hereditary family is the union of power sets of its bases. An example of a hereditary family is the family of independent sets of a *graph* or *matroid*.

Families of signed sets: Let X be an r-set $\{x_1, \dots, x_r\}$. Let $y_1, \dots, y_r \in \mathbb{N}$. We call the set $\{(x_1, y_1), \dots, (x_r, y_r)\}$ a *k-signed r-set* if $\max\{y_i : i \in [r]\} \leq k$. For an integer $k \geq 2$ we define $\mathcal{S}_{X,k}$ to be the family of k-signed r-sets given by

$$\mathcal{S}_{X,k} := \{\{(x_1,y_1),\dots,(x_r,y_r)\} : y_1,\dots,y_r \in [k]\}.$$

So a set A is a member of $\mathcal{S}_{X,k}$ if and only if it is a subset of the Cartesian product $X \times [k] := \{(x,y)\colon x \in X, y \in [k]\}$ satisfying $|A \cap (\{x\} \times [k])| = 1$ for all $x \in X$. We shall set $\mathcal{S}_{\emptyset,k} := \emptyset$. With a slight abuse of notation, for a family $\mathcal{F}$ we define

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}.$$

Families of labeled sets: For $\mathbf{k} := (k_1,\dots,k_n)$ with $k_1,\dots,k_n \in \mathbb{N}$ and $k_1 \le \cdots \le k_n$, we define the family $\mathcal{L}_{\mathbf{k}}$ of *labeled n-sets* by

$$\mathcal{L}_{\mathbf{k}} := \{\{(1,y_1),\dots,(n,y_n)\} : y_i \in [k_i] \text{ for each } i \in [n]\}.$$

Note that $\mathcal{S}_{[n],k} = \mathcal{L}_{(k_1,\dots,k_n)}$ with $k_1 = \cdots = k_n = k$.

An equivalent formulation for $\mathcal{L}_{\mathbf{k}}$ is the Cartesian product $[k_1] \times \cdots \times [k_n] := \{(y_1,\dots,y_n)\colon y_i \in [k_i] \text{ for each } i \in [n]\}$, but it is more convenient to work with n-sets than work with n-tuples (the alternative formulation demands that we change the setting of families of sets to one of sets of n-tuples).

For any $r \in [n]$, we define

$$\mathcal{L}_{\mathbf{k},r} := \left\{\{(x_1,y_{x_1}),\dots,(x_r,y_{x_r})\} \colon \{x_1,\dots,x_r\} \in \binom{[n]}{r}, y_{x_i} \in [k_{x_i}] \text{ for each } i \in [r]\right\},$$

and we set $\mathcal{L}_{\mathbf{k},0} = \emptyset$. Thus, for any $0 \le r \le n$, $\mathcal{L}_{\mathbf{k},r}$ is the family of r-element subsets of the sets in $\mathcal{L}_{\mathbf{k}}$, and $\mathcal{L}_{\mathbf{k},n} = \mathcal{L}_{\mathbf{k}}$. We also define $\mathcal{L}_{\mathbf{k},\le r} := \bigcup_{i=0}^{r} \mathcal{L}_{\mathbf{k},i}$.

Families of permutations: For an r-set $X := \{x_1,\dots,x_r\}$, we define $\mathcal{S}^*_{X,k}$ to be the special sub-family of $\mathcal{S}_{X,k}$ given by

$$\mathcal{S}^*_{X,k} := \{\{(x_1,y_1),\dots,(x_r,y_r)\} : y_1,\dots,y_r \text{ are } \textit{distinct} \text{ elements of } [k]\}.$$

Note that $\mathcal{S}^*_{X,k} \ne \emptyset$ if and only if $r \le k$. With a slight abuse of notation, for a family $\mathcal{F}$ we define $\mathcal{S}^*_{\mathcal{F},k}$ to be the special sub-family of $\mathcal{S}_{\mathcal{F},k}$ given by

$$\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}^*_{F,k}.$$

An *r-partial permutation of a set N* is a pair (A,f) where $A \in \binom{N}{r}$ and $f\colon A \to N$ is an injection. An $|N|$-partial permutation of N is simply called a *permutation of N*. Clearly, the family of permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{[n],n}$, and the family of r-partial permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{\binom{[n]}{r},n}$.

Let X be as above. $\mathcal{S}^*_{X,k}$ can be interpreted as the family of permutations of sets in $\binom{[k]}{r}$: consider the bijection $\beta\colon \mathcal{S}^*_{X,k} \to \{(A,f)\colon A \in \binom{[k]}{r}, f\colon A \to A \text{ is a bijection}\}$ defined by $\beta(\{(x_1,a_1),\dots,(x_r,a_r)\}) := (\{a_1,\dots,a_r\}, f)$ where, for $b_1 < \cdots < b_r$ such that $\{b_1,\dots,b_r\} = \{a_1,\dots,a_r\}$, $f(b_i) := a_i$ for $i = 1,\dots,r$. $\mathcal{S}^*_{X,k}$ can also be interpreted as the sub-family $\mathcal{X} := \{(A,f)\colon A \in \binom{[k]}{r}, f\colon A \to [r] \text{ is a bijection}\}$ of the family of r-partial permutations of $[k]$: consider an obvious bijection from $\mathcal{S}^*_{X,k}$ to $\mathcal{S}^*_{\binom{[k]}{r},r}$ and another one from $\mathcal{S}^*_{\binom{[k]}{r},r}$ to $\mathcal{X}$.

2 Intersecting Sub-Families of $\binom{[n]}{r}$ and $2^{[n]}$

In this section we take t, r and n to be positive integers such that $t \leq r \leq n$.

The study of intersecting families took off with the publication of [28], which features the following classical result, known as the Erdős-Ko-Rado (EKR) Theorem.

Theorem 2.1 (EKR Theorem [28])**.** *If $r \leq n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

This means that for $r \leq n/2$, $\binom{[n]}{r}$ has the star property, because the bound $\binom{n-1}{r-1}$ is the size of any star of $\binom{[n]}{r}$. Note that if $r > n/2$, then any two r-element subsets of $[n]$ must intersect, and hence $\binom{[n]}{r}$ is an intersecting family (also note it is a non-trivial one, so $\binom{[n]}{r}$ does not have the star property in this case).

