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Extremal t-intersecting sub-families of hereditary families

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Abstract

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if any two sets in $\mathcal{A}$ contain at least t common elements. A t-intersecting family is said to be *trivial* if there are at least t elements common to all its sets. A family $\mathcal{H}$ is said to be *hereditary* if all subsets of any set in $\mathcal{H}$ are in $\mathcal{H}$.

For a finite family $\mathcal{F}$, let $\mathcal{F}^{(s)}$ be the family of s-element sets in $\mathcal{F}$, and let $\mu(\mathcal{F})$ be the size of a smallest set in $\mathcal{F}$ that is not a subset of any other set in $\mathcal{F}$. For any two integers r and t with $1 \leqslant t < r$, we determine an integer $n_0(r, t)$ such that, for any non-empty subset S of $\{t, t+1, \ldots, r\}$ and any finite hereditary family $\mathcal{H}$ with $\mu(\mathcal{H}) \geqslant n_0(r, t)$, the largest t-intersecting sub-families of the union $\bigcup_{s \in S} \mathcal{H}^{(s)}$ are trivial. The special case $\mathcal{H} = 2^{[n]}$ yields a classical theorem of Erdős, Ko and Rado. On the basis of the complete intersection theorem of Ahlswede and Khachatrian, we conjecture that the smallest such $n_0(r, t)$ is $(t + 1)(r - t + 1) + 1$, and we show that this is true if $\mathcal{H}$ is *compressed*.

We apply our main result to obtain new results on t-intersecting families of *signed sets*, *permutations* and *separated sets*. This work supports some open conjectures.

1. *Introduction*

1.1. *Notation and definitions*

Throughout this paper, unless otherwise stated, we shall use small letters such as x to denote elements of a set or positive integers, capital letters such as X to denote sets, and *calligraphic* letters such as $\mathcal{F}$ to denote *families* (that is, sets whose members are sets themselves). Unless otherwise stated, it is to be assumed that sets and families represented in this way are *finite*.

The set of positive integers $\{1, 2, \ldots\}$ is denoted by $\mathbb{N}$. For $m, n \in \mathbb{N}$, $m \leqslant n$, the set $\{i \in \mathbb{N} : m \leqslant i \leqslant n\}$ is denoted by $[m, n]$; for $m = 1$, we also write $[n]$. For a set X, the *power set* $\{A : A \subseteq X\}$ of X is denoted by 2^X, and the sub-families $\{Y \subseteq X : |Y| = r\}$ and $\{Y \subseteq X : |Y| \leqslant r\}$ are denoted by $\binom{X}{r}$ and $\binom{X}{\leqslant r}$, respectively. An *r-set* is a set of size r.

For a family $\mathcal{F}$, we define $\mathcal{F}^{(r)} := \{F \in \mathcal{F} : |F| = r\}$ and $\mathcal{F}^{(\leqslant r)} := \{F \in \mathcal{F} : |F| \leqslant r\}$. Also, we define $U(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} F$ and, for any subset V of $U(\mathcal{F})$, we define $\mathcal{F}\langle V\rangle := \{F \in \mathcal{F} : V \subseteq F\}$. We call $\mathcal{F}\langle V\rangle$ a *t-star of $\mathcal{F}$* if $|V| = t$ and $\mathcal{F}\langle V\rangle \neq \emptyset$. We may call a 1-star simply a *star*.

A family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ have a non-empty intersection. More generally, $\mathcal{A}$ is said to be *t-intersecting* if the intersection of any two sets in $\mathcal{A}$ has size at least t. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if the sets in $\mathcal{A}$ have a common t-subset; otherwise, $\mathcal{A}$ is said to be *non-trivial*. Note that a t-star of a family $\mathcal{F}$ is a maximal trivial t-intersecting sub-family of $\mathcal{F}$.

We say that a set M is *$\mathcal{F}$-maximal* if M is not a subset of any set in $\mathcal{F}\backslash\{M\}$. We define

$$\mu(\mathcal{F}) := \min\{|F| : F \in \mathcal{F}, F \text{ is } \mathcal{F}\text{-maximal}\}.$$

A family $\mathcal{F}$ is said to be

– a *hereditary family* (or an *ideal* or a *downset*) if all subsets of any set in $\mathcal{F}$ are in $\mathcal{F}$;
– an *antichain* or a *Sperner family* if all sets in $\mathcal{F}$ are $\mathcal{F}$-maximal;
– *uniform* if the sets in $\mathcal{F}$ are of the same size, and *r-uniform* if $\mathcal{F} = \mathcal{F}^{(r)}$.

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Thus a uniform family is an antichain, whereas a hereditary family $\mathcal{H} \neq \{\emptyset\}$ is not.

We will refer to a uniform sub-family $\mathcal{F}^{(r)}$ of a family $\mathcal{F}$ as a *level of* $\mathcal{F}$ or, more precisely, the *rth level of* $\mathcal{F}$.

A family $\mathcal{F}$ is said to be *compressed* if $U(\mathcal{F})$ has a total ordering of its elements induced by a relation $\preceq$ such that the following holds:

$$\{u_1, \ldots, u_r\} \in \mathcal{F}^{(r)} \text{ and } U(\mathcal{F}) \ni v_i \preceq u_i \text{ for } i = 1, \ldots, r \quad \Longrightarrow \quad \{v_1, \ldots, v_r\} \in \mathcal{F}^{(r)}.$$

1.2. *Extremal t-intersecting sub-families of $\binom{[n]}{r}$, $\binom{[n]}{\leqslant r}$ and $2^{[n]}$*

In the seminal paper [18], Erdős, Ko and Rado initiated the study of intersecting families, which has yielded a vast amount of beautiful results (the survey papers [15] and [20] are recommended) and is still a very active field of research. The first of two classical theorems proved in that paper is that, if $n \geqslant 2r$, then the size of an *extremal* (meaning largest) intersecting sub-family of $\binom{[n]}{r}$ is $\binom{n-1}{r-1}$, which is the size of a star of $\binom{[n]}{r}$. There are various proofs of this theorem, two of which are particularly short and beautiful: Katona's [31] using the *cycle method* and Daykin's [12] using a fundamental result known as the Kruskal–Katona theorem [30, 33]. Hilton and Milner [25] determined the size of a largest non-trivial intersecting sub-family of $\binom{[n]}{r}$, and consequently they established that the extremal intersecting sub-families are the stars if $n > 2r$.

THEOREM 1.1 (Erdős, Ko and Rado [18]; Hilton and Milner [25]). *Let $n \geqslant 2r$, $r \geqslant 2$. Let $\mathcal{A}$ be an intersecting sub-family of $\binom{[n]}{r}$. Then $|\mathcal{A}| \leqslant \binom{n-1}{r-1}$. Moreover, if the bound is attained and $n > 2r$, then $\mathcal{A}$ is a star of $\binom{[n]}{r}$.*

The following is the second classical result in [18].

THEOREM 1.2 (Erdős, Ko and Rado [18]). *For $t \leqslant r$, there exists $n_0(r,t) \in \mathbb{N}$ such that, for all $n \geqslant n_0(r,t)$, the extremal t-intersecting sub-families of $\binom{[n]}{r}$ are the t-stars of $\binom{[n]}{r}$.*

In view of the above facts, we say that a family $\mathcal{F}$ is *t-EKR* if the set of largest t-intersecting sub-families of $\mathcal{F}$ contains a t-star, and *strictly t-EKR* if the set of largest t-intersecting sub-families of $\mathcal{F}$ contains *only* t-stars. We may call a 1-EKR family simply *EKR*.

Erdős, Ko and Rado also illustrated the fact that $\binom{[n]}{r}$ is not t-EKR for a range of small values of n. For $t \geqslant 15$, Frankl [19] showed that the smallest $n_0(r,t)$ for which Theorem 1.2 holds is $(r-t+1)(t+1)+1$, and that $\binom{[n]}{r}$ is t-EKR but not strictly so if $n = (r-t+1)(t+1)$. Subsequently, Wilson [45] proved that, for any $1 \leqslant t \leqslant r$ and $n \geqslant (r-t+1)(t+1)$, $\binom{[n]}{r}$ is t-EKR. Frankl [19] conjectured that, for any $1 \leqslant t \leqslant r \leqslant n$, the size of an extremal t-intersecting sub-family of $\binom{[n]}{r}$ is $\max\{|\{A \in \binom{[n]}{r} : |A \cap [t+2i]| \geqslant t+i\}| : i \in \{0\} \cup [r-t]\}$. A proof of this long-standing conjecture together with a complete characterisation of the extremal structures was finally obtained by Ahlswede and Khachatrian [2]. The following is part of their result.

THEOREM 1.3 (Ahlswede and Khachatrian [2]). *Let $1 \leqslant t < r < n$. Then:*
(i) *$\binom{[n]}{r}$ is t-EKR if and only if $n \geqslant (r-t+1)(t+1)$;*
(ii) *$\binom{[n]}{r}$ is strictly t-EKR if and only if $n > (r-t+1)(t+1)$.*

Ahlswede, Bey, Engel and Khachatrian [1] considered the extremal problem for t-intersecting sub-families of $\binom{[n]}{\leqslant r}$. They made a conjecture which, similarly to that of Frankl mentioned above, says that the size of an extremal t-intersecting sub-family of $\binom{[n]}{r}$ is $\max\{|\{A \in \binom{[n]}{\leqslant r} : |A \cap [t+2i]| \geqslant t+i\}| : i \in \{0\} \cup [r-t]\}$. They also provided some evidence for their conjecture. Note that, by Theorem 1.3, the conjecture is true for $n \geqslant (r-t+1)(t+1)$.

Erdős, Ko and Rado [18] pointed out the simple fact that $2^{[n]}$ is EKR, and they asked what is the size of an extremal t-intersecting sub-family of $2^{[n]}$ for $t \geqslant 2$. The answer in a complete form was given by Katona [29].

THEOREM 1.4 (Katona [29]). *Let $t \geqslant 2$. Let $\mathcal{A}$ be a largest t-intersecting sub-family of $2^{[n]}$.*
(i) *If $n + t = 2l$, then $\mathcal{A} = \{A \subseteq [n] : |A| \geqslant l\}$.*
(ii) *If $n + t = 2l + 1$, then $\mathcal{A}$ is isomorphic to the family $\{A \subseteq [n] : |A \cap [n-1]| \geqslant l\}$.*

1.3. *Extremal-type conjectures on intersecting sub-families of hereditary families*

The power set 2^X of a set X is the simplest example of a hereditary family, but there are various other interesting examples, such as the family of independent sets of a *graph* or *matroid*. Clearly, if $\mathcal{H}$ is a hereditary family and $X_1, \ldots, X_k$ are the $\mathcal{H}$-maximal sets in $\mathcal{H}$, then $\mathcal{H} = 2^{X_1} \cup \ldots \cup 2^{X_k}$; in other words, a hereditary family is a union of power sets. Also note that any union of power sets is hereditary.

The following is an outstanding open problem in extremal set theory.

CONJECTURE 1.5 (Chvátal [10]). If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ is EKR.

Recall that $2^{[n]}$ is EKR; so the conjecture is true if there is only one $\mathcal{H}$-maximal set in $\mathcal{H}$. Chvátal [11] verified his conjecture for the case when $\mathcal{H}$ is *compressed*. Snevily [42] took this result a significant step forward by verifying the conjecture for $\mathcal{H}$ *compressed with respect to an element u of $U(\mathcal{H})$* (that is, $h \in H \in \mathcal{H}, u \notin H \Rightarrow (H \backslash \{h\}) \cup \{u\} \in \mathcal{H}$). Many other results have been inspired by this conjecture, and the PhD dissertation [40] is dedicated to it.

Before turning our attention to uniform intersecting sub-families of hereditary families, which are the theme of this paper, we recall the following. A *graph G* is a pair (V, E) with $E \subseteq \binom{V}{2}$, and a set $I \subseteq V$ is said to be an *independent set of G* if $\{i, j\} \notin E$ for any $i, j \in I$.

Let $\mathcal{I}_G$ denote the family of all independent sets of a graph G. Holroyd and Talbot [28] made the following interesting but apparently very difficult conjecture.

CONJECTURE 1.6 (Holroyd and Talbot [28]). If G is a graph with $\mu(\mathcal{I}_G) \geqslant 2r$, then ${\mathcal{I}_G}^{(r)}$ is EKR, and strictly so if $\mu(\mathcal{I}_G) > 2r$.

Clearly, the family $\mathcal{I}_G$ is a hereditary family. In [8], the following generalisation of Conjecture 1.6 is suggested.

CONJECTURE 1.7 (Borg [8]). If $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geqslant 2r$, then $\mathcal{H}^{(r)}$ is EKR, and strictly so if $\mu(\mathcal{H}) > 2r$.

