A short proof of a cross-intersection theorem of Hilton

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Abstract

Families $\mathcal{A}_1, ..., \mathcal{A}_k$ of sets are said to be *cross-intersecting* if $A_i \cap A_j \neq \emptyset$ for any $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $i \neq j$. A nice result of Hilton that generalises the Erdős-Ko-Rado (EKR) Theorem says that if $r \leq n/2$ and $\mathcal{A}_1$, ..., $\mathcal{A}_k$ are cross-intersecting sub-families of $\binom{[n]}{r}$, then

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{r} & \text{if } k \leq \frac{n}{r}; \\ k\binom{n-1}{r-1} & \text{if } k \geq \frac{n}{r}, \end{cases}$$

and the bounds are best possible. We give a short proof of a slightly stronger version. For this purpose, we extend Daykin's proof of the EKR Theorem to obtain the following improvement of the EKR Theorem: if $r \leq n/2$, $\mathcal{A} \subseteq \binom{[n]}{r}$, $\mathcal{A}^* := \{A^* \in \mathcal{A} \colon A^* \cap A \neq \emptyset \text{ for all } A \in \mathcal{A}\}$ and $\mathcal{A}' := \mathcal{A} \backslash \mathcal{A}^*$, then

$$|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'| \leq \binom{n-1}{r-1}.$$

1 Introduction

A family $\mathcal{A}$ of sets is said to be *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. Families $\mathcal{A}_1, ..., \mathcal{A}_k$ are said to be *cross-intersecting* if $A_i \cap A_j \neq \emptyset$ for any $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $i \neq j$. For $r, m \in [n] := \{1, 2, ..., n\}$, let $\mathcal{S}_{n,r,m}$ be the *star family* $\{A \in \binom{[n]}{r} \colon m \in A\}$, where $\binom{[n]}{r} = \{A \subset [n] \colon |A| = r\}$.

The following is a classical result in the literature.

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Theorem 1.1 (Erdős, Ko, Rado [3], Hilton, Milner [5]) *If $r \leq n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1},$$

and if $n > 2r$, then equality holds iff $\mathcal{A} = \mathcal{S}_{n,r,m}$ for some $m \in [n]$.

The bound was proved by Erdős, Ko and Rado, and the extremal case was established later by Hilton and Milner as part of a more general result. Two alternative short and beautiful proofs of the Erdős-Ko-Rado (EKR) Theorem were obtained by Katona [6] and Daykin [2]. In his proof, Katona introduced an elegant technique called the *cycle method*. Daykin's proof is based on a fundamental result known as the Kruskal-Katona Theorem [7, 8] (stated in the next section).

The KK Theorem was also used by Hilton in the proof of the following generalisation of the EKR Theorem.

Theorem 1.2 (Hilton [4]) *Let $r \leq n/2$ and $k \geq 2$. Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-intersecting sub-families of $\binom{[n]}{r}$, where $\mathcal{A}_1 \neq \emptyset$. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{r} & \text{if } k \leq \frac{n}{r}; \\ k\binom{n-1}{r-1} & \text{if } k \geq \frac{n}{r}. \end{cases}$$

If equality holds, then
(i) $\mathcal{A}_1 = \binom{[n]}{r}$ and $\mathcal{A}_i = \emptyset$, $i = 2, ..., k$, if $k < n/r$;
(ii) $|\mathcal{A}_i| = \binom{n-1}{r-1}$, $i = 1, ..., k$, if $k > n/r$;
(iii) $\mathcal{A}_1$, ..., $\mathcal{A}_k$ are as in (i) or (ii) if $k = n/r > 2$.

By setting $k > n/r$ and $\mathcal{A}_1 = ... = \mathcal{A}_k$ in the above result, we clearly obtain the EKR Theorem.

For $\mathcal{A} \subseteq 2^{[n]} := \{A \colon A \subseteq [n]\}$, let $\overline{\mathcal{A}} := \{[n] \backslash A \colon A \in \mathcal{A}\}$, $\mathcal{A}^* := \{A^* \in \mathcal{A} \colon A^* \cap A \neq \emptyset \text{ for all } A \in \mathcal{A}\}$ and $\mathcal{A}' := \mathcal{A} \backslash \mathcal{A}^*$. Let $\mathcal{A}^{(r)} := \{A \in \mathcal{A} \colon |A| = r\}$ and $\partial_r \mathcal{A} := \{B \colon B \subset A \text{ for some } A \in \mathcal{A}, |B| = r\}$.