In order to prove Theorem 2.1, Erdős, Ko and Rado [28] introduced a method known as *compression* or *shifting*; see [32] for a survey on the uses of this powerful technique in extremal set theory. There are various proofs of Theorem 2.1, two of which are particularly short and beautiful: Katona's proof [40], which featured an elegant argument known as the *cycle method*, and Daykin's proof [22] using another fundamental result known as the Kruskal-Katona Theorem [41, 44]. Hilton and Milner [37] proved that for $r \leq n/2$, the family $\mathcal{N}_{n,r} := \left\{A \in \binom{[n]}{r} : 1 \in A, A \cap [2, r+1] \neq \emptyset\right\} \cup \{[2, r+1]\}$ is a largest non-trivial intersecting sub-family of $\binom{[n]}{r}$, and since the size of $\mathcal{N}_{n,r}$ is $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$, it follows that if $r < n/2$, then the stars of $\binom{[n]}{r}$ are the largest intersecting sub-families of $\binom{[n]}{r}$, i.e. $\binom{[n]}{r}$ has the strict star property. Note that if $r = n/2$, then any sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ satisfying $|\mathcal{A} \cap \{A, [2r] \backslash A\}| = 1$ for all $A \in \binom{[n]}{r}$ is an intersecting sub-family of $\binom{[n]}{r}$ of size $\frac{1}{2}\binom{n}{r} = \frac{1}{2}\binom{2r}{r} = \binom{2r-1}{r-1}$, and hence one of maximum size (an example of such a family $\mathcal{A}$ is $\mathcal{N}_{2r,r}$, so $\binom{[n]}{r}$ does not have the strict star property if $r = n/2$).

Also in [28], Erdős, Ko and Rado initiated the study of t-intersecting families. They proved that for $t < r$, there exists an integer $n_0(r,t)$ such that for all $n \geq n_0(r,t)$, the largest t-intersecting sub-families of $\binom{[n]}{r}$ are the t-stars (which are of size $\binom{n-t}{r-t}$). For $t \geq 15$, Frankl [31] showed that the smallest such $n_0(r,t)$ is $(r-t+1)(t+1)+1$ and that if $n = (r-t+1)(t+1)$, then $\binom{[n]}{r}$ still has the t-star property but not the strict t-star property. Subsequently, using algebraic means, Wilson [58] proved that $\binom{[n]}{r}$ has the t-star property for any t and $n \geq (r-t+1)(t+1)$. Frankl [31] conjectured that among the largest t-intersecting sub-families of $\binom{[n]}{r}$ there is always at least one of the families $\left\{A \in \binom{[n]}{r} : |A \cap [t+2i]| \geq t+i\right\}$, $i = 0, 1, \ldots, r-t$. A remarkable proof of this long-standing conjecture together with a complete characterisation of the extremal structures was finally obtained by Ahlswede and Khachatrian [1] by means of the *compression* technique introduced in [28].

Theorem 2.2 ([1])**.** *Let $\mathcal{A}$ be a largest t-intersecting sub-family of $\binom{[n]}{r}$.*
(i) *If $(r-t+1)(2+\frac{t-1}{i+1}) < n < (r-t+1)(2+\frac{t-1}{i})$ for some $i \in \{0\} \cup \mathbb{N}$ - where, by convention, $(t-1)/i = \infty$ if $i = 0$ - then $\mathcal{A} = \{A \in \binom{[n]}{r} : |A \cap X| \geq t+i\}$ for some $X \in \binom{[n]}{t+2i}$.*
(ii) *If $t \geq 2$ and $(r-t+1)(2+\frac{t-1}{i+1}) = n$ for some $i \in \{0\} \cup \mathbb{N}$, then $\mathcal{A} = \{A \in \binom{[n]}{r} : |A \cap X| \geq t+j\}$ for some $j \in \{i, i+1\}$ and $X \in \binom{[n]}{t+2j}$.*

It is worth mentioning that in [2] Ahlswede and Khachatrian went on to determine the largest non-trivial t-intersecting sub-families of $\binom{[n]}{r}$.

Erdős, Ko and Rado [28] pointed out the simple fact that $2^{[n]}$ has the star property (indeed, for any set A in an intersecting sub-family $\mathcal{A}$ of $2^{[n]}$, the complement $[n]\backslash A$ cannot be in $\mathcal{A}$, and hence the size of $\mathcal{A}$ is at most $\frac{1}{2}|2^{[n]}| = 2^{n-1}$, i.e. the size of a star of $2^{[n]}$); note that there are many non-trivial intersecting sub-families of $2^{[n]}$ of maximum size 2^{n-1} (such as $\{A \subseteq [n] \colon |A \cap [3]| \geq 2\}$), so $2^{[n]}$ does not have the strict star property. They also asked what the size of a largest t-intersecting sub-family of $2^{[n]}$ is for $t \geq 2$. The answer in a complete form was given by Katona [42].

Theorem 2.3 ([42])**.** *Let $t \geq 2$, and let $\mathcal{A}$ be a largest t-intersecting sub-family of* $2^{[n]}$.
(i) *If $n+t = 2l$ then $\mathcal{A} = \{A \subseteq [n] \colon |A| \geq l\}$.*
(ii) *If $n+t = 2l+1$ then $\mathcal{A} = \{A \subseteq [n] \colon |A \cap X| \geq l\}$ for some $X \in \binom{[n]}{n-1}$.*

It is interesting that for $n > t \geq 2$, $2^{[n]}$ does not have the t-star property.

Many other beautiful results were inspired by the seminal paper [28], as are the results we present in the subsequent sections.

3 Intersecting Sub-Families of Hereditary Families

Recall that $2^{[n]}$ has the star property. Also recall that the power set of a set X is the simplest example of a hereditary family since 2^X is a hereditary family with only one base (X). An outstanding open problem in extremal set theory is the following conjecture (see [14] for a more general conjecture).