Note that Theorem 1.1 solves the special case $\mathcal{H} = 2^{[n]}$.

2. *The main result, some consequences and a conjecture*

Conjecture 1.5 cannot be generalised to the t-intersection case; indeed, if $n > t \geqslant 2$ and $\mathcal{H} = 2^{[n]}$ then, by Theorem 1.4, $\mathcal{H}$ is not t-EKR. In view of Theorem 1.2, it is natural to question whether this can be done for Conjecture 1.7 or, more precisely, whether there exists an integer $n_0(r, t)$

such that $\mathcal{H}^{(r)}$ is t-EKR for any hereditary $\mathcal{H}$ with $\mu(\mathcal{H}) \geqslant n_0(r,t)$. Our main result, given by Theorem 2.1 below and proved in Section 4, gives more than an affirmative answer to this question. For $t \leqslant r$, we set

$$n_0^*(r,t) := (r-t)\binom{3r-2t-1}{t+1} + r.$$

THEOREM 2.1. *If $t \leqslant r$, $\emptyset \neq S \subseteq [t,r]$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geqslant n_0^*(r,t)$, then $\bigcup_{s\in S} \mathcal{H}^{(s)}$ is strictly t-EKR.*

Since $n_0^*(r,t)$ increases with r, Theorem 2.1 can be rephrased as follows.

THEOREM 2.1 (Rephrased). *If $\mathcal{H}$ is a hereditary family, $t \leqslant r \leqslant \max\{p \in \mathbb{N} : n_0^*(p,t) \leqslant \mu(\mathcal{H})\}$ and $\emptyset \neq S \subseteq [t,r]$, then $\bigcup_{s\in S} \mathcal{H}^{(s)}$ is strictly t-EKR.*

Note that Theorem 1.2 follows from the case $\mathcal{H} = 2^{[n]}$ and $S = \{r\}$ in Theorem 2.1. Also note that Theorem 1.3 illustrates the fact that we cannot do without some condition $\mu(\mathcal{H}) \geqslant n_0(r,t)$.

REMARK 1. The motivation behind establishing the result for any *union* of levels of a hereditary family $\mathcal{H}$ within a certain range is that this general form cannot be immediately deduced from the result for just one level of $\mathcal{H}$ (that is, the case $S = \{r\}$). As revealed in the example below, the reason is simply that, if T is a t-set such that $\mathcal{H}^{(s)}\langle T\rangle$ ($s \in [t,r]$) is a largest t-star of $\mathcal{H}^{(s)}$ then, for $p \neq s$ ($p \in [t,r]$), $\mathcal{H}^{(p)}\langle T\rangle$ not only may not be a largest t-star of the level $\mathcal{H}^{(p)}$ but may be smaller than some non-trivial t-intersecting sub-family of $\mathcal{H}^{(p)}$.

EXAMPLE 1. Consider $t = 1$, $r = 4$, $S = [3,4]$. Let $M_1, \dots, M_m$ be distinct sets such that their total intersection $X := M_1 \cap \dots \cap M_m$ satisfies $X = M_i \cap M_j$ for any $i, j \in [m]$, $i \neq j$. Let M_0 be a set that is disjoint from $M_1 \cup \dots \cup M_m$. Let $\mathcal{H}_1 := 2^{M_0}$, $\mathcal{H}_2 := 2^{M_1} \cup \dots \cup 2^{M_m}$. Let $\mathcal{H}$ be the hereditary family $\mathcal{H}_1 \cup \mathcal{H}_2$. Suppose $|X| = 3$ and $|M_1| = \dots = |M_m| < |M_0|$. Note that $\mu(\mathcal{H}) = |M_1|$. Let $w \in M_0$ and $x \in X$. Hence, for any $s \in S$, $\mathcal{L}_{1,s} := {\mathcal{H}_1}^{(s)}\langle\{w\}\rangle$ has size $\binom{|M_0|-1}{s-1}$ and is a largest star of ${\mathcal{H}_1}^{(s)}$, and $\mathcal{L}_{2,s} := {\mathcal{H}_2}^{(s)}\langle\{x\}\rangle$ has size $m\binom{|M_1|-1}{s-1} + (4-s)(1-m)$ (that is, $|\mathcal{L}_{2,3}| = m\binom{|M_1|-1}{2} + 1 - m$ and $|\mathcal{L}_{2,4}| = m\binom{|M_1|-1}{3}$) and is a largest star of ${\mathcal{H}_2}^{(s)}$; clearly at least one of $\mathcal{L}_{1,s}$ and $\mathcal{L}_{2,s}$ is a largest star of $\mathcal{H}^{(s)}$. For each $i \in [m]$, let $y_i \in M_i \backslash X$ and $A_i := (X\backslash\{x\}) \cup \{y_i\}$. Let $\mathcal{A}$ be the non-trivial intersecting sub-family $\{A \in {\mathcal{H}_2}^{(3)} : x \in A,\ A \cap (X\backslash\{x\}) \neq \emptyset\} \cup \{A_i : i \in [m]\}$ of ${\mathcal{H}_2}^{(3)}$. Thus $|\mathcal{A}| = m\left(\binom{|M_1|-1}{2} - \binom{|M_1|-3}{2}\right) + 1$. Now suppose $|M_0| = 4000$, $|M_1| = n_0^*(4,1) = 112$ and $m = 40\,000$. Then $|\mathcal{L}_{1,4}| = 10\,650\,673\,999 > |\mathcal{L}_{2,4}| = 8\,872\,600\,000$, and hence $\mathcal{L}_{1,4}$ is a largest star of $\mathcal{H}^{(4)}$ (so, by Theorem 2.1, $\mathcal{L}_{1,4}$ is in fact an extremal intersecting sub-family of $\mathcal{H}^{(4)}$). However, $|\mathcal{L}_{1,3}| = 7\,994\,001 < |\mathcal{A}| = 8\,760\,001$. This proves the claim in Remark 1.

What we have just demonstrated is in fact one of the central difficulties arising from any EKR-type problem for hereditary families. In the proof of Theorem 2.1, we overcome this obstacle by showing that, for any non-trivial t-intersecting sub-family $\mathcal{A}$ of the union, we can construct a t-star that is larger than $\mathcal{A}$ (and that is not necessarily a largest t-star); this is the crucial idea presented here. Many other proofs of EKR-type results (such as Theorem 2.7 below) are based on determining at least one largest t-star; as in the case of each theorem mentioned in Subsection 1.2, the setting is often symmetrical to the extent that all t-stars are of the same size and of a known size.

We now present some immediate consequences of Theorem 2.1, the first of which is actually a special case of the theorem.

COROLLARY 2.2. *Conjecture* 1.7 *is true if* $\mu(\mathcal{H}) \geqslant n_0^*(r,1)$.

COROLLARY 2.3. *Conjecture* 1.5 *is true if* $\mathcal{H} = \mathcal{J}^{(\leqslant r)}$ *for some hereditary family* $\mathcal{J}$ *with* $\mu(\mathcal{J}) \geqslant n_0^*(r,1)$.

Proof. Let $\mathcal{J}$ be a hereditary family with $\mu(\mathcal{J}) \geqslant n_0^*(r,1)$. Let $S = [r]$. By Theorem 2.1 with $t = 1$, $\bigcup_{s \in S} \mathcal{J}^{(s)}$ is strictly EKR. The result follows since $\mathcal{J}^{(\leqslant r)} = \bigcup_{s \in S} \mathcal{J}^{(s)}$. □

COROLLARY 2.4. *Let* $\mathcal{H}$ *be a hereditary sub-family of* $2^{\mathbb{N}}$. *For* $n \in \mathbb{N}$, *let* $\mathcal{H}_n := \mathcal{H} \cap 2^{[n]}$. *Suppose* $\mu(\mathcal{H}_n) \to \infty$ *as* $n \to \infty$. *Then, for any* $t \leqslant r$, *there exists* $n_0(\mathcal{H}, r, t) \in \mathbb{N}$ *such that, for any non-empty* $S \subseteq [t, r]$ *and any* $n \geqslant n_0(\mathcal{H}, r, t)$, $\bigcup_{s \in S} {\mathcal{H}_n}^{(s)}$ *is strictly t-EKR.*

Proof. Since $\mathcal{H}$ is hereditary, $\mathcal{H}_n$ is hereditary for all $n \in \mathbb{N}$. Having $\mu(\mathcal{H}_n) \to \infty$ as $n \to \infty$ means that for any $m \in \mathbb{N}$ there exists $n_1(\mathcal{H}, m) \in \mathbb{N}$ such that $\mu(\mathcal{H}_n) \geqslant m$ for all $n \geqslant n_1(\mathcal{H}, m)$. The result now follows from Theorem 2.1 by setting $n_0(\mathcal{H}, r, t) := n_1(\mathcal{H}, n_0^*(r,t))$. □

In the next section, we obtain an inequality that will yield results on ratios of sizes of certain levels of a hereditary family and on sizes of Sperner sub-families of certain unions of levels of a hereditary family. The inequality (given in Lemma 3.1) will have a fundamental role in the proof of Theorem 2.1 and, as we show in the next section, it also happens to be a stepping stone from Theorem 2.1 to the next theorem.

THEOREM 2.5. *If* $t \leqslant r$ *and* $\mathcal{H}$ *is a hereditary family with* $\mu(\mathcal{H}) \geqslant n_0^*(r,t)$, *then the largest t-intersecting Sperner sub-families of* $\mathcal{H}^{(\leqslant r)}$ *are the largest t-stars of* $\mathcal{H}^{(r)}$.

This result is inspired by the fact that Theorems 1.1 and 1.2 were actually proved in the more general context of Sperner sub-families of $\binom{[n]}{\leqslant r}$.

We finally suggest the following uniform version of Conjecture 1.5 and natural generalisation of Conjectures 1.6 and 1.7.

CONJECTURE 2.6. If $t \leqslant r$, $\emptyset \neq S \subseteq [t, r]$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geqslant n_0(r,t) := (t+1)(r-t+1)$, then $\bigcup_{s \in S} \mathcal{H}^{(s)}$ is t-EKR, and strictly so if $\mu(\mathcal{H}) > n_0(r,t)$ or $S \neq \{r\}$.

This claims that Theorem 2.1 remains true if $n_0^*(r,t)$ is replaced by $(t+1)(r-t+1)+1$. Clearly, Theorem 1.3 implies that the conjecture is true for $\mathcal{H} = 2^{[n]}$ and that the lower bound $(t+1)(r-t+1)$ cannot be replaced by a smaller one. In Section 5, we support the conjecture with the following result, the proof of which is in fact based on Theorem 1.3.

THEOREM 2.7. *Conjecture* 2.6 *is true if* $\mathcal{H}$ *is compressed.*

In Section 6, we apply Theorem 2.1 to obtain new results on t-intersecting families of *signed sets*, *permutations* and *separated sets*.

3. *A Sperner-type inequality for hereditary families, and some corollaries*

For any pair of families $\mathcal{A}$ and $\mathcal{F}$, let

$$\partial_{\mathcal{F}}^{(s)}\mathcal{A} := \{F \in \mathcal{F}^{(s)} : \text{there exists an } A \in \mathcal{A} \text{ such that } A \subseteq F \text{ or } F \subseteq A\}.$$

For $\mathcal{A} \subseteq \binom{[n]}{r}$ and $r < n$, the following holds:

$$|\partial_{2^{[n]}}^{(r+1)}\mathcal{A}| \geqslant \frac{n-r}{r+1}|\mathcal{A}|.$$

This is called a *local LYM* inequality; see [**5**, p. 12]. Sperner [**43**] determined this inequality in order to prove his classical result that Sperner sub-families of $2^{[n]}$ have size at most $\binom{n}{\lfloor n/2 \rfloor}$. The lemma below generalises the above inequality to one for sub-families of hereditary families. The lemma and the subsequent corollaries will lead us to Theorems 2.1 and 2.5.

