The Erdős-Ko-Rado properties of set systems defined by double partitions

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29th May 2008

Abstract

Let $\mathcal{F}$ be a family of subsets of a finite set V. The *star of* $\mathcal{F}$ *at* $v \in V$ is the sub-family $\{A \in \mathcal{F} : v \in A\}$. We denote the sub-family $\{A \in \mathcal{F} : |A| = r\}$ by $\mathcal{F}^{(r)}$.

A *double partition* $\mathcal{P}$ of a finite set V is a partition of V into *large sets* that are in turn partitioned into *small sets*. Given such a partition, the family $\mathcal{F}(\mathcal{P})$ *induced by* $\mathcal{P}$ is the family of subsets of V whose intersection with each large set is either contained in just one small set or empty.

Our main result is that, if one of the large sets is trivially partitioned (that is, into just one small set) and $2r$ is not greater than the *least* cardinality of any *maximal* set of $\mathcal{F}(\mathcal{P})$, then no intersecting sub-family of $\mathcal{F}(\mathcal{P})^{(r)}$ is larger than the largest star of $\mathcal{F}(\mathcal{P})^{(r)}$. We also characterise the cases when every extremal intersecting sub-family of $\mathcal{F}(\mathcal{P})^{(r)}$ is a star of $\mathcal{F}(\mathcal{P})^{(r)}$.

Keywords: Erdős-Ko-Rado; intersecting family; double partition.

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1 Introduction

For positive integers m and n such that $m < n$, we denote the set $\{m, m+1, ..., n\}$ by $[m, n]$, and if $m = 1$ then we also write $[n]$. For a set X, the *power set* $\{A \subset X\}$ of X is denoted by 2^X, and the set $\{Y \subseteq X : |Y| = r\}$ is denoted by $\binom{X}{r}$.

Let V be a set, and let $\mathcal{F} \subseteq 2^V$. For $v \in V$, denote by $\mathcal{F}_v$ the *star* at v, that is, the sub-family $\{A \in \mathcal{F} : v \in A\}$. If $\mathcal{F}_v = \mathcal{F}$ (that is, $v \in \bigcap_{F \in \mathcal{F}} F$), then v is said to be a *centre of* $\mathcal{F}$. If $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ then $\mathcal{F}$ is said to be *centred*; otherwise $\mathcal{F}$ is said to be *non-centred.*

A family $\mathcal{F}$ is *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in F$. Then $\mathcal{F}$ is said to have the *Erdős-Ko-Rado property* if there is a star that is an extremal (that is, largest) intersecting sub-family of $\mathcal{F}$, and to have the *strict Erdős-Ko-Rado property* if there is a star that is strictly larger than any non-centred intersecting sub-family of $\mathcal{F}$. We abbreviate, saying that $\mathcal{F}$ is *EKR* or *strictly EKR*, respectively. Furthermore, for an integer r we say that $\mathcal{F}$ is (*strictly*) r-*EKR* if the uniform sub-family $\mathcal{F}^{(r)} := \{A \in \mathcal{F} : |A| = r\}$ is non-empty and (strictly) EKR.

The above notation permits the use of both subscript and superscript: that is, $\mathcal{F}_v^{(r)} := \{A \in \mathcal{F} : v \in A, |A| = r\}$.

A family $\mathcal{F}$ is said to be *hereditary* if any subset of any set in $\mathcal{F}$ is also in $\mathcal{F}$. Chvátal's conjecture [5], which is one of the central conjectures in extremal set theory, claims that any hereditary family is EKR. In this paper we are concerned with the r-EKR properties of hereditary families, denoted by $\mathcal{F}(\mathcal{P})$, that have the following particular structure.

A *double partition* $\mathcal{P}$ of a finite set V is a partition of V into *large sets* V_i $(0 \leq i \leq k)$ (the *top partition*), each partitioned into a_i *small sets* $V_{i1}, \ldots, V_{ia_i}$. The family $\mathcal{F}(\mathcal{P})$ *induced by* $\mathcal{P}$ is the family of subsets of V whose intersection with each large set is either contained in just one small set or empty.

Let $\mathcal{P}$ be a double partition. Throughout the paper, we shall assume that for $0 \leq i \leq k$ the small sets V_{ij} are presented in non-increasing order of size: $|V_{i1}| \geq |V_{i2}| \geq \ldots \geq |V_{ia_i}| \geq 1$. The elements of each small set V_{ij} are given some arbitrary ordering and denoted by $v_{ij1}, \ldots, v_{ijn_{ij}}$ where $|V_{ij}| = n_{ij}$.