Conjecture 3.1 ([19])**.** *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ has the star property.*

Chvátal [20] verified this conjecture for the case when $\mathcal{H}$ is *left-compressed* (i.e. $\mathcal{H} \subseteq 2^{[n]}$ and $(H\backslash\{j\}) \cup \{i\} \in \mathcal{H}$ whenever $1 \leq i < j \in H \in \mathcal{H}$ and $i \notin H$). Snevily [54] took this result (together with results in [53, 55]) a significant step forward by verifying Conjecture 3.1 for the case when $\mathcal{H}$ is *compressed with respect to an element x of $U(\mathcal{H})$* (i.e. $(H\backslash\{h\}) \cup \{x\} \in \mathcal{H}$ whenever $h \in H \in \mathcal{H}$ and $x \notin H$).

Theorem 3.2 ([54])**.** *If a hereditary family $\mathcal{H}$ is compressed with respect to an element x of $U(\mathcal{H})$, then $\mathcal{H}$ has the star property at $\{x\}$.*

A generalisation is proved in [14] by means of an alternative self-contained argument. Snevily's proof of Theorem 3.2 makes use of the following interesting result of Berge [5] (a proof of which is also provided in [4, Chapter 6]).

Theorem 3.3 ([5])**.** *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ is a disjoint union of pairs of disjoint sets, together with $\emptyset$ if $|\mathcal{H}|$ is odd.*

This result was also motivated by Conjecture 3.1 as it has the following immediate consequence.

Corollary 3.4. *If $\mathcal{A}$ is an intersecting sub-family of a hereditary family $\mathcal{H}$, then*

$$|\mathcal{A}| \leq \frac{1}{2}|\mathcal{H}|.$$

Proof. For any pair of disjoint sets, at most only one set can be in an intersecting family $\mathcal{A}$. By Theorem 3.3, the result follows. $\square$

A special case of Theorem 3.2 is a result of Schönheim [53] which says that Conjecture 3.1 is true if the bases of $\mathcal{H}$ have a common element, and this follows immediately from Corollary 3.4 and the following fact.

Proposition 3.5 ([53]). *If the bases of a hereditary family $\mathcal{H}$ have a common element x, then*

$$|\mathcal{H}(x)| = \frac{1}{2}|\mathcal{H}|.$$

Proof. Partition $\mathcal{H}$ into $\mathcal{A} := \mathcal{H}(x)$ and $\mathcal{B} := \{B \in \mathcal{H} \colon x \notin B\}$. If $A \in \mathcal{A}$ then $A \backslash \{x\} \in \mathcal{B}$; so $|\mathcal{A}| \leq |\mathcal{B}|$. If $B \in \mathcal{B}$ then $B \subseteq C$ for some base C of $\mathcal{H}$, and hence $B \cup \{x\} \in \mathcal{A}$ since $x \in C$; so $|\mathcal{B}| \leq |\mathcal{A}|$. Thus $|\mathcal{A}| = |\mathcal{B}| = \frac{1}{2}|\mathcal{H}|$. $\square$

Many other results and problems have been inspired by Conjecture 3.1 or are related to it; see [21, 51, 57].

Conjecture 3.1 cannot be generalised to the t-intersection case. Indeed, if $n > t \geq 2$ and $\mathcal{H} = 2^{[n]}$, then by Theorem 2.3, $\mathcal{H}$ does not have the t-star property.

We now turn our attention to uniform intersecting sub-families of hereditary families, or rather intersecting sub-families of levels of hereditary families. For any hereditary family $\mathcal{H}$, let $\mu(\mathcal{H})$ denote the size of a smallest base of $\mathcal{H}$.

A *graph* G is a pair (V,E) with $E \subseteq \binom{V}{2}$, and a set $I \subseteq V$ is said to be an *independent set of* G if $\{i, j\} \notin E$ for any $i, j \in I$. Let $\mathcal{I}_G$ denote the family of all independent sets of a graph G. Clearly $\mathcal{I}_G$ is a hereditary family. Holroyd and Talbot [39] made a nice conjecture which claims that if G is a graph and $\mu(\mathcal{I}_G) \geq 2r$, then ${\mathcal{I}_G}^{(r)}$ has the star property, and ${\mathcal{I}_G}^{(r)}$ has the strict star property if $\mu(\mathcal{I}_G) > 2r$. In [11] the author conjectured that this is true for any hereditary family and that in general the following holds.

Conjecture 3.6 ([11]). *If $1 \leq t \leq r$, $\emptyset \neq S \subseteq [t,r]$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq (t+1)(r-t+1)$, then:*
(i) $\bigcup_{s\in S} \mathcal{H}^{(s)}$ *has the t-star property;*
(ii) $\bigcup_{s\in S} \mathcal{H}^{(s)}$ *has the strict t-star property if either $\mu(\mathcal{H}) > (t+1)(r-t+1)$ or $S \neq \{r\}$.*

Note that Theorem 2.2 solves the special case when $\mathcal{H} = 2^{[n]}$ and tells us that we cannot improve the condition $\mu(\mathcal{H}) \geq (t+1)(r-t+1)$. The author [11] proved that this conjecture is true if $\mu(\mathcal{H})$ is sufficiently large.

Theorem 3.7 ([11]). *Conjecture* 3.6 *is true if $\mu(\mathcal{H}) \geq (r-t)\binom{3r-2t-1}{t+1} + r$.*

The motivation behind establishing this result for any union of levels of a hereditary family $\mathcal{H}$ within a certain range is that this general form cannot be immediately deduced from the result for just one level of $\mathcal{H}$ (i.e. the case $S = \{r\}$ in Conjecture 3.6). As demonstrated in Example 1 in [11], the reason is simply that if T is a t-set such that $\mathcal{H}^{(s)}(T)$ ($s \in [t,r]$) is a largest t-star of the level $\mathcal{H}^{(s)}$, then for $p \neq s$ ($p \in [t,r]$), $\mathcal{H}^{(p)}(T)$ not only may not be a largest t-star of the level $\mathcal{H}^{(p)}$ but may be smaller than some non-trivial t-intersecting sub-family of $\mathcal{H}^{(p)}$. This is in fact one of the central difficulties arising from any EKR-type problem for hereditary families. In the proof of Theorem 3.7, this obstacle was overcome by showing that for any non-trivial t-intersecting sub-family $\mathcal{A}$ of the union, we can construct a t-star that is larger than $\mathcal{A}$ (and that is not necessarily a largest t-star). Many other proofs of EKR-type results are based on determining at least one largest t-star; as in the case of each theorem mentioned in Section 2, the setting is often symmetrical to the extent that all t-stars are of the same size and of a known size.