Lemma 3.1. *If $\mathcal{H}$ is hereditary, $\mathcal{A} \subseteq \mathcal{H}^{(p)}$ and $p < q \leqslant \mu(\mathcal{H})$, then*

$$|\partial_{\mathcal{H}}^{(q)}\mathcal{A}| \geqslant \frac{\binom{\mu(\mathcal{H})-p}{q-p}}{\binom{q}{q-p}}|\mathcal{A}|.$$

Proof. For any $A \in \mathcal{A}$, let M_A be some $\mathcal{H}$-maximal set in $\mathcal{H}$ such that $A \subset M_A$. Hence $|M_A| \geqslant \mu(\mathcal{H})$, and $\binom{M_A}{q} \subseteq \mathcal{H}^{(q)}$ since $\mathcal{H}$ is hereditary. Therefore

$$\begin{aligned}\binom{\mu(\mathcal{H})-p}{q-p}|\mathcal{A}| &\leqslant \sum_{A\in\mathcal{A}}\binom{|M_A|-p}{q-p} = \sum_{A\in\mathcal{A}}\left|(\partial_{\mathcal{H}}^{(q)}\{A\}) \cap \binom{M_A}{q}\right| \\ &\leqslant \sum_{A\in\mathcal{A}}|\partial_{\mathcal{H}}^{(q)}\{A\}| = \sum_{B\in\partial_{\mathcal{H}}^{(q)}\mathcal{A}}|\partial_{\mathcal{A}}^{(p)}\{B\}| \leqslant \sum_{B\in\partial_{\mathcal{H}}^{(q)}\mathcal{A}}\binom{q}{p} \\ &= \binom{q}{q-p}|\partial_{\mathcal{H}}^{(q)}\mathcal{A}|,\end{aligned}$$

and hence the result. □

Corollary 3.2. *If $\mathcal{H}$ is hereditary and $p < q \leqslant \mu(\mathcal{H})$, then*

$$|\mathcal{H}^{(q)}| \geqslant \frac{\binom{\mu(\mathcal{H})-p}{q-p}}{\binom{q}{q-p}}|\mathcal{H}^{(p)}|.$$

Proof. This follows immediately from Lemma 3.1 as $\partial_{\mathcal{H}}^{(q)}\mathcal{H}^{(p)} \subseteq \mathcal{H}^{(q)}$. □

We point out that Lemma 3.1 and Hall's marriage theorem [**24**] also yield the following strong corollary which, however, we will not need to apply here.

Corollary 3.3. *Let $\mathcal{H}$ be a hereditary family, and let $p < q \leqslant \mu(\mathcal{H}) - p$. Then there exists an injection $f : \mathcal{H}^{(p)} \to \mathcal{H}^{(q)}$ such that $A \subset f(A)$ for all $A \in \mathcal{H}^{(p)}$. If $q < \mu(\mathcal{H}) - p$ then f is not a bijection.*

Corollary 3.4. *Let $\mathcal{H}$ be a hereditary family with $\mu(\mathcal{H}) \geqslant 2r$, and let $\mathcal{A}$ be a Sperner sub-family of $\mathcal{H}^{(\leqslant r)}$ such that $\mathcal{A} \cap \mathcal{H}^{(\leqslant r-1)} \neq \emptyset$. Then $|\partial_{\mathcal{H}}^{(r)} \mathcal{A}| > |\mathcal{A}|$.*

Proof. Set $m := \min\{|A| : A \in \mathcal{A}\}$. Thus $\bigcup_{s=m}^{r} \mathcal{A}^{(s)}$ is a partition for $\mathcal{A}$. Since $\mathcal{A} \cap \mathcal{H}^{(\leqslant r-1)} \neq \emptyset$, it follows that $m \leqslant r-1$. Take ${}_1\mathcal{A} := (\mathcal{A} \backslash \mathcal{A}^{(m)}) \cup \partial_{\mathcal{H}}^{(m+1)} \mathcal{A}^{(m)}$. Since $\mathcal{A}$ is Sperner, we have $(\partial_{\mathcal{H}}^{(m+1)} \mathcal{A}^{(m)}) \cap \mathcal{A} = \emptyset$, and hence $|{}_1\mathcal{A}| > |\mathcal{A}|$ since $|\partial_{\mathcal{H}}^{(m+1)} \mathcal{A}^{(m)}| > |\mathcal{A}^{(m)}|$ by Lemma 3.1. Also note that ${}_1\mathcal{A}$ is Sperner. Repeating the same procedure $r-m-1$ more times, we obtain a family ${}_q\mathcal{A} \subset \mathcal{H}^{(r)}$ (where $q = r-m$) such that $|{}_q\mathcal{A}| > |\mathcal{A}|$. Clearly, ${}_q\mathcal{A} = \partial_{\mathcal{H}}^{(r)} \mathcal{A}$. □

Corollary 3.5. *Let $\mathcal{H}$ be a hereditary family with $\mu(\mathcal{H}) \geqslant 2r$, and let $\mathcal{A}$ be a largest t-intersecting Sperner sub-family of $\mathcal{H}^{(\leqslant r)}$. Then $\mathcal{A} \subset \mathcal{H}^{(r)}$.*

Proof. Suppose $\mathcal{A} \cap \mathcal{H}^{(\leqslant r-1)} \neq \emptyset$. Trivially, $\partial_{\mathcal{H}}^{(r)} \mathcal{A}$ is a t-intersecting Sperner sub-family of $\mathcal{H}^{(r)}$. By Corollary 3.4, $|\partial_{\mathcal{H}}^{(r)} \mathcal{A}| > |\mathcal{A}|$, which is a contradiction. □

Proof of Theorem 2.5. The result is trivial if $t = r$, so we assume $t < r$. Let $\mathcal{H}$ be a hereditary family with $\mu(\mathcal{H}) \geqslant n_0^*(r,t)$. Therefore $\mu(\mathcal{H}) > 2r$. Let $\mathcal{A}$ be a largest t-intersecting Sperner sub-family of $\mathcal{H}^{(\leqslant r)}$. By Corollary 3.5, we then have $\mathcal{A} \subset \mathcal{H}^{(r)}$, and hence $\mathcal{A}$ is a largest t-intersecting sub-family of $\mathcal{H}^{(r)}$. By Theorem 2.1 with $S = \{r\}$, $\mathcal{A}$ is a t-star of $\mathcal{H}^{(r)}$. □

4. *Proof of the main result*

Lemma 4.1. *Let $r \geqslant t+1$ and $\emptyset \neq S \subseteq [t+1, r]$. Let $\mathcal{H}$ be a hereditary family with $\mu(\mathcal{H}) \geqslant r+1$. Suppose $\emptyset \neq \mathcal{A} \subseteq \bigcup_{s \in S} \mathcal{H}^{(s)}$ such that, for some $J \subseteq U(\mathcal{H})$, $|A \cap J| \geqslant t+1$ for all $A \in \mathcal{A}$. Then there exists a $T \in \binom{J}{t}$ such that*

$$|\mathcal{A}| < \frac{r-t}{\mu(\mathcal{H}) - r} \binom{|J|}{t+1} \left| \bigcup_{s \in S} \mathcal{H}^{(s)} \langle T \rangle \right|.$$

Proof. Choose $I_0 \in \binom{J}{t+1}$ such that

$$\sum_{s \in S} |\mathcal{H}^{(s)} \langle I \rangle| \leqslant \sum_{s \in S} |\mathcal{H}^{(s)} \langle I_0 \rangle| \quad \text{for all } I \in \binom{J}{t+1}.$$

Choose $i_0 \in I_0$, and let $T := I_0 \backslash \{i_0\}$. Let $R := \{s \in S : \mathcal{H}^{(s)} \langle I_0 \rangle \neq \emptyset\}$.
Given that $\emptyset \neq \mathcal{A} \subseteq \bigcup_{s \in S} \mathcal{H}^{(s)}$ and $|A \cap J| \geqslant t+1$ for all $A \in \mathcal{A}$, we have

$$\begin{aligned} 1 \leqslant |\mathcal{A}| = \left| \bigcup_{I \in \binom{J}{t+1}} \mathcal{A}\langle I \rangle \right| &\leqslant \sum_{I \in \binom{J}{t+1}} |\mathcal{A}\langle I \rangle| \leqslant \sum_{I \in \binom{J}{t+1}} \sum_{s \in S} |\mathcal{H}^{(s)} \langle I \rangle| \\ &\leqslant \sum_{I \in \binom{J}{t+1}} \sum_{s \in S} |\mathcal{H}^{(s)} \langle I_0 \rangle| = \binom{|J|}{t+1} \sum_{s \in R} |\mathcal{H}^{(s)} \langle I_0 \rangle|. \end{aligned} \tag{4.1}$$

Let $\mathcal{I} := \{H \backslash I_0 : H \in \mathcal{H}\langle I_0 \rangle\}$. Since $\mathcal{H}$ is hereditary, $\mathcal{I}$ is hereditary.
If $M \in \mathcal{H}\langle I_0 \rangle$ and $N \in \mathcal{H}$ such that $M \subseteq N$, then $N \in \mathcal{H}\langle I_0 \rangle$, and hence $N = M$ if M is $\mathcal{H}\langle I_0 \rangle$-maximal. Thus the $\mathcal{H}\langle I_0 \rangle$-maximal sets in $\mathcal{H}\langle I_0 \rangle$ are also $\mathcal{H}$-maximal, and hence, since (4.1) gives us $\mathcal{H}\langle I_0 \rangle \neq \emptyset$, we have $\mu(\mathcal{H}\langle I_0 \rangle) \geqslant \mu(\mathcal{H})$. Now clearly $\mu(\mathcal{I}) = \mu(\mathcal{H}\langle I_0 \rangle) - |I_0| =$

$\mu(\mathcal{H}\langle I_0\rangle) - t - 1$. Therefore

$$\mu(\mathcal{I}) \geqslant \mu(\mathcal{H}) - t - 1. \tag{4.2}$$

Note that (4.1) implies $R \neq \emptyset$. Let $s \in R$, and let $p := s - |I_0| = s - t - 1$, $q := p + 1 = s - t$. Given that $\mu(\mathcal{H}) \geqslant r + 1$, it follows by (4.2) that

$$\mu(\mathcal{I}) \geqslant (r+1) - t - 1 \geqslant s - t = q.$$

Therefore, by Corollary 3.2, we have

$$|\mathcal{I}^{(q)}| \geqslant \frac{\mu(\mathcal{I}) - p}{q}|\mathcal{I}^{(p)}|,$$

and hence, since $|\mathcal{I}^{(p)}| = |\mathcal{H}^{(s)}\langle I_0\rangle|$ and $|\mathcal{I}^{(q)}| = |\mathcal{H}^{(s+1)}\langle I_0\rangle|$ (by definition of $\mathcal{I}$, p and q),

$$\begin{aligned}
|\mathcal{H}^{(s+1)}\langle I_0\rangle| &\geqslant \frac{\mu(\mathcal{I}) - p}{q}|\mathcal{H}^{(s)}\langle I_0\rangle| \\
&\geqslant \frac{(\mu(\mathcal{H}) - t - 1) - (s - t - 1)}{s - t}|\mathcal{H}^{(s)}\langle I_0\rangle| \quad \text{(by (4.2))} \\
&= \frac{\mu(\mathcal{H}) - s}{s - t}|\mathcal{H}^{(s)}\langle I_0\rangle| \\
&\geqslant \frac{\mu(\mathcal{H}) - r}{r - t}|\mathcal{H}^{(s)}\langle I_0\rangle|.
\end{aligned} \tag{4.3}$$

Let $\mathcal{B} := \{A\backslash\{i_0\} : A \in \mathcal{H}^{(s+1)}\langle I_0\rangle\}$. Note that, for all $B \in \mathcal{B}$, we have $T \subset B$, $|B| = s$ and, since $\mathcal{H}$ is hereditary, $B \in \mathcal{H}$; so $\mathcal{B} \subseteq \mathcal{H}^{(s)}\langle T\rangle$. Since $\mathcal{H}^{(s)}\langle I_0\rangle \neq \emptyset$ (as $s \in R$) and $\mathcal{H}^{(s)}\langle I_0\rangle \subseteq \mathcal{H}^{(s)}\langle T\rangle\backslash\mathcal{B}$, we actually have $\mathcal{B} \subsetneq \mathcal{H}^{(s)}\langle T\rangle$ and hence $|\mathcal{B}| < |\mathcal{H}^{(s)}\langle T\rangle|$. Thus, since $|\mathcal{B}| = |\mathcal{H}^{(s+1)}\langle I_0\rangle|$, we have $|\mathcal{H}^{(s+1)}\langle I_0\rangle| < |\mathcal{H}^{(s)}\langle T\rangle|$. From this strict inequality and (4.3) (which gives us $|\mathcal{H}^{(s)}\langle I_0\rangle| \leqslant ((r-t)/(\mu(\mathcal{H}) - r))|\mathcal{H}^{(s+1)}\langle I_0\rangle|$), we immediately obtain

$$|\mathcal{H}^{(s)}\langle I_0\rangle| < \frac{r - t}{\mu(\mathcal{H}) - r}|\mathcal{H}^{(s)}\langle T\rangle|. \tag{4.4}$$

Finally, by (4.1) and (4.4), we have

$$\begin{aligned}
|\mathcal{A}| &\leqslant \binom{|J|}{t+1}\sum_{s\in R}|\mathcal{H}^{(s)}\langle I_0\rangle| \\
&< \binom{|J|}{t+1}\sum_{s\in R}\frac{r - t}{\mu(\mathcal{H}) - r}|\mathcal{H}^{(s)}\langle T\rangle| \\
&= \frac{r - t}{\mu(\mathcal{H}) - r}\binom{|J|}{t+1}\sum_{s\in S}|\mathcal{H}^{(s)}\langle T\rangle|,
\end{aligned}$$

which establishes the result since $\sum_{s\in S}|\mathcal{H}^{(s)}\langle T\rangle| = |\bigcup_{s\in S}\mathcal{H}^{(s)}\langle T\rangle|$. □

Proof of Theorem 2.1. If $S = \{t\}$, then the result is trivial; so we consider $t < r$ and $S \subseteq [t, r]$ such that $S \cap [t+1, r] \neq \emptyset$. Let $\mathcal{H}$ be a hereditary family with $\mu(\mathcal{H}) \geqslant n_0^*(r, t)$. Let $\mathcal{A}$ be a (non-empty) non-trivial t-intersecting sub-family of $\bigcup_{s\in S}\mathcal{H}^{(s)}$.