We denote $\sum_{i=0}^{k} |V_{i1}|$ by $\alpha(\mathcal{P})$ and $\sum_{i=0}^{k} |V_{ia_i}|$ by $\mu(\mathcal{P})$. Thus $\alpha(\mathcal{P})$ is the largest cardinality of any set in $\mathcal{F}(\mathcal{P})$, while $\mu(\mathcal{P})$ is the *least* cardinality of any *maximal* set of $\mathcal{F}(\mathcal{P})$.

The case $V = V_0, a_0 = 1$ gives $\mathcal{F}(\mathcal{P}) = 2^V$ and $\mathcal{F}(\mathcal{P})^{(r)} = \binom{V}{r}$. If $|V|/2 < r < |V|$, then $\binom{V}{r}$ is intersecting and hence not EKR. For $r \leq |V|/2$, the solution to our problem in this simple case is the following classical result.

Theorem 1.1 (Erdős, Ko, Rado [8], Hilton, Milner [16]) *Let X be a finite set, and let $r \leq |X|/2$. Then:*
(i) $\binom{X}{r}$ is EKR;
(ii) $\binom{X}{r}$ is strictly EKR if and only if $r < |X|/2$.

Part (i) was proved by Erdős, Ko and Rado, and part (ii) was established later by Hilton and Milner as part of a more general result. The Erdős-Ko-Rado Theorem inspired a wealth of results in extremal set theory; the survey paper by Deza and Frankl [6] is recommended.

The EKR problem for the case when the small sets are singletons has attracted much attention. The following is a well-known important result that was stated by Meyer [20] and proved in different ways by Deza and Frankl [6] and Bollobás and Leader [3].

Theorem 1.2 (Meyer [20]; Deza, Frankl [6]; Bollobás, Leader [3]) *Let $\mathcal{P}$ be a double partition of V into k large sets each of cardinality $t \geq 2$, where each small set is a singleton. Then (i) $\mathcal{F}(\mathcal{P})$ is r-EKR $(1 \leq r \leq k)$ and (ii) strictly so unless $t = 2$ and $r = k \geq 3$.*

Other proofs of this result are found in [4, 7, 10]. The special case $r = k$ was also treated in [1, 13, 19, 21]. Holroyd, Spencer and Talbot [17] showed that Theorem 1.2(i) still holds if the cardinalities of the large sets are not all the same but at least 2. The case $r = k$ of this extension implies that $\mathcal{F}(\mathcal{P})$ is $\alpha(\mathcal{P})$-EKR for any $\mathcal{P}$.

We now state two theorems that will both be generalised in this paper: the first one is for the case when all small sets are again singletons but at least one large set is also a singleton, and the second one is for the case when the small sets are not necessarily singletons but there are just two large sets.

Theorem 1.3 (Bey [2]; Holroyd, Spencer, Talbot [17]) *Let $\mathcal{P}$ be a double partition of V into k large sets, where at least one large set is a singleton and each small set is a singleton. Then $\mathcal{F}(\mathcal{P})$ is r-EKR if $r \leq k/2$.*

Theorem 1.4 (Holroyd, Talbot [18]) *Let $\mathcal{P}$ be the double partition $V = V_0 \cup V_1$ with $a_1 > 1$.*
(i) If $r \leq \mu(\mathcal{P})/2$, then $\mathcal{F}(\mathcal{P})$ is r-EKR;
(ii) if $r < \mu(\mathcal{P})/2$, then $\mathcal{F}(\mathcal{P})$ is strictly r-EKR.

Suppose no large set is trivially partitioned, that is, $a_i > 1$ for each $i \in \{0, 1, ..., k\}$. Then the problem immediately reduces to one with $a_i = 1$ for some i if and only if $k \leq 1$ as in the result above; see [18]. We mention that, if $k = 1$, then, for the "reduced" problem with say $a_0 = 1$, the family

$\{V_0 \cup V_{11}, ..., V_0 \cup V_{1a_1}\}$ of maximal sets of $\mathcal{F}(\mathcal{P})$ is what is commonly referred to as a *sunflower* or *delta-system.* The Erdős-Rado Theorem [9] is an example of a well-known result about sunflowers. Sunflowers are used in the *kernel method* introduced by Hajnal and Rothschild [14]; a brief review together with another application of this method is given in [11].