An interesting immediate consequence of Theorem 3.7 is that the union of the first $r \geq t$ levels of a hereditary family $\mathcal{H}$ has the strict t-star property if $\mu(\mathcal{H})$ is sufficiently larger than r.

Corollary 3.8 ([11])**.** *If $1 \leq t \leq r$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq (r-t)\binom{3r-2t-1}{t+1} + r$, then $\bigcup_{s=0}^{r} \mathcal{H}^{(s)}$ has the strict t-star property.*

Proof. Let $\mathcal{A}$ be a t-intersecting sub-family of $\bigcup_{s=0}^{r} \mathcal{H}^{(s)}$. Then no set in $\mathcal{A}$ is of size less than t, so $\mathcal{A} \subseteq \bigcup_{s \in S} \mathcal{H}^{(s)}$ with $S = [t,r]$. The result follows by Theorem 3.7. $\square$

This means that for the special case $t = 1$, we have the following.

Corollary 3.9 ([11])**.** *Conjecture 3.1 is true if $\mathcal{H} = \bigcup_{s=0}^{r} \mathcal{J}^{(s)}$ for some $r \in \mathbb{N}$ and some hereditary family $\mathcal{J}$ with $\mu(\mathcal{J}) \geq \frac{3}{2}(r-1)^2(3r-4) + r$.*

The following extension of Theorem 2.2 for $n \geq (t+1)(r-t+1)$ was also proved in [11].

Theorem 3.10 ([11])**.** *Conjecture 3.6 is true if $\mathcal{H}$ is left-compressed.*

4 Intersecting Families of Signed Sets

The 'signed sets' terminology was introduced in [10] for a setting that can be re-formulated as $\mathcal{S}_{\binom{[n]}{r},k}$, and the general formulation $\mathcal{S}_{\mathcal{F},k}$ was introduced in [13], the theme of which is the following conjecture.

Conjecture 4.1 ([13])**.** *For any family $\mathcal{F}$ and any $k \geq 2$,*
(i) *$\mathcal{S}_{\mathcal{F},k}$ has the star property;*
(ii) *$\mathcal{S}_{\mathcal{F},k}$ does not have the strict star property only if $k = 2$ and there exist at least three elements u_1, u_2, u_3 of $U(\mathcal{F})$ such that $\mathcal{F}(u_1) = \mathcal{F}(u_2) = \mathcal{F}(u_3)$ and $\mathcal{S}_{\mathcal{F},2}((u_1,1))$ is a largest star of $\mathcal{S}_{\mathcal{F},2}$.*

The converse of (ii) is true, and the proof is simply that $\{A \in \mathcal{S}_{\mathcal{F},2} \colon |A \cap \{(u_1,1),(u_2,1),(u_3,1)\}| \geq 2\}$ is a non-trivial intersecting sub-family of $\mathcal{S}_{\mathcal{F},2}$ that is as large as $\mathcal{S}_{\mathcal{F},2}((u_1,1))$.

In [14] a similarity between the intersection problem for hereditary families and the one presented above is demonstrated, and in fact a conjecture generalising both Conjecture 3.1 and the above conjecture is suggested.

Recall that a family $\mathcal{F}$ is said to be compressed with respect to an element x of $U(\mathcal{F})$ if $(F \setminus \{u\}) \cup \{x\} \in \mathcal{F}$ whenever $u \in F \in \mathcal{F}$ and $x \notin F$. The following is the main result in the paper featuring the above conjecture.

Theorem 4.2 ([13])**.** *Conjecture* 4.1 *is true if $\mathcal{F}$ is compressed with respect to an element x of $U(\mathcal{F})$, and $\mathcal{S}_{\mathcal{F},k}$ has the star property at $\{(x,1)\}$.*

Since $\binom{[n]}{r}$ is compressed with respect to any element of $[n]$, the above result has the following immediate consequence, which is a well-known result that was first stated by Meyer [50] and proved in different ways by Deza and Frankl [25], Bollobás and Leader [10], Engel [27] and Erdős et al. [29].

Theorem 4.3 ([10, 25, 27, 29])**.** *Let $r \in [n]$ and let $k \geq 2$. Then:*
(i) *$\mathcal{S}_{\binom{[n]}{r},k}$ has the star property;*
(ii) *if $(r,k) \neq (n,2)$ then $\mathcal{S}_{\binom{[n]}{r},k}$ has the strict star property.*

Thus the size of an intersecting sub-family of $\mathcal{S}_{\binom{[n]}{r},k}$ is at most $\binom{n-1}{r-1}k^{r-1}$, i.e. the size of any star of $\mathcal{S}_{\binom{[n]}{r},k}$. Berge [6] and Livingston [49] had proved (i) and (ii), respectively, for the special case $\mathcal{F} = \{[n]\}$ (other proofs are found in [36, 52]).

In [13] Conjecture 4.1 is also verified for the case when $\mathcal{F}$ is uniform and has the star property; Holroyd and Talbot [39] had essentially proved part (i) of the conjecture for such a family $\mathcal{F}$ in a graph-theoretical context.

The t-intersection problem for sub-families of $\mathcal{S}_{[n],k}$ has also been solved. Frankl and Füredi were the first to investigate it. In [33] they conjectured that among the largest t-intersecting sub-families of $\mathcal{S}_{[n],k}$ there is always one of the families $\mathcal{A}_i := \{A \in \mathcal{S}_{[n],k} \colon |A \cap ([t+2i] \times [1])| \geq t+i\}$, $i = 0,1,2,\ldots$, and they proved that if $k \geq t+1 \geq 16$, then $\mathcal{A}_0$ is extremal and hence $\mathcal{S}_{[n],k}$ has the star property. The conjecture was proved independently by Ahlswede and Khachatrian [3] and Frankl and Tokushige [34] (Kleitman [43] had long established this result for $k = 2$). As in Theorem 2.2, Ahlswede and Khachatrian [3] also determined the extremal structures.