We first show that there exists a set $J \subset U(\mathcal{H})$ of size at most $3r - 2t - 1$ such that $|A \cap J| \geqslant t + 1$ for all $A \in \mathcal{A}$ (this idea was used in [**18**] for the proof of Theorem 1.2). If $\mathcal{A}$ is $(t+1)$-intersecting, then we just take J to be an arbitrary set in $\mathcal{A}$. Hence suppose $\mathcal{A}$ is not $(t+1)$-intersecting. Then there exist $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| = t$. Since $\mathcal{A}$ is a non-trivial t-intersecting family, there exists an $A_3 \in \mathcal{A}$ such that $A_1 \cap A_2 \nsubseteq A_3$, and hence $|A_1 \cap A_2 \cap A_3| \leqslant t - 1$. Take J to be $A_1 \cup A_2 \cup A_3$. Therefore $|A \cap J| \geqslant t$ for all $A \in \mathcal{A}$. Suppose there exists an $A \in \mathcal{A}$ such that $|A \cap J| = t$. Then $t \geqslant |A \cap (A_1 \cup A_2)| = |A \cap A_1| + |A \cap A_2| - |A \cap A_1 \cap A_2| \geqslant 2t - |A \cap A_1 \cap A_2|$, and hence $|A \cap A_1 \cap A_2| \geqslant t$. Also $|A \cap A_1 \cap A_2| \leqslant$

$|A \cap J| = t$; so $|A \cap A_1 \cap A_2| = |A \cap J|$, and hence $A \cap J = A \cap A_1 \cap A_2$ (as $A_1 \cap A_2 \subset J$). Thus we have $t \leqslant |A \cap A_3| = |A \cap (A_3 \cap J)| = |(A \cap J) \cap A_3| = |(A \cap A_1 \cap A_2) \cap A_3| \leqslant |A_1 \cap A_2 \cap A_3|$, which contradicts $|A_1 \cap A_2 \cap A_3| \leqslant t-1$. Therefore $|A \cap J| \geqslant t+1$ for all $A \in \mathcal{A}$. Now $|J| = |A_1 \cup A_2| + |A_3| - |A_3 \cap (A_1 \cup A_2)|$. Since $|A_1 \cup A_2| = 2r - |A_1 \cap A_2| = 2r - t$ and $|A_3 \cap (A_1 \cup A_2)| = |A_3 \cap A_1| + |A_3 \cap A_2| - |A_3 \cap A_2 \cap A_1| \geqslant 2t - |A_1 \cap A_2 \cap A_3| \geqslant 2t - (t-1) = t+1$, we obtain $|J| \leqslant (2r-t) + r - (t+1) = 3r - 2t - 1$.

Since we established the existence of a set J such that $|A \cap J| \geqslant t+1$ for all $A \in \mathcal{A}$, we may assume that $S \subseteq [t+1, r]$. Since $\mu(\mathcal{H}) \geqslant n_0^*(r,t)$, it follows by Lemma 4.1 that, for some $T \in \binom{J}{t}$,

$$\begin{aligned}
|\mathcal{A}| &< \frac{r-t}{\mu(\mathcal{H}) - r}\binom{|J|}{t+1}\left|\bigcup_{s\in S}\mathcal{H}^{(s)}\langle T\rangle\right| \\
&\leqslant \frac{r-t}{n_0^*(r,t) - r}\binom{3r-2t-1}{t+1}\left|\bigcup_{s\in S}\mathcal{H}^{(s)}\langle T\rangle\right| \\
&= \left|\bigcup_{s\in S}\mathcal{H}^{(s)}\langle T\rangle\right|.
\end{aligned}$$

We have therefore shown that, for any non-trivial t-intersecting sub-family $\mathcal{A}$ of $\bigcup_{s\in S}\mathcal{H}^{(s)}$, there exists a trivial t-intersecting sub-family of $\bigcup_{s\in S}\mathcal{H}^{(s)}$ that is a larger than $\mathcal{A}$. The result follows. □

5. *Proof of Theorem* 2.7

The proof that we now present is based on the *compression* (also known as *shifting*) technique, which was introduced in [**18**]. Frankl's survey paper [**20**] gives an excellent account of the efficacy of this technique in extremal set theory.

For $i, j \in [n]$, the *compression operation* $\Delta_{i,j} : 2^{2^{[n]}} \to 2^{2^{[n]}}$ is defined by

$$\Delta_{i,j}(\mathcal{A}) := \{\delta_{i,j}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : \delta_{i,j}(A) \in \mathcal{A}\},$$

where $\delta_{i,j} : 2^{[n]} \to 2^{[n]}$ is defined by

$$\delta_{i,j}(A) = \begin{cases} (A\backslash\{j\}) \cup \{i\} & \text{if } i \notin A \text{ and } j \in A, \\ A & \text{otherwise.} \end{cases}$$

Note that $|\Delta_{i,j}(\mathcal{A})| = |\mathcal{A}|$.

If $i < j$, then $\Delta_{i,j}$ is said to be a *left-compression*. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be *left-compressed* if $\Delta_{i,j}(\mathcal{F}) = \mathcal{F}$ for any left-compression $\Delta_{i,j}$.

The following lemma captures some well-known fundamental properties of compressions.

LEMMA 5.1. *Let $t < n$. Let $\mathcal{H}$ be a left-compressed sub-family of $2^{[n]}$. Suppose that $\mathcal{A}$ is a non-empty t-intersecting sub-family of $\bigcup_{s\in S}\mathcal{H}^{(s)}$, where $S \subseteq [t, n]$.*

(i) *If $1 \leqslant i < j \leqslant n$, then $\Delta_{i,j}(\mathcal{A})$ is a t-intersecting sub-family of $\bigcup_{s\in S}\mathcal{H}^{(s)}$.*

(ii) *If $\mathcal{A}$ is left-compressed and $s \in S$, then $|A \cap B \cap [2s-t]| \geqslant t$ for any $A, B \in \mathcal{A}^{(s)}$.*

Proof. Let $1 \leqslant i < j \leqslant n$. Since $\mathcal{H}$ is left-compressed and $\mathcal{A} \subseteq \bigcup_{s\in S}\mathcal{H}^{(s)}$, it is straightforward that $\Delta_{i,j}(\mathcal{A}) \subseteq \bigcup_{s\in S}\mathcal{H}^{(s)}$. Let $A, B \in \Delta_{i,j}(\mathcal{A})$. If $A, B \in \mathcal{A}$, then $|A \cap B| \geqslant t$ as $\mathcal{A}$ is t-intersecting. Suppose $A, B \notin \mathcal{A}$. Then, for some $C, D \in \mathcal{A}$, $A = \delta_{i,j}(C) \neq C$ and $B = \delta_{i,j}(D) \neq D$. Hence $|A \cap B| = |C \cap D| \geqslant t$. Finally, suppose without loss of generality that

$A \in \mathcal{A}$ and $B \notin \mathcal{A}$. Then $\delta_{i,j}(A) \in \mathcal{A}$ and $B = \delta_{i,j}(E) \neq E$ for some $E \in \mathcal{A}$. Thus $|A \cap B| \geqslant |\delta_{i,j}(A) \cap E| \geqslant t$. Hence (i).

Suppose that $\mathcal{A}$ is left-compressed. Let $s \in S$ such that $\mathcal{A}^{(s)} \neq \emptyset$, and let $A, B \in \mathcal{A}^{(s)}$. Therefore $|A \cap B| \geqslant t$. Let $X := (A \cap B) \cap [2s-t]$, $Y := (A \cap B) \backslash [2s-t]$, $Z := [2s-t] \backslash (A \cup B)$. If $Y = \emptyset$, then $X = A \cap B$ and hence $|X| \geqslant t$. Now consider $Y \neq \emptyset$. Let $p := |Y|$. Since

$$\begin{aligned}|Z| &= 2s - t - |(A \cup B) \cap [2s-t]| \geqslant 2s - t - |X| - |A \backslash B| - |B \backslash A| \\ &= 2s - t - |X| - |A \backslash (X \cup Y)| - |B \backslash (X \cup Y)| \\ &= 2s - t - |X| - 2(s - |X| - |Y|) = 2|Y| + |X| - t \\ &= |Y| + |Y \cup X| - t = p + |A \cap B| - t \geqslant p,\end{aligned}$$

$\binom{Z}{p} \neq \emptyset$. Let $W \in \binom{Z}{p}$. Let $C := (B \backslash Y) \cup W$. Let $y_1, \dots, y_p$ be the elements of Y, and let $w_1, \dots, w_p$ be those of W. Therefore $C = \delta_{w_1,y_1} \circ \dots \circ \delta_{w_p,y_p}(B)$. Note that $\delta_{w_1,y_1}, \dots, \delta_{w_p,y_p}$ are left-compressions as $W \subseteq [2s-t]$ and $Y \subseteq [n] \backslash [2s-t]$. Since $\mathcal{A}$ is left-compressed, $C \in \mathcal{A}$. Thus $|A \cap C| \geqslant t$ as $\mathcal{A}$ is t-intersecting. Now clearly $|A \cap C| = |X|$, and so $|X| \geqslant t$. Hence (ii). □

Lemma 5.2. *Let $\mathcal{F} \subseteq 2^{[n]}$ be left-compressed. Let $Z \subseteq [n]$, $1 \leqslant i < j \leqslant n$, and $Y := \delta_{i,j}(Z)$. Then $|\mathcal{F}\langle Z \rangle| \leqslant |\mathcal{F}\langle Y \rangle|$.*

Proof. Suppose $Y \neq Z$. Setting $W := Z \cap Y$, we therefore have $Z = W \cup \{j\} \neq W$ and $Y = W \cup \{i\} \neq W$. Let $\mathcal{D} := \{F \in \mathcal{F}\langle Z \rangle : i \notin F\}$ and $\mathcal{E} := \{F \in \mathcal{F}\langle Y \rangle : j \notin F\}$. Since $\mathcal{F}$ is left-compressed, we have $\Delta_{i,j}(\mathcal{D}) \subseteq \mathcal{E}$, and hence $|\mathcal{E}| \geqslant |\mathcal{D}|$. Thus $|\mathcal{F}\langle Y \rangle| - |\mathcal{F}\langle Z \rangle| \geqslant 0$ as $|\mathcal{F}\langle Y \rangle| - |\mathcal{F}\langle Z \rangle| = (|\mathcal{F}\langle W \cup \{i,j\} \rangle| + |\mathcal{E}|) - (|\mathcal{F}\langle W \cup \{i,j\} \rangle| + |\mathcal{D}|) = |\mathcal{E}| - |\mathcal{D}|$. □

Corollary 5.3. *Let $\mathcal{F} \subseteq 2^{[n]}$ be left-compressed. Let $Z \subseteq [n]$ and $Y := [|Z|]$. Then $|\mathcal{F}\langle Z \rangle| \leqslant |\mathcal{F}\langle Y \rangle|$.*

Proof. Clearly, we can construct a composition of operations $\delta_{i,j}$, $i < j$, that gives Y when applied to Z. Thus the result follows by repeated application of Lemma 5.2. □

Next we present the key tool for obtaining Theorem 2.7 from Theorem 1.3.

Lemma 5.4. *Let $\mathcal{F}$ be a left-compressed sub-family of $2^{[n]}$ such that $[n] \notin \mathcal{F}$. Let $\mathcal{E} := \{F \in \mathcal{F} : n \notin F\}$. Then $\mu(\mathcal{E}) \geqslant \mu(\mathcal{F})$.*

Proof. Let $M \in \mathcal{E}$ be $\mathcal{E}$-maximal. Suppose $|M| < \mu(\mathcal{F})$. Then there exists an $N \in \mathcal{F}$ such that $n \in N$ and $M \subsetneq N$. Let $X := [n] \backslash N$. Since $[n] \notin \mathcal{F}$, $X \neq \emptyset$. Let $x \in X$ and $L := \delta_{x,n}(N) = (N \backslash \{n\}) \cup \{x\}$. Given that $\mathcal{F}$ is left-compressed, $L \in \mathcal{F}$. Since $n \notin L$, $L \in \mathcal{E}$. Now $M \subsetneq L$, but this is a contradiction since M is $\mathcal{E}$-maximal; so $|M| \geqslant \mu(\mathcal{F})$. Hence result. □

The remaining lemmas will be used for obtaining the strict t-EKR part of Theorem 2.7.