The main contribution of the present paper is to develop the method used in [18], to allow us to prove quite a general result concerning double partitions. Before proceeding, we note that there is a considerable difference between the case when there is a set V_i that is not further partitioned (that is, V_i is both a large and a small set, so $a_i = 1$) and the case where this is not so. This requires the following modification of our notation.

Suppose that for some non-empty $S \subseteq \{0\} \cup [k]$ and for all $i \in S$, $a_i = 1$. Then replacing the large sets V_i, $i \in S$, by the single large set $\bigcup_{i \in S} V_i$ does not alter $\mathcal{F}(\mathcal{P})$. Thus we adopt the following convention: The set V_0 is the unique large set that is trivially partitioned (i.e., also a small set), and also the only large set that is allowed to be empty. We say that $\mathcal{P}$ is *anchored* if $V_0 \neq \emptyset$, and *unanchored* if $V_0 = \emptyset$. A double partition that is given to be unanchored may, if convenient, be described by a top partition $V = \bigcup_{j=1}^{k} V_i$ and the empty V_0 ignored.

The *width* of a double partition $\mathcal{P}$ is the number of non-trivially partitioned large sets.

Our main theorem concerns anchored double partitions and is as follows.

Theorem 1.5 *Let $\mathcal{P}$ be an anchored double partition of width $k > 0$. Let $1 \leq r \leq \mu(\mathcal{P})/2$. Then:*
(i) $\mathcal{F}(\mathcal{P})$ is r-EKR;
(ii) $\mathcal{F}(\mathcal{P})$ fails to be strictly r-EKR if and only if $2r = \mu(\mathcal{P}) = \alpha(\mathcal{P})$, $3 \leq |V_0| \leq r$, and $k = 1$.

Clearly, this result significantly generalises Theorems 1.3 and 1.4 (recall that Theorem 1.4 follows immediately from the statement of Theorem 1.5 with $k = 1$). We remark that, unlike Theorem 1.2, this result in general does not hold for $\mu(\mathcal{P})/2 < r < \alpha(\mathcal{P})$; examples can be constructed easily, especially for anchored partitions of width 1 (see also [18]).

Removing the anchor condition from Theorem 1.5 seems to make the problem much harder. However, in the special case of an unanchored double partition of width 3 where all the V_{ij} have the same cardinality, we have the following result.

Theorem 1.6 *Let $\mathcal{P}$ be an unanchored double partition of width 3 such that every small set is of size c. Then $\mathcal{F}(\mathcal{P})$ is strictly r-EKR for all $r \leq \mu(\mathcal{P})/2 = 3c/2$.*

2 Crossing sets

Let $\mathcal{Y} := \{X_0, X_1, \ldots, X_l\}$ be a family of disjoint non-empty finite sets, $Y := \bigcup_{i=0}^{l} X_i$, $x_i := |X_i| \quad (0 \le i \le l)$, $y := |Y|$. A subset A of Y is a *crossing set* of $\mathcal{Y}$ if $A \cap X_i \neq \emptyset \quad (0 \le i \le l)$. We denote by $\mathcal{C}(\mathcal{Y})$ the family of crossing sets of $\mathcal{Y}$; thus, for $l+1 \le m \le y$, $\mathcal{C}(\mathcal{Y})^{(m)}$ is the family of crossing m-sets of $\mathcal{Y}$. We denote $|\mathcal{C}(\mathcal{Y})^{(m)}|$ by $(x_0, \ldots, x_l)^{(m)}$ or, where the x_i are clear from context, by $\mathbf{y}^{(m)}$. These numbers mimic the behaviour of the binomial coefficients $\binom{y}{m}$ in some respects; in particular, they have the following property.

Lemma 2.1 *If $l+1 \le m < y/2$ and $m < n \le y-m$, then*

$$\mathbf{y}^{(m)} \le \mathbf{y}^{(n)}$$

with equality if and only if $n = y - m$ and $l = 0$.