Theorem 4.4 ([3])**.** *Let $1 \leq t \leq n$ and $k \geq 2$. Let m be the largest integer such that $t+2m < \min\{n+1, t+2\frac{t-1}{k-2}\}$ (by convention, $\frac{t-1}{k-2} = \infty$ if $k = 2$).*
(i) *If $(k,t) \neq (2,1)$ and $\frac{t-1}{k-2}$ is not integral, then $\mathcal{A}$ is a largest t-intersecting sub-family of $\mathcal{S}_{[n],k}$ if and only if*

$$\mathcal{A} = \{A \in \mathcal{S}_{[n],k} \colon |A \cap X| \geq t+m\}$$

for some $X \in \mathcal{S}_{Y,k}$ with $Y \in \binom{[n]}{t+2m}$.
(ii) *If $(k,t) \neq (2,1)$ and $\frac{t-1}{k-2}$ is integral, then $\mathcal{A}$ is a largest t-intersecting sub-family of $\mathcal{S}_{[n],k}$ if and only if*

$$\mathcal{A} = \{A \in \mathcal{S}_{[n],k} \colon |A \cap X| \geq t+j\}$$

for some $j \in \{m, m+1\}$ and some $X \in \mathcal{S}_{Y,k}$ with $Y \in \binom{[n]}{t+2j}$.
(iii) *If $(k,t) = (2,1)$, then $\mathcal{A}$ is a largest t-intersecting sub-family of $\mathcal{S}_{[n],k}$ if and only if for any $y_1,\ldots,y_n \in [2]$, exactly one of $\{(1,y_1),\ldots,(n,y_n)\}$ and $\{(1,3-y_1),\ldots,(n,3-y_n)\}$ is in $\mathcal{A}$.*

Note that (iii) follows trivially from the fact that for any set $A := \{(1,y_1),\ldots,(n,y_n)\}$ in $\mathcal{S}_{[n],2}$, $\{(1,3-y_1),\ldots,(n,3-y_n)\}$ is the only set in $\mathcal{S}_{[n],2}$ that does not intersect A. The rest of the theorem is highly non-trivial!

What led to Theorem 4.4 was the accomplishment of Theorem 2.2. The following is an immediate consequence of Theorem 4.4.

Corollary 4.5. *Let $1 \le t \le n$ and $k \ge 2$. Then:*
(i) *$\mathcal{S}_{[n],k}$ has the t-star property if and only if $k \ge t+1$;*
(ii) *$\mathcal{S}_{[n],k}$ has the strict t-star property if and only if $k \ge t+2$.*

We point out that Bey and Engel [9] extended Theorem 4.4 by determining the size of a largest non-trivial t-intersecting sub-family of $\mathcal{S}_{[n],k}$ (see Examples 10, 11 and Lemma 18 in [9]).

Note that $\mathcal{S}_{[n],k} = \mathcal{S}_{\binom{[n]}{r},k}$ with $r = n$. For the case $t \le r < n$, Bey [8] proved the following.

Theorem 4.6 ([8])**.** *Let $1 \le t \le r < n$. $\mathcal{S}_{\binom{[n]}{r},k}$ has the t-star property if and only if $n \ge \frac{(r-t+k)(t+1)}{k}$.*

Thus, if $t \le r < n$ and $n \ge \frac{(r-t+k)(t+1)}{k}$, then the size of a t-intersecting sub-family of $\mathcal{S}_{\binom{[n]}{r},k}$ is at most $\binom{n-t}{r-t}k^{r-t}$, i.e. the size of any t-star of $\mathcal{S}_{\binom{[n]}{r},k}$. From Corollary 4.5 and Theorem 4.6 we immediately obtain the following.

Corollary 4.7. *For any $1 \le t \le r \le n$ and $k \ge t+1$, $\mathcal{S}_{\binom{[n]}{r},k}$ has the t-star property.*

To the best of the author's knowledge, no complete t-intersection theorem for $\mathcal{S}_{\binom{[n]}{r},k}$ has been obtained.

For the case when $\mathcal{F}$ is any family, the author [15] suggested the following general conjecture.

Conjecture 4.8 ([15])**.** *For any integer $t \ge 1$, there exists an integer $k_0(t)$ such that for any $k \ge k_0(t)$ and any family $\mathcal{F}$, $\mathcal{S}_{\mathcal{F},k}$ has the t-star property.*

In view of Corollary 4.7, we conjecture that the smallest $k_0(t)$ is $t+1$. In [15] it is actually conjectured that for some integer $k'_0(t)$, $\mathcal{S}_{\mathcal{F},k}$ has the strict t-star property for any $\mathcal{F}$, and hence, in view of Corollary 4.5(ii), we conjecture that the smallest $k'_0(t)$ is $t+2$. Note that Conjecture 4.1 claims that the smallest values of $k_0(1)$ and $k'_0(1)$ are 2 and 3, respectively. The author [15] proved the following relaxation of the statement of Conjecture 4.8.

Theorem 4.9 ([15])**.** *For any integers r and t with $1 \le t < r$, let $k_0(r,t) := \binom{r}{t}\binom{r}{t+1}$. For any $k \ge k_0(r,t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) \le r$, $\mathcal{S}_{\mathcal{F},k}$ has the strict t-star property.*

The general idea behind the proof of this result is similar to that behind the proof of Theorem 3.7, described in Section 3.

Corollary 4.10. *Conjecture 4.1 is true if $k \ge \alpha(\mathcal{F})\binom{\alpha(\mathcal{F})}{2}$.*

5 Intersecting Families of Labeled Sets

Consider the family $\mathcal{L}_{\mathbf{k}}$, $\mathbf{k} = (k_1,\dots,k_n)$, of *labeled n-sets*. If $k_1 = 1$ then all the sets in $\mathcal{L}_{\mathbf{k}}$ contain the point $(1,1)$ and hence $\mathcal{L}_{\mathbf{k}}$ has the strict star property. Berge [6] proved that for any $\mathbf{k}$, $\mathcal{L}_{\mathbf{k}}$ has the star property, and hence the size of an intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$ is at most the size $\frac{1}{k_1}|\mathcal{L}_{\mathbf{k}}| = k_2k_3\dots k_n$ of the star $\mathcal{L}_{\mathbf{k}}((1,1))$, as this is clearly a largest star (since $k_1 \le \cdots \le k_n$). We shall reproduce the remarkably short proof of this result.