Lemma 5.5. *Let $r \geqslant t$. Let $\mathcal{F}$ be a family such that $\binom{M}{s} \subseteq \mathcal{F}$ for some $s \in [t, r]$ and some set M with $|M| \geqslant \max\{2r - t, 2s - t + 1\}$. Let $T \in \binom{M}{t}$, and let $\mathcal{A}$ be a t-intersecting sub-family of $\mathcal{F}^{(\leqslant r)}$ such that $\mathcal{A}$ contains $\mathcal{B} := \{B \in \binom{M}{s} : T \subseteq B\}$. Then $\mathcal{A} \subseteq \mathcal{F}\langle T \rangle$.*

Proof. Let F be a set in $\mathcal{F}^{(\leqslant r)}$ not containing T. If we show that $|B \cap F| \leqslant t-1$ for some $B \in \mathcal{B}$, then since $\mathcal{A}$ is t-intersecting and contains $\mathcal{B}$, we get $F \notin \mathcal{A}$, and the result follows.

Let L be a largest subset of M such that $T \subseteq L$ and $|L \cap F| \leqslant t-1$ (such a set L exists as $T \nsubseteq F$). If $|F \cap M| \leqslant t-1$, then $L = M$, and hence $|L| \geqslant 2r - t \geqslant r$. If instead $|F \cap M| \geqslant t$, then $L = (M \backslash F) \cup K$ for some set K that is in $\binom{F \cap M}{t-1}$ and contains $F \cap T$, and hence

$$|L| = |M| - |F \cap M| + |K| \geqslant \max\{2r-t, 2s-t+1\} - r + (t-1) = \begin{cases} r & \text{if } s = r, \\ r-1 & \text{if } s \leqslant r-1. \end{cases}$$

Thus $\binom{L}{s} \neq \emptyset$. Let $B \in \binom{L}{s}$ such that $T \subseteq B$. Then $|B \cap F| \leqslant |L \cap F| \leqslant t-1$. Hence the result. □

LEMMA 5.6. *Let $t \leqslant r$, $S \subseteq [t+1, r]$ and $p := \min\{s \in S\}$. Let $\mathcal{H} \subseteq 2^{[n]}$ be a hereditary family with $\mu(\mathcal{H}) > \max\{2r-t, 2p-t+1\}$. Let $\mathcal{A}$ be a t-intersecting sub-family of $\bigcup_{s \in S} \mathcal{H}^{(s)}$. Suppose that, for some $\{i,j\} \in \binom{[n]}{2}$, $\Delta_{i,j}(\mathcal{A})$ is a largest t-star of $\bigcup_{s \in S} \mathcal{H}^{(s)}$. Then $\mathcal{A}$ is a largest t-star of $\bigcup_{s \in S} \mathcal{H}^{(s)}$.*

We base the proof of the above lemma on the following simple-but-useful result.

LEMMA 5.7. *Suppose $\emptyset \neq \mathcal{F} \subseteq \binom{X}{q}$, $2q < |X|$, such that, for any $A \in \mathcal{F}$ and $B \in \binom{X \backslash A}{q}$, $B \in \mathcal{F}$. Then $\mathcal{F} = \binom{X}{q}$.*

Proof. Let $A_1 \in \mathcal{F}$. Let A_2 be an arbitrary set in $\binom{X}{q}$ that intersects A_1 in exactly $q-1$ elements. Since $|X| \geqslant 2q+1$, we can choose $A_3 \in \binom{X}{q}$ such that A_3 is disjoint from $A_1 \cup A_2$. By the assumption of the proposition, we have $A_3 \in \mathcal{F}$, which in turn implies $A_2 \in \mathcal{F}$. By repeated application of this step, we get that any set in $\binom{X}{q}$ is also in $\mathcal{F}$. □

Proof of Lemma 5.6. Let $\mathcal{G} := \bigcup_{s \in S} \mathcal{H}^{(s)}$ and $\mathcal{D} := \Delta_{i,j}(\mathcal{A})$. Given that $\mathcal{D}$ is a t-star of $\mathcal{G}$, $\mathcal{D} = \mathcal{G}\langle T \rangle$ for some t-subset T of some $\mathcal{H}$-maximal set $N \in \mathcal{H}$. Let $\mathcal{N}$ be the t-star $\{A \in \binom{N}{p} : T \subset A\}$ of $\binom{N}{p}$. Since $\mathcal{H}$ is hereditary, $\mathcal{N} \subseteq \mathcal{D}^{(p)}$. Also, $\mathcal{N} \neq \emptyset$ as $t < p < \mu(\mathcal{H}) \leqslant |N|$.

If $\mathcal{A} = \mathcal{D}$, then there is nothing to prove (as we are given that $\mathcal{D}$ is a largest t-star).

Suppose $\mathcal{A} \neq \mathcal{D}$. Then there exists a set $A \in \mathcal{A}$ such that $j \in A$, $i \notin A$, $\delta_{i,j}(A) \in \Delta_{i,j}(\mathcal{A}) \backslash \mathcal{A}$ and $A \notin \Delta_{i,j}(\mathcal{A})$; so $T \nsubseteq A$ (as otherwise, since $\mathcal{A} \subset \mathcal{G}$, we get $A \in \mathcal{G}\langle T \rangle$, contradicting $A \notin \Delta_{i,j}(\mathcal{A}) = \mathcal{G}\langle T \rangle$). Since $\Delta_{i,j}(\mathcal{A}) = \mathcal{G}\langle T \rangle$, we have $T \subset \delta_{i,j}(A)$, and (together with $T \nsubseteq A$) this implies $i \in T, j \notin T$. Let $R := T \backslash \{i\} \cup \{j\} = \delta_{j,i}(T)$, $L := \delta_{j,i}(N)$ (towards the end of the proof we discover that $L \neq N$, that is, $i \in N$, $j \notin N$).

Suppose $\mathcal{A}^{(p)}$ has a member A_0 not containing R. Then, by definition of R and the equality $\Delta_{i,j}(\mathcal{A}) = \mathcal{G}\langle T \rangle$, we must have $T \subset A_0$ and $j \notin A_0$. Let $M \in \mathcal{H}$ be an $\mathcal{H}$-maximal set such that $A_0 \subset M$. Let $K := M \backslash (T \cup \{j\})$ and $q := p - t$. Then, given that $\mu(\mathcal{H}) \geqslant 2p - t + 2$, we have $2(p-t) < 2p - 2t + 1 \leqslant \mu(\mathcal{H}) - (t+1) \leqslant |M| - (t+1)$ and hence $2q < |K|$. Let $\mathcal{K} := \binom{K \cup T}{p}$ and $\mathcal{B} := \{A \backslash T : A \in \mathcal{A}^{(p)} \cap \mathcal{K}\langle T \rangle\}$; so $\mathcal{B} \subseteq \binom{K}{q}$, and $\mathcal{B} \neq \emptyset$ as $A_0 \backslash T \in \mathcal{B}$. We will actually arrive at the equality $\mathcal{B} = \binom{K}{q}$.

Let $B \in \mathcal{B}$ and $C \in \binom{K \backslash B}{q}$ (C exists as $2q < |K|$). By definition of $\mathcal{B}$, the set $D := B \cup T$ is in $\mathcal{A}^{(p)}$. We have $C \subset M \backslash D$ and $T \subset M$ (as $T \subset A_0 \subset M$). Since $\mathcal{H}$ is hereditary and the set $E := C \cup T$ is a p-subset of $M \in \mathcal{H}$, $E \in \mathcal{H}^{(p)}\langle T \rangle$; so $E \in \mathcal{D}$. Suppose $E \notin \mathcal{A}$; then $\delta_{j,i}(E) \in \mathcal{A}$ and $|D \cap \delta_{j,i}(E)| = |T \backslash \{i\}| = t-1$, which is a contradiction as $\mathcal{A}$ is t-intersecting and contains also D. Therefore $E \in \mathcal{A}$ and hence $C \in \mathcal{B}$. Thus, by Lemma 5.7 (with $\mathcal{F} = \mathcal{B}$, $X = K$), $\mathcal{B} = \binom{K}{q}$.

Now $\mathcal{B} = \binom{K}{q}$ implies $\{A \in \binom{K \cup T}{p} : T \subset A\} \subseteq \mathcal{A}^{(p)}$ (by definition of $\mathcal{B}$). Since $|K \cup T| \geqslant |M| - 1 \geqslant \mu(\mathcal{H}) - 1 \geqslant \max\{2r - t, 2p - t + 1\}$, we therefore get $\mathcal{A} \subseteq \mathcal{H}\langle T\rangle$ by Lemma 5.5. Given that $\Delta_{i,j}(\mathcal{A}) = \mathcal{G}\langle T\rangle$, we should then have $\mathcal{A} = \Delta_{i,j}(\mathcal{A})$, but this contradicts $\mathcal{A} \neq \mathcal{D}$.

Therefore, all sets in $\mathcal{A}^{(p)}$ contain R. Thus any set P in $\mathcal{N}$ is a p-set in $\Delta_{i,j}(\mathcal{A})\backslash\mathcal{A}$ (as $\mathcal{N} \subseteq \mathcal{D}^{(p)} = \mathcal{H}^{(p)}\langle T\rangle$) and hence $P = \delta_{i,j}(Q) \neq Q$, where $Q = P\backslash\{i\} \cup \{j\} \in \mathcal{A}$. This clearly means that $\mathcal{R} := \{A \in \binom{L}{p} : R \subset A\}$ is a sub-family of $\mathcal{A}^{(p)}$ (and the sub-family $\mathcal{N}$ of $\Delta_{i,j}(\mathcal{A})$ is the result of the compression $\Delta_{i,j}$ on $\mathcal{R}$). By Lemma 5.5, $\mathcal{A} \subseteq \mathcal{G}\langle R\rangle$. Since $\mathcal{D}$ is a largest t-star of $\mathcal{G}$ and $|\mathcal{D}| = |\mathcal{A}|$, it follows that $\mathcal{A} = \mathcal{G}\langle R\rangle$ and that $\mathcal{G}\langle R\rangle$ is a largest t-star of $\mathcal{G}$ as required. □

Proof of Theorem 2.7. Fix $t \in \mathbb{N}$. If $r = t$, then the result is trivial. Thus we assume $r > t$, and we prove the result by induction on r.

Let $\mathcal{H}$ be a compressed hereditary family with $\mu(\mathcal{H}) \geqslant (t+1)(r-t+1)$. Let $n := |U(\mathcal{H})|$. It is easy to see that $\mathcal{H}$ is isomorphic to a left-compressed family $\mathcal{H}' \subseteq 2^{[n]}$ with $U(\mathcal{H}) = [n]$; so we may assume $\mathcal{H} = \mathcal{H}'$. Let $S \subseteq [t, r]$ and $\mathcal{G} := \bigcup_{s \in S} \mathcal{H}^{(s)}$. Let $\mathcal{A}$ be an extremal t-intersecting sub-family of $\mathcal{G}$. If $t \in S$ and $\mathcal{A}^{(t)}$ has a member A, then, since $\mathcal{A}$ is t-intersecting, all sets in $\mathcal{A}$ must contain A, and hence $\mathcal{A}$ can only be a t-star of $\mathcal{G}$. Therefore we assume $S \subseteq [t+1, r]$. Let $T := [t]$, and let $\mathcal{T} := \mathcal{G}\langle T\rangle$. For $m \in \mathbb{N}$, let $n_1(m) := (t+1)(m-t+1)$. Thus $\mu(\mathcal{H}) \geqslant n_1(r)$ and hence $n \geqslant n_1(r)$.

We first consider $n = n_1(r)$; so $\mu(\mathcal{H}) = n$ and hence $[n] \in \mathcal{H}$. Thus, since $\mathcal{H}$ is hereditary, $\mathcal{H}^{(m)} = \binom{[n]}{m}$ for all $m \in [n]$. For $s \in S$, $n_1(s) \leqslant n_1(r)$. By Theorem 1.3, $|\mathcal{A}^{(s)}| \leqslant |\mathcal{T}^{(s)}|$ for all $s \in S$, and hence $|\mathcal{A}| \leqslant |\mathcal{T}|$. This proves that $\mathcal{G}$ is t-EKR.