We will show that Theorem 1.2 follows from the next result, the proof of which will be a slight extension of Daykin's proof of the EKR Theorem.

Theorem 1.3 *Let $r \leq n/2$ and $s \leq n - r$. Let $\mathcal{A}$ be an intersecting sub-family of $\binom{[n]}{r}$. Then*

$$|\partial_s \overline{\mathcal{A}}| \geq \frac{\binom{n-1}{s}}{\binom{n-1}{r-1}} |\mathcal{A}|,$$

and if $s < n - r$, then equality holds iff $\mathcal{A} = \mathcal{S}_{n,r,m}$ for some $m \in [n]$.

As a consequence of the above result, we have the following extension of Theorem 1.1.

Corollary 1.4 *Let $r \leq n/2$, and let $\mathcal{A} \subseteq \binom{[n]}{r}$. Then*

$$|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'| \leq \binom{n-1}{r-1},$$

and if $n > 2r$ and $\mathcal{A}^* \neq \emptyset$, then equality holds iff $\mathcal{A} = \mathcal{S}_{n,r,m}$ for some $m \in [n]$.

Proof. By definition, $\mathcal{A}' \subseteq \binom{[n]}{r} \backslash (\mathcal{A}^* \cup \partial_r \overline{\mathcal{A}^*})$. So by Theorem 1.3 with $r = s$,

$$|\mathcal{A}'| \leq \binom{n}{r} - (|\mathcal{A}^*| + |\partial_r \overline{\mathcal{A}^*}|) \leq \binom{n}{r} - \left(|\mathcal{A}^*| + \frac{(n-r)|\mathcal{A}^*|}{r}\right) = \binom{n}{r} - \frac{n|\mathcal{A}^*|}{r},$$

and if $n > 2r$ and $\mathcal{A}^* \neq \emptyset$, then equality holds iff $\mathcal{A}^* = \mathcal{S}_{n,r,m}$ for some $m \in [n]$. Now $|\mathcal{A}'| \leq \binom{n}{r} - \frac{n|\mathcal{A}^*|}{r}$ implies $\frac{r}{n}|\mathcal{A}'| \leq \binom{n-1}{r-1} - |\mathcal{A}^*|$. Hence result. $\square$

Note that Theorem 1.1 is the special case $\mathcal{A} = \mathcal{A}^*$ in the above corollary. We will show that this corollary leads to Theorem 1.2 and the following refinement.

Theorem 1.5 (Extension of Theorem 1.2) *Suppose equality holds in Theorem 1.2.*
(I) If $k > n/r$, then $\mathcal{A}_1 = ... = \mathcal{A}_k$, $\mathcal{A}_1$ is intersecting and $|\mathcal{A}_1| = \binom{n-1}{r-1}$; and if moreover $n > 2r$, then $\mathcal{A}_1 = \mathcal{S}_{n,r,m}$ for some $m \in [n]$.
(II) If $k = n/r > 2$, then $\mathcal{A}_1$, ..., $\mathcal{A}_k$ are as in (I) or Theorem 1.2(i).

2 Proofs

We first prove Theorems 1.2 and 1.5 from Corollary 1.4, and we prove Theorem 1.3 later. We need the following result, which is often useful for determining the structure of extremal intersecting families. The proof is an easy exercise, but we shall give it for completeness.

Proposition 2.1 *Let $\emptyset \neq \mathcal{A} \subseteq \binom{[n]}{r}$, $2r < n$, such that, for any $A \in \mathcal{A}$ and $B \in \binom{[n]\backslash A}{r}$, $B \in \mathcal{A}$. Then $\mathcal{A} = \binom{[n]}{r}$.*

Proof. Let $A \in \mathcal{A}$. Let B be an arbitrary set in $\binom{[n]}{r}$ that intersects A in $r-1$ elements. Since $n \geq 2r+1$, we can choose $C \in \binom{[n]}{r}$ such that C is disjoint from $A \cup B$. By the assumption of the proposition, we have $C \in \mathcal{A}$, which in turn implies $B \in \mathcal{A}$. Repeated application of this step gives us that any set in $\binom{[n]}{r}$ is also in $\mathcal{A}$. $\square$