Let mod* be the usual modulo operation with the exception that for any integer a, $a \bmod^* a$ is a instead of 0. For any integer q, let $\theta^q_{\mathbf{k}} : \mathcal{L}_{\mathbf{k}} \to \mathcal{L}_{\mathbf{k}}$ be the *translation operation* defined by

$$\theta^q_{\mathbf{k}}(A) := \{(a,(b+q) \bmod^* k_a) \colon (a,b) \in A\},$$

and define $\Theta^q_{\mathbf{k}} : 2^{\mathcal{L}_{\mathbf{k}}} \to 2^{\mathcal{L}_{\mathbf{k}}}$ by

$$\Theta^q_{\mathbf{k}}(\mathcal{F}) := \{\theta^q_{\mathbf{k}}(A) \colon A \in \mathcal{F}\}.$$

Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$. For any $A \in \mathcal{A}$ and $q \in [k_1 - 1]$, we have $\theta^q_{\mathbf{k}}(A) \cap A = \emptyset$ and hence $\theta^q_{\mathbf{k}}(A) \notin \mathcal{A}$. Therefore $\mathcal{A}, \Theta^1_{\mathbf{k}}(\mathcal{A}),\dots,\Theta^{k_1-1}_{\mathbf{k}}(\mathcal{A})$ are k_1 disjoint sub-families of $\mathcal{L}_{\mathbf{k}}$. So $k_1|\mathcal{A}| \le |\mathcal{L}_{\mathbf{k}}|$ and hence $|\mathcal{A}| \le \frac{1}{k_1}|\mathcal{L}_{\mathbf{k}}|$.

Livingston [49] proved that for $3 \le k_1 = \cdots = k_n$, $\mathcal{L}_{\mathbf{k}}$ has the strict star property. Using the shifting technique (see [32]) in an inductive argument, the author [12] extended Livingston's result for the case when $3 \le k_1 \le \cdots \le k_n$. The above results sum up as follows.

Theorem 5.1 ([6, 12, 49])**.** *Let* $1 \le k_1 \le \cdots \le k_n$ *and let* $\mathbf{k} := (k_1,\dots,k_n)$. *Then:*
(i) $\mathcal{L}_{\mathbf{k}}$ *has the star property at* $\{(1,1)\}$*;*
(ii) *if* $k_1 \neq 2$ *then* $\mathcal{L}_{\mathbf{k}}$ *has the strict star property.*

If $k_1 = 2$ then $\mathcal{L}_{\mathbf{k}}$ may not have the strict star property; indeed, if $k_1 = k_2 = k_3$ then $\{A \in \mathcal{L}_{\mathbf{k}} \colon |A \cap \{(1,1),(2,1),(3,1)| \ge 2\}$ is a non-trivial intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$ whose size is $\frac{1}{k_1}|\mathcal{L}_{\mathbf{k}}|$ (i.e. the maximum).

Recall that $\mathcal{S}_{[n],k} = \mathcal{L}_{(k_1,\dots,k_n)}$ with $k_1 = \cdots = k_n = k$. The same argument used in [12] to extend Livingston's result [49] gives the following extension of part (the sufficiency conditions) of Corollary 4.5 and generalisation of Theorem 5.1 with $k_1 \ge 2$.

Theorem 5.2. *Let* $2 \le t+1 \le k_1 \le \cdots \le k_n$ *and let* $\mathbf{k} := (k_1,\dots,k_n)$. *Then:*
(i) $\mathcal{L}_{\mathbf{k}}$ *has the t-star property at* $\{(1,1),\dots,(t,1)\}$*;*
(ii) *if* $k_1 \ge t+2$ *then* $\mathcal{L}_{\mathbf{k}}$ *has the strict t-star property.*

As we can see from Theorem 4.4 and Corollary 4.5, $\mathcal{L}_{\mathbf{k}}$ may not have the t-star property when $2 \le k_1 \le t$. Recall that for the case $k_1 = \cdots = k_n$, the extremal structures are given in Theorem 4.4, and they are all non-trivial when $2 \le k_1 \le t$.

The intersection problem for the families $\mathcal{L}_{\mathbf{k},r}$, $r = 1,\dots,n$, has also been treated to a significant extent. Note that $\mathcal{S}_{\binom{[n]}{r},k} = \mathcal{L}_{(k_1,\dots,k_n),r}$ with $k_1 = \cdots = k_n = k$. Using the shifting technique (see [32]) in an inductive argument, Holroyd, Spencer and Talbot [38] extended Theorem 4.3(i) as follows.

Theorem 5.3 ([38])**.** *Let* $2 \le k_1 \le \cdots \le k_n$ *and let* $\mathbf{k} := (k_1,\dots,k_n)$. *Then for any* $r \in [n]$, $\mathcal{L}_{\mathbf{k},r}$ *has the star property at* $\{(1,1)\}$.

The proof of their result can be easily extended to obtain that $\mathcal{L}_{\mathbf{k},r}$ has the strict star property if $(r,k_1) \neq (n,2)$ (see, for example, the proof of [12, Theorem 1.4]). The case $k_1 = 1$ proved to be harder, and Bey [7] solved it by applying the idea of *generating sets* introduced in [1].

Theorem 5.4 ([7]). *Let* $1 = k_1 = \cdots = k_m < k_{m+1} \leq \cdots \leq k_n$ *and let* $\mathbf{k} := (k_1, \dots, k_n)$. *Let* $p := \lfloor (m+1)/2 \rfloor$, *and for each* $i \in [p]$, *let* $\mathcal{A}_i := \{A \in \mathcal{L}_{\mathbf{k},r} \colon (1,1) \in A, i \leq |A \cap \{(1,1), \dots, (m,1)\}| \leq m-i\} \cup \{A \in \mathcal{L}_{\mathbf{k},r} \colon |A \cap \{(1,1), \dots, (m,1)\}| \geq m-i+1\}$. *Then one of the families* $\mathcal{A}_1, \dots, \mathcal{A}_p$ *is a largest intersecting sub-family of* $\mathcal{L}_{\mathbf{k},r}$.