Proof. For each $A \in \mathcal{C}(\mathcal{Y})^{(m)}$ there are $\binom{y-m}{n-m}$ sets $B \in \mathcal{C}(\mathcal{Y})^{(n)}$ that contain A (since every n-subset of Y containing A is also a crossing set). Moreover, any such set B has at most $\binom{n}{m}$ subsets that belong to $\mathcal{C}(\mathcal{Y})^{(m)}$. Counting in two ways the pairs (A, B) with $A \in \mathcal{C}(\mathcal{Y})^{(m)}, B \in \mathcal{C}(\mathcal{Y})^{(n)}$, we obtain

$$\mathbf{y}^{(m)} \binom{y-m}{n-m} \le \mathbf{y}^{(n)} \binom{n}{m}. \tag{1}$$

Since $\binom{n}{m} = \binom{n}{n-m}$, the inequality holds under the stated conditions and is strict when $n < y - m$.

Now consider the case $n = y-m$. If $l = 0$, then $\mathbf{y}^{(m)} = \binom{x_0}{m} = \binom{x_0}{n} = \mathbf{y}^{(n)}$; so assume $l \ge 1$. We shall show that the inequality (1) is strict by finding some $B \in \mathcal{C}(\mathcal{Y})^{(n)}$ having an m-subset A such that $A \notin \mathcal{C}(\mathcal{Y})^{(m)}$.

There exists $X_i \in \mathcal{Y}$ such that $|X_i| \le y/2$. Let $z := |X_i|$. Choose $B \in \mathcal{C}(\mathcal{Y})^{(n)}$ such that $|B \cap X_i|$ is as small as possible; that is, $|B \cap X_i| = \max\{1, n - |Y \backslash X_i|\} = \max\{1, n - y + z\}$. Then, since $m < y/2$, we conclude

$$|B \cap (Y \backslash X_i)| = \min\{n-1, y-z\} \ge \min\{n-1, y/2\} \ge m.$$

Thus there exists $A \subseteq B \cap (Y \backslash X_i)$ with $|A| = m$. Then $A \notin \mathcal{C}(\mathcal{Y})^{(m)}$, as required. □

Remark 2.2 We note that (1) still holds if we replace $\mathcal{C}(\mathcal{Y})^{(m)}$ by any subset $\mathcal{M}$ of $\mathcal{C}(\mathcal{Y})^{(m)}$ and $\mathcal{C}(\mathcal{Y})^{(n)}$ by $\mathcal{N} := \{B \in \mathcal{C}(\mathcal{Y})^{(n)} \colon A \subset B \text{ for some } A \in \mathcal{M}\}$. Thus, by Hall's Marriage Theorem [15], there is an injection $f \colon \mathcal{C}(\mathcal{Y})^{(m)} \to \mathcal{C}(\mathcal{Y})^{(n)}$ such that $A \subset f(A)$ for any $A \in \mathcal{C}(\mathcal{Y})^{(m)}$.

Let $l+1 \leq r \leq y$ and $v \in X_0$. The *crossing r-star* with *centre v* is the sub-family of $\mathcal{C}(\mathcal{Y})^{(r)}$ consisting of those sets containing v, and is denoted by $\mathcal{C}(\mathcal{Y})_v^{(r)}$. A family $\mathcal{F}$ of crossing sets of $\mathcal{Y}$ is said to be *strongly intersecting* if, for any $A, B \in \mathcal{F}$, $A \cap B \cap X_0 \neq \emptyset$.

We now prove an 'EKR-type' theorem for strongly intersecting families of crossing sets. (The proof is actually the most technically complex part of proving Theorem 1.5.)

Theorem 2.3 *Let $\mathcal{Y} := \{X_0, \ldots, X_k\}$ be a family of disjoint non-empty sets and let $Y := \bigcup_{i=0}^{k} X_i$, $2 \leq k+1 \leq r \leq |Y|/2$. Then:*
(i) the crossing r-stars with centres in X_0 are extremal strongly intersecting sub-families of $\mathcal{C}(\mathcal{Y})^{(r)}$;
(ii) these are the only extremal such families, unless $3 \leq |X_0| \leq r = |Y|/2$ and $k = 1$.

Proof. Let $\mathcal{F}$ be a strongly intersecting sub-family of $\mathcal{C}(\mathcal{Y})^{(r)}$. A necessary condition for it to be extremal is that it be a maximal such family, and we may therefore assume this. Let $\mathcal{G} := \{A \cap X_0 \colon A \in \mathcal{F}\}$; then by maximality, $\mathcal{F} = \{A \in \mathcal{C}(\mathcal{Y})^{(r)} \colon A \cap X_0 \in \mathcal{G}\}$.