Suppose $S \neq \{r\}$. Since $\mathcal{A}$ is extremal, we actually have $|\mathcal{A}| = |\mathcal{T}|$ and hence $|\mathcal{A}^{(s)}| = |\mathcal{T}^{(s)}|$ for all $s \in S$. Fix $p \in S\backslash\{r\}$. Since $n_1(p) < n_1(r)$ and $|\mathcal{A}^{(p)}| = |\mathcal{T}^{(p)}|$, it follows by Theorem 1.3 that $\mathcal{A}^{(p)} = \{A \in \binom{[n]}{p} : Z \subset A\}$ for some t-subset Z of $[n]$. By Lemma 5.5, $\mathcal{A} \subseteq \mathcal{H}\langle Z\rangle$. Therefore $\mathcal{A}$ is a t-star of $\mathcal{G}$, and hence $\mathcal{G}$ is strictly t-EKR.

We now consider $n > n_1(r)$ and proceed by induction on n. If $[n] \in \mathcal{H}$, then the result follows by Theorem 1.3 as in the case $n = n_1(r)$ above; so we consider $[n] \notin \mathcal{H}$.

We start by applying left-compressions $\Delta_{i,j}$ to $\mathcal{A}$ until we obtain a left-compressed family $\mathcal{B}$; so $|\mathcal{B}| = |\mathcal{A}|$. By Lemma 5.1(i), $\mathcal{B}$ is a t-intersecting sub-family of $\mathcal{G}$. Moreover, by Lemma 5.1(ii), $|A \cap B \cap [2r-t]| \geqslant t$ for any $A, B \in \mathcal{B}$, and hence, since $n > n_1(r) \geqslant 2r - t$,

$$|A \cap B \cap [n-1]| \geqslant t \quad \text{for any } A, B \in \mathcal{B}. \tag{5.1}$$

Let $\mathcal{B}_1 := \{B \in \mathcal{B} : n \notin B\}$ and $\mathcal{B}_2 := \{B\backslash\{n\} : n \in B \in \mathcal{B}\}$. Define $\mathcal{H}_1$ and $\mathcal{H}_2$ similarly. Hence $\mathcal{B}_1 \subset \bigcup_{s \in S} {\mathcal{H}_1}^{(s)}$ and $\mathcal{B}_2 \subset \bigcup_{s \in S} {\mathcal{H}_2}^{(s-1)}$. By (5.1), $\mathcal{B}_1$ and $\mathcal{B}_2$ are t-intersecting. It is straightforward that, since $\mathcal{H}$ is a left-compressed hereditary sub-family of $2^{[n]}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are left-compressed hereditary sub-families of $2^{[n-1]}$. By Lemma 5.4, we have $\mu(\mathcal{H}_1) \geqslant \mu(\mathcal{H})$, and hence $\mu(\mathcal{H}_1) \geqslant n_1(r)$. Since $U(\mathcal{H}) = [n]$, $\mathcal{H}_2 \neq \emptyset$. Therefore, similarly to (4.2), we have $\mu(\mathcal{H}_2) \geqslant \mu(\mathcal{H}) - 1$, and hence $\mu(\mathcal{H}_2) > n_1(r-1)$. We can now apply the inductive hypothesis to obtain the following:

- for each $s \in S$, there exists a $T_{1,s} \in \binom{[n-1]}{t}$ such that $|{\mathcal{B}_1}^{(s)}| \leqslant |{\mathcal{H}_1}^{(s)}\langle T_{1,s}\rangle|$ and ${\mathcal{H}_1}^{(s)}\langle T_{1,s}\rangle$ is an extremal t-intersecting sub-family of ${\mathcal{H}_1}^{(s)}$;
- for each $s \in S$, there exists a $T_{2,s} \in \binom{[n-1]}{t}$ such that $|{\mathcal{B}_2}^{(s-1)}| \leqslant |{\mathcal{H}_2}^{(s-1)}\langle T_{2,s}\rangle|$ and ${\mathcal{H}_2}^{(s-1)}\langle T_{2,s}\rangle$ is an extremal t-intersecting sub-family of ${\mathcal{H}_2}^{(s-1)}$.

For $s \in S$, since $\mathcal{H}^{(s)}$ is left-compressed, ${\mathcal{H}_1}^{(s)}$ and ${\mathcal{H}_2}^{(s-1)}$ are left-compressed, and hence by Corollary 5.3, $|{\mathcal{H}_1}^{(s)}\langle T_{1,s}\rangle| \leqslant |{\mathcal{H}_1}^{(s)}\langle T\rangle|$ and $|{\mathcal{H}_2}^{(s-1)}\langle T_{2,s}\rangle| \leqslant |{\mathcal{H}_2}^{(s-1)}\langle T\rangle|$. Thus we have

$$\begin{aligned}|\mathcal{B}^{(s)}| = |{\mathcal{B}_1}^{(s)}| + |{\mathcal{B}_2}^{(s-1)}| &\leqslant |{\mathcal{H}_1}^{(s)}\langle T_{1,s}\rangle| + |{\mathcal{H}_2}^{(s-1)}\langle T_{2,s}\rangle| \\ &\leqslant |{\mathcal{H}_1}^{(s)}\langle T\rangle| + |{\mathcal{H}_2}^{(s-1)}\langle T\rangle| = |\mathcal{T}^{(s)}|\end{aligned}$$

for each $s \in S$, and hence $|\mathcal{B}| \leqslant |\mathcal{T}|$ as $|\mathcal{B}| = \sum_{s\in S} |\mathcal{B}^{(s)}|$ and $|\mathcal{T}| = \sum_{s\in S} |\mathcal{T}^{(s)}|$. This proves that $\mathcal{G}$ is t-EKR as $|\mathcal{A}| = |\mathcal{B}|$.

We now prove the strict t-EKR part. Suppose $\mu(\mathcal{H}) > n_1(r)$ or $S \neq \{r\}$. Taking $p := \min\{s \in S\}$, we then have $\mu(\mathcal{H}) \geqslant \max\{n_1(r), n_1(p)+1\}$, and so $\mu(\mathcal{H}_1) \geqslant \max\{n_1(r), n_1(p)+1\}$ (as $\mu(\mathcal{H}_1) \geqslant \mu(\mathcal{H})$ by Lemma 5.4). Since $\mathcal{A}$ is extremal and $|\mathcal{B}| = |\mathcal{A}|$, $\mathcal{B}$ is extremal and hence $|\mathcal{B}| = |\mathcal{T}|$. Thus, for all $s \in S$, $|{\mathcal{B}_1}^{(s)}| = |{\mathcal{H}_1}^{(s)}\langle T_{1,s}\rangle|$ and hence, by the above, ${\mathcal{B}_1}^{(s)}$ is an extremal t-intersecting sub-family of ${\mathcal{H}_1}^{(s)}$. Let us focus on $s = p$. By the inductive hypothesis, there exists a $T' \in \binom{[n-1]}{t}$ such that ${\mathcal{B}_1}^{(p)} = {\mathcal{H}_1}^{(p)}\langle T'\rangle$ and ${\mathcal{H}_1}^{(p)}\langle T'\rangle$ is an extremal t-intersecting sub-family of ${\mathcal{H}_1}^{(p)}$. Therefore $|{\mathcal{H}_1}^{(p)}\langle T'\rangle| = |{\mathcal{H}_1}^{(p)}\langle T\rangle|$ because, as is clear from the above, ${\mathcal{H}_1}^{(p)}\langle T\rangle$ is also extremal. We therefore establish that ${\mathcal{H}_1}^{(p)}\langle T'\rangle \neq \emptyset$ by showing that ${\mathcal{H}_1}^{(p)}\langle T\rangle \neq \emptyset$.

Clearly, $[\mu(\mathcal{H}_1)] \in \mathcal{H}_1$ since $\mathcal{H}_1$ is left-compressed. Thus, since $T \subset [p] \subset [\mu(\mathcal{H}_1)]$ and $\mathcal{H}_1$ is hereditary, $[p] \in {\mathcal{H}_1}^{(p)}\langle T\rangle$. Hence ${\mathcal{H}_1}^{(p)}\langle T'\rangle \neq \emptyset$ as claimed.

Let $A \in {\mathcal{H}_1}^{(p)}\langle T'\rangle$, and let M be an $\mathcal{H}_1$-maximal set in $\mathcal{H}_1$ such that $A \subset M$. Then $T' \in \binom{M}{t}$. Given that $\mu(\mathcal{H}_1) \geqslant n_1(r)$, we have $|M| > 2r - t$. Since $\mathcal{H}_1$ is hereditary, $\binom{M}{p} \subseteq {\mathcal{H}_1}^{(p)}$, and so $\{B \in \binom{M}{p} : T' \subset B\} \subseteq {\mathcal{H}_1}^{(p)}\langle T'\rangle$. Since ${\mathcal{H}_1}^{(p)}\langle T'\rangle = {\mathcal{B}_1}^{(p)} \subseteq \mathcal{B}$ and $\mu(\mathcal{H}) \geqslant \max\{n_1(r), n_1(p)+1\} > \max\{2r-t, 2p-t+1\}$, it follows by Lemma 5.5 that $\mathcal{B} \subseteq \mathcal{H}\langle T'\rangle$. Since $\mathcal{B}$ is an extremal t-intersecting sub-family of $\mathcal{G}$, $\mathcal{B} = \mathcal{G}\langle T'\rangle$ and $\mathcal{B}$ is a largest t-star of $\mathcal{G}$. By Lemma 5.6, $\mathcal{A}$ is a largest t-star of $\mathcal{G}$. This proves that $\mathcal{G}$ is strictly t-EKR. □

6. *Applications of the main result*

6.1. *Extremal t-intersecting families of signed sets*

Let X be an n-set $\{x_1, \ldots, x_n\}$. If $y_1, \ldots, y_n \in \mathbb{N}$ and $|\{y_1, \ldots, y_n\}| \leqslant k$, then we call the set $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ a *k-signed n-set*. For $k \geqslant 2$, we define $\mathcal{S}_{X,k}$ to be the family of k-signed n-sets given by

$$\mathcal{S}_{X,k} := \{\{(x_1, a_1), \ldots, (x_n, a_n)\} : a_1, \ldots, a_n \in [k]\}.$$

The *Cartesian product* $X \times Y$ of sets X and Y is the set $\{(x, y) : x \in X, y \in Y\}$. Hence $\mathcal{S}_{X,k} = \{A \in \binom{X\times[k]}{|X|} : |A \cap (\{x\} \times [k])| = 1 \text{ for all } x \in X\}$.

For a family $\mathcal{F}$ of sets, we define

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F\in\mathcal{F}} \mathcal{S}_{F,k}.$$

The 'signed sets' terminology was introduced in [6] for a setting that can be re-formulated as $\mathcal{S}_{\binom{[n]}{r},k}$, and the general formulation $\mathcal{S}_{\mathcal{F},k}$ was introduced in [7], the theme of which is the following conjecture.

Conjecture 6.1 (Borg [7]). Let $\mathcal{F}$ be any family, and let $k \geqslant 2$. Then:

(i) $\mathcal{S}_{\mathcal{F},k}$ is EKR;
(ii) $\mathcal{S}_{\mathcal{F},k}$ is not strictly EKR if and only if $k = 2$ and there exist at least three elements u_1, u_2, u_3 of $U(\mathcal{F})$ such that $\mathcal{F}\langle\{u_1\}\rangle = \mathcal{F}\langle\{u_2\}\rangle = \mathcal{F}\langle\{u_3\}\rangle$ and $\mathcal{S}_{\mathcal{F},2}\langle\{(u_1, 1)\}\rangle$ is a largest star of $\mathcal{S}_{\mathcal{F},2}$.

The main result in the same paper [7] is that the above conjecture is true if $\mathcal{F}$ is compressed with respect to some element of $U(\mathcal{F})$. This generalises a well-known result that was first stated by Meyer [39] and proved in different ways by Deza and Frankl [15], Bollobás and Leader [6], Engel [16] and Erdős *et al.* [17], and that can be perfectly described as saying that the conjecture is true for $\mathcal{F} = \binom{[n]}{r}$. We point out that Berge [4] and Livingston [38] had proved (i) and (ii), respectively, for the special case $\mathcal{F} = \{[n]\}$ (other proofs are found in [23, 41]).

The t-intersection problem for sub-families of $\mathcal{S}_{[n],k}$ has also been solved. Frankl and Füredi [21] were the first to investigate it, and they conjectured that an extremal t-intersecting sub-family of $\mathcal{S}_{[n],k}$ has size $\max\{|\{A \in \mathcal{S}_{[n],k} : |A \cap ([t+2i] \times [1])| \geqslant t+i\}| : i \in \{0\} \cup \mathbb{N}\}$. The conjecture claims that $\mathcal{S}_{[n],k}$ is t-EKR if $k \geqslant t+1$, and they showed that this is true if $t \geqslant 15$. A result of Kleitman [32], which is known to be equivalent to Theorem 1.4, had long established the truth of the conjecture for the special case $k = 2$. After the complete intersection theorem [2] was established, Ahlswede and Khachatrian [3] and Frankl and Tokushige [22] were able to prove this conjecture independently and by different methods; Ahlswede and Khachatrian also determined the extremal structures.