Bey [7] also showed that when $r \leq n/2$ in the above theorem, $\mathcal{L}_{\mathbf{k},r}$ has the star property at $(1,1)$ (this is also proved in [38], and in [16] it is shown that $\mathcal{L}_{\mathbf{k},r}$ has the strict star property if $r < n/2$).

For the case when k_1 can be any positive integer but n is sufficiently large, Theorem 3.7 gives us the following t-intersection result.

Theorem 5.5. *Let* $1 \leq t \leq r$ *and let* $n \geq (r-t)\binom{3r-2t-1}{t+1} + r$. *Let* $1 \leq k_1 \leq \cdots \leq k_n$ *and let* $\mathbf{k} := (k_1, \dots, k_n)$. *Then:*
(i) $\mathcal{L}_{\mathbf{k},r}$ *has the* t*-star property at* $\{(1,1), \dots, (t,1)\}$.
(ii) $\mathcal{L}_{\mathbf{k},r}$ *has the strict* t*-star property.*

Proof. Let $\mathcal{H} := \mathcal{L}_{\mathbf{k},\leq n}$. Then clearly $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) = n$. Thus, by Theorem 3.7 (with $S = \{r\}$), $\mathcal{H}^{(r)}$ has the strict t-star property. Part (ii) follows since $\mathcal{H}^{(r)} = \mathcal{L}_{\mathbf{k},r}$. This in turn proves (i) since the family $\mathcal{L}_{\mathbf{k},r}(T)$ with $T := \{(1,1), \dots, (t,1)\}$ is clearly a largest t-star of $\mathcal{L}_{\mathbf{k},r}$. □

We mention that Erdős, Seress, and Székely [30] determined non-trivial t-intersecting sub-families of $\mathcal{L}_{\mathbf{k},r}$ of maximum size for the case when n is sufficiently large.

Finally, for the family $\mathcal{L}_{\mathbf{k},\leq n}$ of all labeled sets defined on the n-tuple $\mathbf{k}$, we have the following immediate consequence of Theorems 3.2 and 5.3.

Theorem 5.6. *For any* $1 \leq k_1 \leq \cdots \leq k_n$, $\mathcal{L}_{(k_1,\dots,k_n),\leq n}$ *has the star property at* $\{(1,1)\}$.

Proof. Let $\mathbf{k} := (k_1, \dots, k_n)$. If $k_1 = 1$ then $\mathcal{L}_{\mathbf{k},\leq n}$ is compressed with respect to $(1,1)$ and hence, since $\mathcal{L}_{\mathbf{k},\leq n}$ is hereditary, the result follows by Theorem 3.2. Now suppose $k_1 \geq 2$. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{L}_{\mathbf{k},\leq n}$. So $\emptyset \notin \mathcal{A}$. By Theorem 5.3, $|\mathcal{A}^{(r)}| \leq |\mathcal{L}_{\mathbf{k},r}((1,1))|$ for all $r \in [n]$. Thus, we have $|\mathcal{A}| = \sum_{r=1}^{n} |\mathcal{A}^{(r)}| \leq \sum_{r=1}^{n} |\mathcal{L}_{\mathbf{k},r}((1,1))| = |\mathcal{L}_{\mathbf{k},\leq n}((1,1))|$. □

The above fact was also observed in [7], and it implies that the size of an intersecting sub-family of $\mathcal{L}_{\mathbf{k},\leq n}$ is at most $\frac{1}{k+1}|\mathcal{L}_{\mathbf{k},\leq n}|$, i.e. the size of the star $\mathcal{L}_{\mathbf{k},\leq n}((1,1))$ (indeed, the k_1+1 families $\mathcal{L}_{\mathbf{k},\leq n}((1,1)), \dots, \mathcal{L}_{\mathbf{k},\leq n}((1,k_1))$ and $\mathcal{L}_{(k_2,\dots,k_n),\leq n-1}$ partition $\mathcal{L}_{\mathbf{k},\leq n}$ and are of the same size). In view of the above-mentioned fact that $\mathcal{L}_{\mathbf{k},r}$ has the strict star property when $k_1 \geq 2$ and $(r,k_1) \neq (n,2)$ (in particular, when $1 \leq r \leq n-1$), one can go on to show that $\mathcal{L}_{\mathbf{k},\leq n}$ has the strict star property if $k_1 \geq 2$. If $k_1 = 1$ then $\mathcal{L}_{\mathbf{k},\leq n}$ may not have the strict star property; indeed, if $k_1 = k_2 = k_3 = 1$ then $\{A \in \mathcal{L}_{\mathbf{k},\leq n} \colon |A \cap \{(1,1),(2,1),(3,1)\}| \geq 2\}$ is a non-trivial intersecting sub-family that is as large as the largest star $\mathcal{L}_{\mathbf{k},\leq n}((1,1))$.

To the best of the author's knowledge, no general t-intersection theorem for $\mathcal{L}_{\mathbf{k},\leq n}$ is known.

6 Intersecting Families of Permutations and Partial Permutations

In [23, 24] the study of intersecting permutations was initiated. Deza and Frankl [24] showed that $\mathcal{S}^*_{[n],n}$ has the star property. So the size of an intersecting sub-family of $\mathcal{S}^*_{[n],n}$ is at most $(n-1)!$. The argument of the proof of this result is the same translation argument, given in the previous section, that yields Berge's intersection result for labeled sets [6], and it also gives us that for $n \leq k$, $\mathcal{S}^*_{[n],k}$ has the star property (recall that $\mathcal{S}^*_{[n],k} = \emptyset$ if $n > k$). Indeed, it gives us that for any intersecting sub-family $\mathcal{A}$ of $\mathcal{S}^*_{[n],k}$, $k|\mathcal{A}| \leq |\mathcal{S}^*_{[n],k}| = \frac{k!}{(k-n)!}$ and hence $|\mathcal{A}| \leq \frac{(k-1)!}{(k-n)!}$.