Thus, for any $P \in \mathcal{G}$ with $|P| = p$ and any crossing $(r-p)$-set Q of $\{X_1, \ldots, X_l\}$, we have $P \cup Q \in \mathcal{F}$ so that

$$|\{A \in \mathcal{F} \colon A \cap X_0 = P\}| = (x_1, \ldots, x_k)^{(r-p)}.$$

Similarly, let $\mathcal{C}(\mathcal{Y})_v^{(r)}$ be a crossing r-star with $v \in X_0$ and let $\mathcal{H} := \{A \cap X_0 \colon A \in \mathcal{C}(\mathcal{Y})_v^{(r)}\}$. For any $P \in \mathcal{H}$ with $|P| = p$ we obtain

$$|\{A \in \mathcal{Y}_v^{(r)} \colon A \cap X_0 = P\}| = (x_1, \ldots, x_k)^{(r-p)}.$$

We shall denote $(x_1, \ldots, x_k)$ by $\mathbf{x}$.

We thus have a weighted Erdős-Ko-Rado problem to solve concerning intersecting families of subsets of X_0.

It is convenient to set $w := x_0, x := y - w$. Observe that for any crossing r-set A of $\mathcal{Y}$, we have $s \leq |A \cap X_0| \leq t$, where $s := \max\{1, r-x\}$, $t := \min\{r-k, w\}$. Thus, partitioning $\mathcal{G}, \mathcal{H}$ by cardinality, and noting that $|\mathcal{H}^{(p)}| = \binom{w-1}{p-1}$, we need to show that

$$\sum_{p=s}^{t} |\mathcal{G}^{(p)}| \mathbf{x}^{(r-p)} \leq \sum_{p=s}^{t} \binom{w-1}{p-1} \mathbf{x}^{(r-p)} \tag{2}$$

and that, if $\mathcal{G}$ is non-centred, then the inequality is strict unless $k = 1$ and $3 \leq w \leq r = |Y|/2$.

To establish (2), it is sufficient to show that:

1. if either $p = t = w$ or $p \leq w/2$, then
   $|\mathcal{G}|^{(p)}\mathbf{x}^{(r-p)} \leq \binom{w-1}{p-1}\mathbf{x}^{(r-p)}$
   (that is, $|\mathcal{G}^{(p)}| \leq \binom{w-1}{p-1}$);

2. if $w/2 < p \leq \min\{t-1, w\}$, then
   $|\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))} \leq \binom{w-1}{p-1}\mathbf{x}^{(r-p)} + \binom{w-1}{w-p-1}\mathbf{x}^{r-(w-p))}$.

Statement 1 follows easily since if $p = w$ then $\binom{w-1}{p-1} = 1$ and $\mathcal{G}^{(p)}$ is either empty or consists of the single set X_0, and if $p \leq w/2$ then $|\mathcal{G}^{(p)}| \leq \binom{w-1}{p-1}$ by Theorem 1.1(i).

We now prove Statement 2. So suppose $w/2 < p \leq \min\{t-1, w\}$. Observe that the strong intersection condition implies that no set in $\mathcal{G}^{(p)}$ can be the complement in X_0 of a set in $\mathcal{G}^{(w-p)}$, and hence $|\mathcal{G}^{(p)}| + |\mathcal{G}^{(w-p)}| \leq \binom{w}{p}$. Thus, for such a pair $p, w-p$:

$$|\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))} \leq \left(\binom{w}{p} - |\mathcal{G}^{(w-p)}|\right)\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))}.$$

Since $k \leq w-p < y/2$ and $w-p < p \leq y-(w-p)$, the conditions of Lemma 2.1 hold with $l = k-1$, $m = r-p$, $n = r-(w-p)$. Therefore, since Theorem 1.1(i) gives us $|\mathcal{G}^{(w-p)}| \leq \binom{w-1}{w-p-1} = \binom{w-1}{p}$ (note that $w-p < w/2$ as $p > w/2$), the maximum value of $|\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))}$ is obtained when $|\mathcal{G}^{(w-p)}| = \binom{w-1}{p} = |\mathcal{H}^{(w-p)}|$ and $|\mathcal{G}^{(p)}| = \binom{w}{p} - \binom{w-1}{p} = \binom{w-1}{p-1} = |\mathcal{H}^{(p)}|$, and, unless $\mathbf{x}^{(r-p)} = \mathbf{x}^{(r-(w-p))}$, this is the only way to achieve the maximum. This already verifies (2) and hence part (i) of the theorem.