Proof of Theorems 1.2, 1.5. Let $\mathcal{A} := \bigcup_{i=1}^k \mathcal{A}_i$. Clearly $\mathcal{A}^* = \bigcup_{i=1}^k \mathcal{A}_i^*$ and $\mathcal{A}' = \bigcup_{i=1}^k \mathcal{A}_i'$. Suppose $\mathcal{A}_i' \cap \mathcal{A}_j' \neq \emptyset$, $i \neq j$. Let $A \in \mathcal{A}_i' \cap \mathcal{A}_j'$. Then there exists $A_i \in \mathcal{A}_i'$ such that $A \cap A_i = \emptyset$, which is a contradiction because $A \in \mathcal{A}_j$. So $\mathcal{A}_i' \cap \mathcal{A}_j' = \emptyset$ for $i \neq j$, and hence $|\mathcal{A}'| = \sum_{i=1}^k |\mathcal{A}_i'|$. Note that Corollary 1.4 gives us $|\mathcal{A}'| + \frac{n}{r}|\mathcal{A}^*| \leq \binom{n}{r}$. So we have

$$\sum_{i=1}^k |\mathcal{A}_i| = \sum_{i=1}^k |\mathcal{A}_i'| + \sum_{i=1}^k |\mathcal{A}_i^*| \leq |\mathcal{A}'| + k|\mathcal{A}^*| \leq \binom{n}{r} + (k - \frac{n}{r})|\mathcal{A}^*|. \quad (1)$$

If $k < \frac{n}{r}$ then $\sum_{i=1}^{k} |\mathcal{A}_i| \leq \binom{n}{r}$, and equality holds iff $\mathcal{A}^* = \emptyset$ and $\mathcal{A} = \mathcal{A}' = \binom{[n]}{r}$. If $A \in \mathcal{A}_1$ and $B \in \binom{[n]\backslash A}{r} \backslash \mathcal{A}_1$, then $B \notin \mathcal{A}_i$, $i = 2, ..., k$, and hence $B \in \binom{[n]}{r} \backslash \mathcal{A}$. Thus, if $\mathcal{A} = \binom{[n]}{r}$ then the conditions of Proposition 2.1 hold for $\mathcal{A}_1$ (recall that $\mathcal{A}_1 \neq \emptyset$), and therefore $\mathcal{A}_1 = \mathcal{A} = \binom{[n]}{r}$. Hence (i).

If $k > \frac{n}{r}$ then, by (1) and Corollary 1.4,

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \binom{n}{r} + (k - \frac{n}{r})\binom{n-1}{r-1} = k\binom{n-1}{r-1},$$

and equality holds iff $\mathcal{A}_1^* = ... = \mathcal{A}_k^* = \mathcal{A}^*$ and $|\mathcal{A}^*| = \binom{n-1}{r-1} = |\mathcal{A}|$. Also by Corollary 1.4, if $|\mathcal{A}^*| = \binom{n-1}{r-1}$ and $n > 2r$, then $\mathcal{A}^* = \mathcal{S}_{n,r,m}$ for some $m \in [n]$. Hence (I).

Suppose $k = \frac{n}{r} > 2$. Then, by (1), $\sum_{i=1}^{k} |\mathcal{A}_i| \leq |\mathcal{A}'| + \frac{n}{r}|\mathcal{A}^*| \leq \binom{n}{r}$. So $|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'| \leq \binom{n-1}{r-1}$. It is immediate by Corollary 1.4 that if $\mathcal{A}^* \neq \emptyset$ then $\mathcal{A}^*$ is as in the case $k > \frac{n}{r}$. If $\mathcal{A}^* = \emptyset$ then $\mathcal{A}$ is as in the case $k < \frac{n}{r}$. Hence (II). $\square$

We now work towards the proof of Theorem 1.3. The proof is based on the two well-known results below, the first of which is a deep and fundamental theorem. For $1 \leq p \leq m$ and $1 \leq q \leq \binom{m}{p}$, denote the family of the first q sets in $\binom{[m]}{p}$ in *colex order* by $\mathcal{C}(p, q)$.

Theorem 2.2 (KK Theorem) *Let $1 \leq s < p \leq m$, and let $\emptyset \neq \mathcal{F} \subseteq \binom{[m]}{p}$. Then*

$$|\partial_s \mathcal{F}| \geq |\partial_s \mathcal{C}(p, |\mathcal{F}|)|.$$

If $p \leq l \leq m$ and $|\mathcal{F}| = \binom{l}{p}$, then equality holds iff $\mathcal{F}$ is isomorphic to $\binom{[l]}{p}$.