The question of whether $\mathcal{S}^*_{[n],n}$ has the strict star property proved to be much more difficult to answer. Cameron and Ku [18] and Larose and Malvenuto [47] independently gave an affirmative answer (other proofs are given in [35, 56]). Larose and Malvenuto [47] also proved the following generalisation (another proof is found in [17]).

Theorem 6.1 ([47]). *For $1 \leq n \leq k$, $\mathcal{S}^*_{[n],k}$ has the strict star property.*

Ku and Leader [46] investigated partial permutations. Using Katona's cycle method [40], they proved that $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the star property for all $r \in [n-1]$ (note that $\mathcal{S}^*_{\binom{[n]}{r},n} = \mathcal{S}^*_{[n],n}$ if $r = n$), and they also showed that $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict star property for all $r \in [8, n-3]$. Naturally, they conjectured that $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict star property for the few remaining values of r too. This was settled by Li and Wang [48] using tools forged by Ku and Leader. So the intersection results for $\mathcal{S}^*_{[n],n}$ and $\mathcal{S}^*_{\binom{[n]}{r},n}$ ($r \in [n-1]$) sum up as follows.

Theorem 6.2 ([18, 46, 47, 48]). *For any $r \in [n]$, $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict star property.*

When it comes to t-intersecting families of permutations, things are of course much harder. Solving a long-standing conjecture of Deza and Frankl [24], Ellis, Friedgut and Pilpel [26] recently managed to prove the following.

Theorem 6.3 ([26]). *For any integer $t \geq 1$, there exists an integer $n_0(t)$ such that for any $n \geq n_0(t)$, $\mathcal{S}^*_{[n],n}$ has the strict t-star property.*

Their remarkable proof is based on eigenvalue techniques and representation theory of the symmetric group. The condition $n \geq n_0(t)$ is necessary. Indeed, let $P_j := \{(i,i)\colon i \in [j]\}$ for any integer $j \geq 1$, and let

$$\mathcal{G}_{n,k,t} := \begin{cases} \{A \in \mathcal{S}_{[n],k}\colon |A \cap P_n| \geq (n+t)/2\} & \text{if } n-t \text{ is even;} \\ \{A \in \mathcal{S}_{[n],k}\colon |A \cap P_{n-1}| \geq (n+t-1)/2\} & \text{if } n-t \text{ is odd.} \end{cases}$$

Deza and Frankl [24] showed that when $t = n-s$ for some $s \geq 3$ and n is sufficiently large (depending on s), $\mathcal{G}_{n,n,t}$ is a largest t-intersecting sub-family of $\mathcal{S}^*_{[n],n}$ and is larger than the t-stars. Brunk and Huczynska [17] extended this result as follows.

Theorem 6.4 ([17, 24]). *For any integers $p \geq 0$ and $q \geq 2$ with $(p,q) \neq (0,2)$, there exists an integer $n^*_0(p,q)$ such that for any $n \geq n^*_0(p,q)$, any largest $(n-q)$-intersecting sub-family of $\mathcal{S}^*_{[n],n+p}$ is a copy of $\mathcal{G}_{n,n+p,n-q}$.*

They also conjectured that for any $n \leq k$ and $k \geq 8$, the extremal structures are similar to those in Theorem 2.2.

Conjecture 6.5 ([17])**.** *Let $1 \leq t \leq n \leq k$ and $k \geq 8$. Let $p := \lfloor (n-t)/2 \rfloor$, and for any integer i with $0 \leq i \leq p$, let $\mathcal{A}_i := \left\{ A \in \mathcal{S}^*_{[n],k} \colon |A \cap P_{t+2i}| \geq t+i \right\}$. Then:*
(i) *one of the families $\mathcal{A}_0, \ldots, \mathcal{A}_p$ is a largest t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$;*
(ii) *any largest t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$ is a copy of one of the families $\mathcal{A}_0, \ldots, \mathcal{A}_p$.*

For the general case when $\mathcal{F}$ is any family, a conjecture for t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$ similar to Conjecture 4.8 was suggested in [15].

Conjecture 6.6 ([15])**.** *For any integer $t \geq 1$, there exists an integer $k_0^*(t)$ such that for any $k \geq k_0^*(t)$ and any family $\mathcal{F}$, $\mathcal{S}^*_{\mathcal{F},k}$ has the strict t-star property.*

Theorem 6.3 solves the special case $\mathcal{F} = \{[n]\}$ and $k = n \geq k_0^*(t)$. The author [15] proved the following relaxation of the statement of the conjecture.

Theorem 6.7 ([15])**.** *For any integers r and t with $1 \leq t < r$, let $k_0^*(r,t) := \binom{r}{t}\binom{3r-2t-1}{\lfloor \frac{3r-2t-1}{2} \rfloor}\frac{r!}{(r-t-1)!} + r + 1$. For any $k \geq k_0^*(r,t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) \leq r$, $\mathcal{S}^*_{\mathcal{F},k}$ has the strict t-star property.*

This is an analogue of Theorem 4.9, and the general idea behind its proof is similar to that behind the proofs of Theorems 3.7 (see Section 3) and 4.9.

By taking $\mathcal{F} = [n]$ and $k \geq k_0^*(n,t)$ in Theorem 6.7, we obtain the following.

Corollary 6.8. *Let $k \geq k_0^*(n,t)$, where $k_0^*(n,t)$ is as in Theorem 6.7. Then $\mathcal{S}^*_{[n],k}$ has the strict t-star property.*

Thus, when k is sufficiently large, the size of a t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$ is at most $\frac{(k-t)!}{(k-n)!}$.

The following t-intersection result for partial permutations is another immediate consequence of Theorem 6.7, obtained by taking $n \geq k_0^*(r,t)$ and $\mathcal{F} = \binom{[n]}{r}$.

Corollary 6.9. *Let $n \geq k_0^*(r,t)$, where $k_0^*(r,t)$ is as in Theorem 6.7. Then $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict t-star property.*

Thus, when n is sufficiently large, the size of a t-intersecting sub-family of $\mathcal{S}^*_{\binom{[n]}{r},n}$ is at most $\binom{n-t}{r-t}\frac{(n-t)!}{(n-r)!}$. This was also proved in [45].

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