We now prove part (ii) of the theorem, and we therefore assume that the bound in (2) is attained. Observe that (unless $|X_0| = 1$, when the theorem is trivial) $p < w/2$ for at least one $p \in [s,t]$. Thus, unless $\mathbf{x}^{(r-(w-p))} = \mathbf{x}^{(r-p)}$, we know from Theorem 1.1(ii) that $\mathcal{G}^{(p)}$ is a star, centred (say) on v, and hence, since every other set of $\mathcal{G}$ must intersect each element of $\mathcal{G}^{(p)}$, $\mathcal{F}$ must be $\mathcal{C}(\mathcal{Y})_v^{(r)}$. So the only possibility for an extremal non-star occurs when:
(a) $\mathbf{x}^{(r-(w-p))} = \mathbf{x}^{(r-p)}$ for every $p \in [s,t]$ with $p < w/2 < w-p$;
(b) there is no $p < w/2$ with $w-p > t$.
By Lemma 2.1, (a) happens only if $2r - w = x$ (that is, $r = |Y|/2$) and $k = 1$. Clearly we also require $w \geq 3$ in order to obtain a non-star for $\mathcal{G}$. Finally, (b) requires $w \leq r$.

Now, if $k = 1$ and $3 \leq w \leq r = |Y|/2$, then we may construct a non-star family $\mathcal{A}$ of crossing r-sets such that $|\mathcal{A}| = |\mathcal{C}(\mathcal{Y})_v^{(r)}|$ (where $v \in X_0$) as follows. Let $\mathcal{B}_1 := \{A \in \mathcal{C}(\mathcal{Y})_v^{(r)} \colon A \cap X_0 = \{v\}\}$, $\mathcal{B}_2 := \{Y \backslash A \colon A \in \mathcal{B}_1\}$. Then define $\mathcal{A} := (\mathcal{C}(\mathcal{Y})_v^{(r)} \backslash \mathcal{B}_1) \cup \mathcal{B}_2$. □

3 Double partitions and compressions

We shall now develop some further notation.

Let $\mathcal{P}$ be a double partition. Recall that, within each large set, the small sets are ordered by size. The set V_0 and the small sets V_{i1}, $1 \leq i \leq k$, are said to be the *floor sets*, while the remaining small sets V_{ij}, $1 \leq i \leq k$, $2 \leq j \leq a_i$, are said to be the *upper sets*. The union of the floor sets is said to be the *floor* and is denoted by F.

We now define *compressions*; see [12] for an excellent survey on the uses of the compression (also known as *shifting*) technique in extremal set theory.

For $i = 1, ..., k$, $j = 2, ..., a_i$, define $\delta_{ij} \colon V \to V$ by $\delta_{ij}(v_{ijp}) := v_{i1p}$ ($p = 1, ..., n_{ij}$), and $\delta_{ij}(v) := v$ otherwise. Thus, each δ_{ij} maps an upper set to the corresponding floor set and leaves all other small sets unaffected.

Let $A \in \mathcal{F}(\mathcal{P})$. We may denote $\{\delta_{ij}(x) \colon x \in A\}$ by $\delta_{ij}(A)$; note that $\delta_{ij}(A) \in \mathcal{F}(\mathcal{P})$. Define the *compression operation* Δ_{ij} on sub-families $\mathcal{A}$ of $\mathcal{F}(\mathcal{P})$ by

$$\Delta_{ij}(\mathcal{A}) := \{\delta_{ij}(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \delta_{ij}(A) \in \mathcal{A}\}.$$

The following lemma outlines the fundamental properties of $\Delta_{ij}(\mathcal{A})$.

Lemma 3.1 *Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{F}(\mathcal{P})$. Then*
(i) $\Delta_{ij}(\mathcal{A}) \subseteq \mathcal{F}(\mathcal{P})$.
(ii) $|\Delta_{ij}(\mathcal{A})| = |\mathcal{A}|$,
(iii) $\Delta_{ij}(\mathcal{A})$ is intersecting,
(iv) if V' is a union of upper sets of $\mathcal{F}(\mathcal{P})$ such that $(A \cap B)\backslash V' \neq \emptyset$ for all $A, B \in \mathcal{A}$, then $(C \cap D)\backslash(V' \cup V_{ij}) \neq \emptyset$ for all $C, D \in \Delta_{ij}(\mathcal{A})$.