Lemma 2.3 *Let $1 \leq s < p \leq m$, and let $\emptyset \neq \mathcal{F} \subseteq \binom{[m]}{p}$. Then*

$$|\partial_s \mathcal{F}| \geq \frac{\binom{m}{s}}{\binom{m}{p}}|\mathcal{F}|,$$

and equality holds iff $\mathcal{F} = \binom{[m]}{p}$.

For easy-to-read proofs of the above results, we refer the reader to [1, Chapter 5] and [1, Chapter 3] respectively. We point out that Lemma 2.3 follows by a short and standard double-counting argument.

Proof of Theorem 1.3. If $s = n - r$ then $\partial_s \overline{\mathcal{A}} = \overline{\mathcal{A}}$ and hence $|\partial_s \overline{\mathcal{A}}| = |\mathcal{A}| = \frac{\binom{n-1}{s}}{\binom{n-1}{r-1}}|\mathcal{A}|$. Now consider $s < n - r$. By Theorem 2.2 (with $p = n - r$, $m = n$ and $\mathcal{F} = \overline{\mathcal{A}}$), we have $|\partial_s \overline{\mathcal{A}}| \geq |\partial_s \mathcal{C}(n - r, |\overline{\mathcal{A}}|)|$.

Suppose $|\overline{\mathcal{A}}| > \binom{n-1}{n-r}$. Then $|\partial_r \mathcal{C}(n - r, |\overline{\mathcal{A}}|)| > \binom{n-1}{r}$, and hence, since $|\mathcal{A}| = |\overline{\mathcal{A}}|$ and $|\partial_r \overline{\mathcal{A}}| \geq |\partial_r \mathcal{C}(n-r, |\overline{\mathcal{A}}|)|$, we get $|\mathcal{A}| + |\partial_r \overline{\mathcal{A}}| > \binom{n-1}{n-r} + \binom{n-1}{r} = \binom{n}{r}$, which is a contradiction because, since $\mathcal{A}$ is intersecting, $\mathcal{A}$ and $\partial_r \overline{\mathcal{A}}$ are disjoint sub-families of $\binom{[n]}{r}$.

Therefore, $|\overline{\mathcal{A}}| \leq \binom{n-1}{n-r}$. By Lemma 2.3 (with $p = n-r$, $m = n-1$ and $\mathcal{F} = \mathcal{C}(n-r, |\overline{\mathcal{A}}|)$) and the above, we have

$$|\partial_s \overline{\mathcal{A}}| \geq |\partial_s \mathcal{C}(n-r, |\overline{\mathcal{A}}|)| \geq \frac{\binom{n-1}{s}}{\binom{n-1}{n-r}} |\mathcal{C}(n-r, |\overline{\mathcal{A}}|)| = \frac{\binom{n-1}{s}}{\binom{n-1}{n-r}} |\overline{\mathcal{A}}| = \frac{\binom{n-1}{s}}{\binom{n-1}{r-1}} |\mathcal{A}|.$$

So we have proved the bound in the theorem. If $\mathcal{A} = \mathcal{S}_{n,r,m}$ for some $m \in [n]$, then clearly the bound is attained. We now prove the converse. So suppose the bound is attained. Then

$$|\partial_s \overline{\mathcal{A}}| = |\partial_s \mathcal{C}(n-r, |\overline{\mathcal{A}}|)| = \frac{\binom{n-1}{s}}{\binom{n-1}{n-r}} |\mathcal{C}(n-r, |\overline{\mathcal{A}}|)|,$$

and Lemma 2.3 tells us that the second equality implies $\mathcal{C}(n-r, |\overline{\mathcal{A}}|) = \binom{[n-1]}{n-r}$. So we have $|\partial_s \overline{\mathcal{A}}| = |\partial_s \mathcal{C}(n-r, |\overline{\mathcal{A}}|)|$ with $|\overline{\mathcal{A}}| = \binom{n-1}{n-r}$, and hence, by Theorem 2.2 (with $p = n-r$, $l = n-1$, $m = n$ and $\mathcal{F} = \overline{\mathcal{A}}$), $\overline{\mathcal{A}}$ is isomorphic to $\binom{[n-1]}{n-r}$. It follows that $\mathcal{A}$ is isomorphic to $\mathcal{S}_{n,r,n}$, i.e. $\mathcal{A} = \mathcal{S}_{n,r,m}$ for some $m \in [n]$. $\square$

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