To the best of the author's knowledge, other than the following consequence of Theorem 2.1, no result of 't-EKR' type for $\mathcal{S}_{\mathcal{F},k}$, where $\mathcal{F}$ is some family containing more than one set, has been published yet.

THEOREM 6.2. *If $t \leqslant r$, $S \subseteq [t,r]$ and $\mathcal{F} = \bigcup_{s\in S} \mathcal{H}^{(s)}$ for some hereditary family $\mathcal{H}$ with $\mu(\mathcal{H}) \geqslant n_0^*(r,t)$, then $\mathcal{S}_{\mathcal{F},k}$ is strictly t-EKR for all k.*

Proof. Clearly, for any family $\mathcal{G}$, $\mu(\mathcal{S}_{\mathcal{G},k}) = \mu(\mathcal{G})$, and $\mathcal{S}_{\mathcal{G},k}$ is hereditary if and only if $\mathcal{G}$ is hereditary. Let $\mathcal{F}$ and $\mathcal{H}$ be as in the theorem; so $\mathcal{S}_{\mathcal{F},k} = \mathcal{S}_{\bigcup_{s\in S}\mathcal{H}^{(s)},k} = \bigcup_{s\in S} {\mathcal{S}_{\mathcal{H},k}}^{(s)}$ and $\mathcal{S}_{\mathcal{H},k}$ is hereditary. The result now follows by Theorem 2.1. □

Note that the above result with $\mathcal{H} = 2^{[n]}$ $(n \geqslant n_0^*(r,t))$ tells us that the extremal t-intersecting sub-families of $\mathcal{S}_{\binom{[n]}{r},k}$ are the t-stars. It also yields the following.

COROLLARY 6.3. *Conjecture 6.1 is true if $\mathcal{F} = \bigcup_{s\in S} \mathcal{H}^{(s)}$ for some hereditary family $\mathcal{H}$ with $\mu(\mathcal{H}) \geqslant n_0^*(r,1)$ and some $S \subseteq [r]$.*

Proof. The only thing we need to check is that, for $\mathcal{F}$ as in the corollary, the condition in Conjecture 6.1(ii) holds. We prove more by showing that $\mathcal{F}\langle\{u\}\rangle \neq \mathcal{F}\langle\{v\}\rangle$ for any distinct $u, v \in U(\mathcal{F})$. Indeed, let $u, v \in U(\mathcal{F})$, $u \neq v$. Then $u \in E$ and $v \in F$ for some $E, F \in \bigcup_{s\in S} \mathcal{H}^{(s)}$. If $\{u,v\} \nsubseteq E \cap F$, then clearly $\mathcal{F}\langle\{u\}\rangle \neq \mathcal{F}\langle\{v\}\rangle$; so suppose $\{u,v\} \subseteq E \cap F$. Let M be an $\mathcal{H}$-maximal set in $\mathcal{H}$ such that $E \subseteq M$. Since $|\{u,v\}| \leqslant |E| \leqslant r$ and $n_0^*(r,1) \leqslant \mu(\mathcal{H}) \leqslant |M|$, we have $2 \leqslant r < n_0^*(r,1) \leqslant |M|$. Hence $E \subsetneq M$. Let $w \in M\backslash E$, and let $D := (E\backslash\{v\}) \cup \{w\}$. Since $D \subset M$ and $\mathcal{H}$ is hereditary, $D \in \mathcal{H}$. Thus, since $|D| = |E|$ and $E \in \bigcup_{s\in S} \mathcal{H}^{(s)} = \mathcal{F}$, we have $D \in \mathcal{F}$. By definition of D, it follows that $D \in \mathcal{F}\langle\{u\}\rangle\backslash\mathcal{F}\langle\{v\}\rangle$, and so $\mathcal{F}\langle\{u\}\rangle \neq \mathcal{F}\langle\{v\}\rangle$. □

6.2. *Extremal t-intersecting families of permutations and partial permutations*

For an n-set $X := \{x_1, \ldots, x_n\}$, we define $\mathcal{S}^*_{X,k}$ to be the special sub-family of $\mathcal{S}_{X,k}$ given by

$$\mathcal{S}^*_{X,k} := \{\{(x_1,a_1), \ldots, (x_n,a_n)\} : a_1, \ldots, a_n \text{ are distinct elements of } [k]\}.$$

Therefore $\mathcal{S}^*_{X,k} = \left\{\{(x_1,a_1), \ldots, (x_n,a_n)\} : \{a_1, \ldots, a_n\} \in \binom{[k]}{n}\right\}$. Note that $\mathcal{S}^*_{X,k} \neq \emptyset$ if and only if $n \leqslant k$.

For a family $\mathcal{F}$, we define $\mathcal{S}^*_{\mathcal{F},k}$ to be the special sub-family of $\mathcal{S}_{\mathcal{F},k}$ given by

$$\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F\in\mathcal{F}} \mathcal{S}^*_{F,k}.$$

An *r-partial permutation of an n-set N* is a pair (A,f), where $A\in\binom{N}{r}$ and $f: A\to N$ is an injection. An n-partial permutation of N is simply called a *permutation of N*. Clearly, the family of permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{[n],n}$, and the family of r-partial permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{\binom{[n]}{r},n}$.

For X as above, $\mathcal{S}^*_{X,k}$ can be interpreted as the family of permutations of sets in $\binom{[k]}{n}$: consider the bijection $\beta : \mathcal{S}^*_{X,k} \to \{(A,f) : A\in\binom{[k]}{n}, f: A\to A \text{ is a bijection}\}$ defined by $\beta(\{(x_1,a_1),\dots,(x_n,a_n)\}) := (\{a_1,\dots,a_n\}, f)$ where, for $b_1<\dots<b_n$ such that $\{b_1,\dots,b_n\} = \{a_1,\dots,a_n\}$, $f(b_i) := a_i$ for $i=1,\dots,n$. Now, $\mathcal{S}^*_{X,k}$ can also be interpreted as the sub-family $\mathcal{X} := \{(A,f) : A\in\binom{[k]}{n}, f: A\to[n] \text{ is an injection}\}$ of the family of n-partial permutations of $[k]$: consider an obvious bijection from $\mathcal{S}^*_{X,k}$ to $\mathcal{S}^*_{\binom{[k]}{n},n}$ and another one from $\mathcal{S}^*_{\binom{[k]}{n},n}$ to $\mathcal{X}$.

In [**13**, **14**], the study of intersecting permutations was initiated. Deza and Frankl [**14**] showed that $\mathcal{S}^*_{[n],n}$ is EKR. Cameron and Ku [**9**] proved that actually $\mathcal{S}^*_{[n],n}$ is strictly EKR; this result was also deduced from a more general result on certain vertex transitive graphs in [**36**].

Ku and Leader [**35**] investigated the intersection problem for partial permutations. They established that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is EKR, and strictly so for $r\in[8,n-3]$. Naturally, they conjectured that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is strictly EKR for all $r\in[n]$, and this was settled by Li and Wang [**37**].

Concerning t-intersecting families of permutations, the most interesting challenge comes from the following open conjecture.

Conjecture 6.4 (Deza and Frankl [**14**]). For any integer $t\geqslant 2$, there exists an $n_0(t)\in\mathbb{N}$ such that, for any $n\geqslant n_0(t)$ and any t-intersecting sub-family $\mathcal{A}$ of $\mathcal{S}^*_{[n],n}$, $|\mathcal{A}|\leqslant(n-t)!$.

In other words, this conjecture claims that $\mathcal{S}^*_{[n],n}$ is t-EKR for n sufficiently large, and hence also suggests the strict t-EKR property. It is worth pointing out that the condition $n\geqslant n_0(t)$ is necessary; [**34**, Example 3.1.1] is a simple illustration of this fact. Ku [**34**] proved an analogue of the statement of the conjecture for partial permutations.

Theorem 6.5 (Ku [**34**]). *Let $2\leqslant t\leqslant r$, and let $\mathcal{F}_n = \binom{[n]}{\leqslant r}$. Then there exists an $n_0(r,t)\in\mathbb{N}$ such that, for all $n\geqslant n_0(r,t)$, the t-stars of ${\mathcal{S}^*_{\mathcal{F}_n,n}}^{(r)} = \mathcal{S}^*_{\binom{[n]}{r},n}$ are among the largest t-intersecting Sperner sub-families of $\mathcal{S}^*_{\mathcal{F}_n,n}$.*

The above result follows from the case $\mathcal{I}=2^{[n]}$ and $k=n$ in the following consequence of Theorem 2.5.

Theorem 6.6. *Let $t\leqslant r$. Let $\mathcal{I}$ be a hereditary family with $\mu(\mathcal{I})\geqslant n_0^*(r,t)$, and let $\mathcal{F} := \mathcal{I}^{(\leqslant r)}$. Then, for $k\geqslant n_0^*(r,t)$, the largest t-intersecting Sperner sub-families of $\mathcal{S}^*_{\mathcal{F},k}$ are the largest t-stars of ${\mathcal{S}^*_{\mathcal{F},k}}^{(r)} = \mathcal{S}^*_{\mathcal{F}^{(r)},k}$.*

Proof. Let $m := n_0^*(r,t)$. Let $\mathcal{G} := \mathcal{I}^{(\leqslant m)}$ and $\mathcal{H} := \mathcal{S}^*_{\mathcal{G},k}$. Since $r\leqslant m$, we have $\mathcal{F} = \mathcal{G}^{(\leqslant r)}$. Now $\mathcal{S}^*_{\mathcal{G}^{(\leqslant r)},k} = {\mathcal{S}^*_{\mathcal{G},k}}^{(\leqslant r)}$, that is, $\mathcal{S}^*_{\mathcal{F},k} = \mathcal{H}^{(\leqslant r)}$. Since $\mathcal{I}$ is hereditary, $\mathcal{G}$ is hereditary and hence $\mathcal{H}$ is hereditary. Clearly $\mu(\mathcal{H}) = \mu(\mathcal{G})$. Hence the result follows by Theorem 2.5 if we show that $\mu(\mathcal{G}) = m$. Since $\mu(\mathcal{I})\geqslant m$, there exists an $\mathcal{I}$-maximal set M in $\mathcal{I}$ of size at least m, and hence, since $\mathcal{I}$ is hereditary, $\emptyset \neq \binom{M}{m} \subseteq \mathcal{I}$. By definition of $\mathcal{G}$, any set in $\binom{M}{m}$ is a $\mathcal{G}$-maximal set in $\mathcal{G}$. Therefore $\mu(\mathcal{G}) = m$ as required. □

From Theorem 2.1, we obtain the following analogue of Theorem 6.2.

THEOREM 6.7. *Let $t \leqslant r$, $S \subseteq [t, r]$ and $\mathcal{F} = \bigcup_{s \in S} \mathcal{I}^{(s)}$ for some hereditary family $\mathcal{I}$ with $\mu(\mathcal{I}) \geqslant n_0^*(r, t)$. Then, for $k \geqslant n_0^*(r, t)$, $\mathcal{S}^*_{\mathcal{F},k}$ is strictly t-EKR.*

Proof. Let $m := n_0^*(r, t)$. Let $\mathcal{G} := \mathcal{I}^{(\leqslant m)}$ and $\mathcal{H} := \mathcal{S}^*_{\mathcal{G},k}$. Since $r \leqslant m$, we have $\mathcal{F} = \bigcup_{s \in S} \mathcal{G}^{(s)}$. Now $\mathcal{S}^*_{\bigcup_{s \in S} \mathcal{G}^{(s)},k} = \bigcup_{s \in S} {\mathcal{S}^*_{\mathcal{G},k}}^{(s)}$, that is, $\mathcal{S}^*_{\mathcal{F},k} = \bigcup_{s \in S} \mathcal{H}^{(s)}$. Since $\mathcal{I}$ is hereditary, $\mathcal{G}$ is hereditary and hence $\mathcal{H}$ is hereditary. As we showed in the preceding proof, $\mu(\mathcal{H}) = \mu(\mathcal{G}) = n_0^*(r, t)$. Therefore the result follows by Theorem 2.1. □

6.3. *Extremal t-intersecting families of separated sets*

For a sequence $\{d_i\}_{i \in \mathbb{N}}$ of non-negative integers, we define

$$\begin{aligned} \mathcal{P}(\{d_i\}_{i \in \mathbb{N}}) &:= \{\{a_1, \dots, a_r\} \subset \mathbb{N} : r \in \mathbb{N}, a_{i+1} > a_i + d_{a_i}, i = 1, \dots, r-1\}, \\ \mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}}) &:= \mathcal{P}(\{d_i\}_{i \in \mathbb{N}}) \cap 2^{[n]}, \\ \mathcal{P}_{n,r}(\{d_i\}_{i \in \mathbb{N}}) &:= \mathcal{P}(\{d_i\}_{i \in \mathbb{N}}) \cap \binom{[n]}{r}. \end{aligned}$$

Holroyd, Spencer and Talbot [27] proved that $\mathcal{P}_{n,r}(\{d_i = d\}_{i \in \mathbb{N}})$ is EKR for any $d, r \in \mathbb{N}$. The author has tackled the wider problem of determining the EKR and strict EKR properties of $\mathcal{P}_{n,r}(\{d_i\}_{i \in \mathbb{N}})$ for the case when $\{d_i\}_{i \in \mathbb{N}}$ is a monotonic non-decreasing sequence with $d_1 > 0$; it turns out that $\mathcal{P}_{n,r}(\{d_i\}_{i \in \mathbb{N}}))$ is also EKR in this case. For the very general case where $\{d_i\}_{i \in \mathbb{N}}$ is any sequence, we prove the following t-intersection result using Theorem 2.1.