Proof. (i) and (ii) are straightforward, and (iii) follows from (iv) by setting $V' = \emptyset$. We now prove (iv).

Let $C, D \in \Delta_{ij}(\mathcal{A})$. If $C \notin \mathcal{A}$, let $A \in \mathcal{A}$ such that $\delta_{ij}(A) = C$. If $D \in \mathcal{A}$, let $G := \delta_{ij}(D)$ (note that in this case $G \in \mathcal{A}$); otherwise, let $B \in \mathcal{A}$ such that $\delta_{ij}(B) = D$.

If at least one of C, D belongs to $\mathcal{A}$, we may assume $D \in \mathcal{A}$. If also $C \in \mathcal{A}$ then $(C \cap D)\backslash(V' \cup V_{ij}) \supseteq (C \cap G)\backslash V'$ (since $G \cap V_{ij} = \emptyset$), and $C, G \in \mathcal{A}$; hence $(C \cap D)\backslash(V' \cup V_{ij}) \neq \emptyset$. If $C \notin \mathcal{A}$ then $(C \cap D)\backslash(V' \cup V_{ij}) \supseteq (A \cap G)\backslash V' \neq \emptyset$.

If $C, D \notin \mathcal{A}$ then $(A \cap B)\backslash V' \neq \emptyset$; moreover, $C \cap D = \delta_{ij}(A \cap B)$ and hence $(C \cap D)\backslash(V' \cup V_{ij}) \neq \emptyset$. □

Lemma 3.2 *Let $\mathcal{A}^* := \Delta_{12} \circ ... \circ \Delta_{1a_1} \circ ... \circ \Delta_{k2} \circ ... \circ \Delta_{ka_k}(\mathcal{A})$, where $\mathcal{A}$ is an intersecting subfamily of $\mathcal{F}(\mathcal{P})$. Then*
(i) $\mathcal{A}^* \subseteq \mathcal{F}(\mathcal{P})$,

(ii) $|\mathcal{A}^*| = |\mathcal{A}|$,
(iii) $\mathcal{A}^*$ *is an intersecting sub-family of* $\mathcal{F}(\mathcal{P})$,
(iv) $A \cap B \cap F \neq \emptyset$ *for any* $A, B \in \mathcal{A}^*$.

Proof. Each part follows by repeated application of the corresponding part of Lemma 3.1. □

Throughout the remainder of the paper, we use $\mathcal{A}^*$ as in the statement of Lemma 3.2.

Let $\mathcal{A} \subseteq \mathcal{F}(\mathcal{P})^{(r)}$ be an intersecting family. By (i) and (ii) of Lemma 3.1, if $\mathcal{A}$ is non-centred and $\Delta_{ij}(\mathcal{A})$ is a star of largest size, then $\mathcal{F}(\mathcal{P})$ is not strictly r-EKR. Thus, in order to demonstrate the strict r-EKR property of $\mathcal{F}(\mathcal{P})$ by considering families that are obtained through compression operations, we must first show that a star of largest size cannot be obtained from a compression operation on a non-centred intersecting family. Now when $\mathcal{P}$ is anchored, then a star with centre in V_0 certainly cannot be obtained through a compression operation Δ_{ij} on any other sub-family of $\mathcal{F}(\mathcal{P})^{(r)}$. Moreover, if $x \in V_0, y \notin V_0$, and $r \leq \mu(\mathcal{P})$, then more sets of $\mathcal{F}(\mathcal{P})^{(r)}$ contain x but not y than contain y but not x, and hence the stars with centres in V_0 are precisely those of maximum size. Thus, for an anchored double partition, a star of largest size can never result from a compression operation on a non-centred intersecting family. However, for the more general case when the double partition may be unanchored, we require the following less trivial result. (In the statement and proof of this lemma, we abbreviate $\mathcal{F}(\mathcal{P})$ to $\mathcal{F}$.)