THEOREM 6.8. *Let $\{d_i\}_{i \in \mathbb{N}}$ be a sequence of non-negative integers, and let $t \leqslant r$. Then there exists $n_0 := n_0(\{d_i\}_{i \in \mathbb{N}}, r, t) \in \mathbb{N}$ such that $n_0 = \min\{n \in \mathbb{N} : \mu(\mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}})) \geqslant n_0^*(r, t)\}$ and, for any $n \geqslant n_0$ and any $S \subseteq [t, r]$, $\bigcup_{s \in S} \mathcal{P}_{n,s}(\{d_i\}_{i \in \mathbb{N}})$ is strictly t-EKR.*

Before proving this result, we illustrate the fact that we cannot do without some condition $n \geqslant n_0(\{d_i\}_{i \in \mathbb{N}}, r, t)$. For example, since $\mathcal{P}_{n,r}(\{d_i = 0\}_{i \in \mathbb{N}}) = \binom{[n]}{r}$, the smallest $n_0(\{d_i = 0\}_{i \in \mathbb{N}}, r, t)$ is $(r - t + 1)(t + 1) + 1$ by Theorem 1.3. To take another example, let $t \geqslant 4$ and let $\mathcal{P} := \mathcal{P}_{2t+5,t+1}(\{d_i = 1\}_{i \in \mathbb{N}})$. For each $j \in [t + 3]$, let $P_j := \{2i - 1 : i \in [j]\}$; so $P_j \in \mathcal{P}^{(j)}$. It is easy to see that $\mathcal{P}\langle P_t \rangle$ is a largest t-star of $\mathcal{P}$. Let $\mathcal{A} := \binom{P_{t+2}}{t+1}$. Clearly, $\mathcal{A}$ is a non-trivial t-intersecting sub-family of $\mathcal{P}$ and $|\mathcal{A}| - |\mathcal{P}\langle P_t \rangle| = (t + 2) - |[2t + 1, 2t + 5]| = t - 3 \geqslant 1$. Thus, for $t \geqslant 4$, the smallest $n_0(\{d_i = 1\}_{i \in \mathbb{N}}, t + 1, t)$ is larger than $2t + 5$, which is the value of the smallest $n_0(\{d_i = 0\}_{i \in \mathbb{N}}, t + 1, t)$ (by Theorem 1.3 as remarked above).

We now work towards the proof of Theorem 6.8, which requires two lemmas about the nature of $\mu(\mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}}))$.

LEMMA 6.9. *Let $\{d_i\}_{i \in \mathbb{N}}$ be a sequence of non-negative integers, and let $m \in \mathbb{N}$. Then there exists $n_0(m) \in \mathbb{N}$ such that $\mu(\mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}})) \geqslant m$ for all $n \geqslant n_0(m)$.*

Proof. The result is trivial if $m = 1$; so suppose $m \geqslant 2$. Let $a_0 := 0$, $a_1 := 1$, $a_2 := 1 + d_1$, and let $a_i := a_{i-1} + \max\{d_j : j \in [a_{i-2} + 1, a_{i-1}]\} + 1$ for $i = 3, \dots, 2m$. Let $n \geqslant n_0(m) :=$

a_{2m}. Let $P \in \mathcal{P}_n := \mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}})$ such that $P \cap [a_{2i'}+1, a_{2i'+2}] = \emptyset$ for some $i' \in \{0\} \cup [m-1]$. It is clear from the choice of the integers $a_0, a_1, \ldots, a_{2m}$ that we then have $P \cup \{a_{2i'+1}\} \in \mathcal{P}_n$, and hence P is not $\mathcal{P}_n$-maximal. Thus any $\mathcal{P}_n$-maximal set in $\mathcal{P}_n$ must intersect $[a_{2i}+1, a_{2i+2}]$ for each $i \in \{0\} \cup [m-1]$. Therefore $\mu(\mathcal{P}_n) \geqslant m$. □

Note that Corollary 2.4 and the above lemma already imply that, for any sequence $\{d_i\}_{i\in\mathbb{N}}$ of non-negative integers and any $t, r \in \mathbb{N}$, $t \leqslant r$, there exists $n_0(\{d_i\}_{i\in\mathbb{N}}, r, t) \in \mathbb{N}$ such that, for any $S \in [t, r]$ and any $n \geqslant n_0(\{d_i\}_{i\in\mathbb{N}}, r, t)$, $\bigcup_{s\in S} \mathcal{P}_{n,s}(\{d_i\}_{i\in\mathbb{N}})$ is strictly t-EKR. To establish the slightly stronger fact given by Theorem 6.8, we need the next lemma, which says that $\{\mu(\mathcal{P}_n)\}_{n\in\mathbb{N}}$ ($\mathcal{P}_n := \mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}})$) is a monotonic non-decreasing sequence.

Lemma 6.10. *Let $\{d_i\}_{i\in\mathbb{N}}$ be a sequence of non-negative integers, and let $n \in \mathbb{N}$. Then $\mu(\mathcal{P}_{n+1}(\{d_i\}_{i\in\mathbb{N}})) \geqslant \mu(\mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}}))$.*

Proof. For any $m \in \mathbb{N}$, let $\mathcal{P}_m := \mathcal{P}_m(\{d_i\}_{i\in\mathbb{N}})$. Suppose $\mu(\mathcal{P}_{n+1}) < \mu(\mathcal{P}_n)$ for some $n \in \mathbb{N}$. Let $1 \leqslant p_1 \leqslant \ldots \leqslant p_{\mu(\mathcal{P}_{n+1})} \leqslant n+1$ such that $P_1 := \{p_1, \ldots, p_{\mu(\mathcal{P}_{n+1})}\}$ is $\mathcal{P}_{n+1}$-maximal. If $p_{\mu(\mathcal{P}_{n+1})} \neq n+1$, then P_1 is a $\mathcal{P}_n$-maximal set in $\mathcal{P}_n$, and hence $\mu(\mathcal{P}_n) \leqslant |P_1|$; but this contradicts $\mu(\mathcal{P}_n) > \mu(\mathcal{P}_{n+1})$ as $|P_1| = \mu(\mathcal{P}_{n+1})$. Thus $p_{\mu(\mathcal{P}_{n+1})} = n+1$. Let $P_2 := P_1 \backslash \{n+1\}$; so $P_2 \in \mathcal{P}_n$. If P_2 is $\mathcal{P}_n$-maximal, then $\mu(\mathcal{P}_n) \leqslant |P_2|$; but this contradicts $|P_2| + 1 = |P_1| = \mu(\mathcal{P}_{n+1}) < \mu(\mathcal{P}_n)$, and so P_2 is not $\mathcal{P}_n$-maximal. Therefore there exists a non-empty set $Q \subseteq [n] \backslash P_2$ such that $P_3 := P_2 \cup Q$ is a $\mathcal{P}_n$-maximal set in $\mathcal{P}_n$. Thus, since $\mu(\mathcal{P}_n) > \mu(\mathcal{P}_{n+1})$, we have $|P_3| > |P_1|$, which implies $|Q| > 1$ as $|P_1| > |P_2|$. Let $q_1, q_2 \in Q$, $q_1 < q_2$. Suppose $q < p_{\mu(\mathcal{P}_n)-1}$ for some $q \in \{q_1, q_2\}$. Then $q + d_q + 1 \leqslant p_{\mu(\mathcal{P}_n)-1}$ (since $q, p_{\mu(\mathcal{P}_n)-1} \in P_3 \in \mathcal{P}_n$), and hence $P_1 \cup \{q\} \in \mathcal{P}_{n+1}$; but this is a contradiction since P_1 is $\mathcal{P}_{n+1}$-maximal. So $q > p_{\mu(\mathcal{P}_n)-1}$ for each $q \in \{q_1, q_2\}$. Since $q_1, q_2 \in P_3 \in \mathcal{P}_n$, $q_1 + d_{q_1} + 1 \leqslant q_2$. Thus, since $q_2 < n+1$, we have $P_2 \cup \{q_1, n+1\} \in \mathcal{P}_{n+1}$, and hence $P_1 \cup \{q_1\} \in \mathcal{P}_{n+1}$; but this is a contradiction since P_1 is $\mathcal{P}_{n+1}$-maximal. We therefore conclude that $\mu(\mathcal{P}_{n+1}) \geqslant \mu(\mathcal{P}_n)$. □

Proof of Theorem 6.8. For $n \in \mathbb{N}$, let $\mathcal{P}_n := \mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}})$. By Lemma 6.9, $\min\{n \in \mathbb{N} : \mu(\mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}})) \geqslant n_0^*(r,t)\}$ exists; let n_0 be this integer. Clearly, $\mathcal{P}(\{d_i\}_{i\in\mathbb{N}})$ is hereditary, and so $\mathcal{P}_n$ is hereditary. If $n \geqslant n_0$, then $\mu(\mathcal{P}_n) \geqslant \mu(\mathcal{P}_{n_0})$ by Lemma 6.10, and hence $\mu(\mathcal{P}_n) \geqslant n_0^*(r,t)$ as $\mu(\mathcal{P}_{n_0}) = n_0^*(r,t)$. The result now follows by Theorem 2.1. □

Finally, let us quickly demonstrate another application of Theorem 2.1 for families whose sets obey a slightly different separation condition. For $1 \leqslant k \leqslant n$, let $\mathcal{C}_{n,k}$ be the family consisting of all subsets A of $[n]$ such that, if $|A| \geqslant 2$, then $k < |a-b| < n-k$ for any $a, b \in A$, $a \neq b$. Solving the problem in [**26**], Talbot [**44**] proved the following result.

Theorem 6.11 (Talbot [**44**]). *For any $n, k, r \in \mathbb{N}$, $\mathcal{C}_{n,k}^{(r)}$ is EKR, and strictly so unless $n = 2r+2$ and $k = 1$.*

Clearly, $\mathcal{C}_{n,k}$ is a hereditary family. It is also easy to see that $\{\mu(\mathcal{C}_{n,k})\}_{n\in\mathbb{N}}$ is a monotonic non-decreasing sequence and that $\mu(\mathcal{C}_{n,k}) \to \infty$ as $n \to \infty$. Thus, for the general t-intersection case, Theorem 2.1 gives us the following.

Theorem 6.12. *For any $k, r \in \mathbb{N}$ and $n \geqslant \min\{m \in \mathbb{N} : \mu(\mathcal{C}_{m,k}) \geqslant n_0^*(r,t)\}$, $\mathcal{C}_{n,k}^{(r)}$ is strictly t-EKR.*

Again, if $t \geqslant 2$, then the above equality does not hold for certain small values of n. Indeed, let $t \geqslant 2$ and $\mathcal{C} := \mathcal{C}_{2t+4,1}$. For each $j \in [t+2]$, let $C_j := \{2i-1 : i \in [j]\}$; so $C_j \in \mathcal{C}^{(j)}$. It is easy to see that $\mathcal{C}^{(t+1)}\langle C_t \rangle$ is a largest t-star of $\mathcal{C}^{(t+1)}$. Let $\mathcal{A} := \binom{C_{t+2}}{t+1}$. Clearly, $\mathcal{A}$ is a non-trivial t-intersecting sub-family of $\mathcal{C}^{(t+1)}$ and $|\mathcal{A}| - |\mathcal{C}^{(t+1)}\langle C_t\rangle| = (t+2) - |[2t+1, 2t+3]| = t-1 \geqslant 1$. Therefore $\mathcal{C}^{(t+1)}$ is not t-EKR.

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