Lemma 3.3 *Let* $\mathcal{P}$ *be a double partition, let* $r \leq \mu(\mathcal{P})/2$, *and suppose that* $\mathcal{A}$ *is an intersecting sub-family of* $\mathcal{F}^{(r)}$ *such that* $\mathcal{A} \neq \Delta_{ij}(\mathcal{A}) = \mathcal{F}_x^{(r)}$ *for some* $x \in V$ *and some compression* Δ_{ij}. *Then* $|V_{ij}| = |V_{i1}|$ *and* $\mathcal{A} = \mathcal{F}_y^{(r)}$, *where* $y \in V_{ij}$ *and* $x = \delta_{ij}(y)$ $(\in V_{i1})$.

We need the following simple lemma, which is often useful for determining the structure of extremal intersecting families.

Lemma 3.4 *Suppose* $\emptyset \neq \mathcal{A} \subseteq \binom{X}{r}$, $2r < n := |X|$, *such that if* $A \in \mathcal{A}$ *and* $B \in \binom{X \backslash A}{r}$ *then* $B \in \mathcal{A}$. *Then* $\mathcal{A} = \binom{X}{r}$.

Proof. Let $A_0 \in \mathcal{A}$ and $B \in \binom{X}{r}$ such that $1 \leq q_0 := |A_0 \cap B| \leq r-1$, i.e. $B \neq A_0$ and $B \notin \binom{X \backslash A}{r}$. It is required to show that $B \in \mathcal{A}$. We claim that the following procedure takes a finite number of steps k, and we first assume the claim is true. For $i = 1, 2, ..., k$, choose $A_i \in \binom{X \backslash A_{i-1}}{r}$ such that $|A_i \cap B|$

is a minimum if i is odd, and $|A_i \cap B|$ is a maximum if i is even, where k is the first even integer that gives $A_k = B$. So $A_i \in \mathcal{A}$ for all $i \in [k]$, and hence we are done.

We now prove the claim. Let $q_i := |A_i \cap B|$ if i is even, and $q_i := r - |A_i \cap B|$ if i is odd. If i is even then $q_i = r - |A_{i-1} \cap B| = q_{i-1}$. If i is odd then $q_i = r - \max\{0, r - ((n - |A_{i-1} \cup B|))\} = \min\{r, n - (2r - |A_{i-1} \cap B|)\} = \min\{r, (n-2r) + q_{i-1}\} > q_{i-1}$. So the claim holds. □

Proof of Lemma 3.3. Let $A^* \in \mathcal{A} \backslash \Delta_{ij}(\mathcal{A})$. So $\delta_{ij}(A^*) \in \Delta_{ij}(\mathcal{A})$. Since $\Delta_{ij}(\mathcal{A}) = \mathcal{F}_x^{(r)}$, $x \in \delta_{ij}(A^*)$. Since $A^* \notin \Delta_{ij}(\mathcal{A}) = \mathcal{F}_x^{(r)}$, $x \notin A^*$. So $x \in \delta_{ij}(A^*) \backslash A^*$. So $x = \delta_{ij}(y)$ for some $y \in A^* \cap V_{ij}$, $j > 1$, and $x \in V_{i1}$.

Let M be any maximal set of $\mathcal{F}$ that contains $A^* \cup V_{ij}$, and let $\mathcal{A}_M := \{A \in \mathcal{A} \cap \mathcal{F}_y^{(r)} \colon A \subset M\}$. Let $N := M \backslash \{y\}$ and $\mathcal{A}'_M := \{A \backslash \{y\} \colon A \in \mathcal{A}_M\} \subseteq \binom{N}{r'}$, where $r' = r - 1 \leq \mu(\mathcal{P})/2 - 1 \leq |M|/2 - 1 = (|M| - 1)/2 - 1/2 < |N|/2$. Suppose $A' \in \mathcal{A}'_M$ and $B' \notin \mathcal{A}'_M$ for some $B' \in \binom{N \backslash A'}{r'}$. Then $A'' := A' \cup \{y\} \in \mathcal{A}_M$, $B'' := B' \cup \{y\} \notin \mathcal{A}$, and $\delta_{ij}(B'') \notin \mathcal{A}$ since $\delta_{ij}(B'') \cap A'' = \emptyset$. So $\delta_{ij}(B'') \in \mathcal{F}_x^{(r)} \backslash \Delta_{ij}(\mathcal{A})$, a contradiction. Therefore, if $A' \in \mathcal{A}'_M$ then $B' \notin \mathcal{A}'_M$ for all $B' \in \binom{N \backslash A'}{r'}$. Also, $A^* \backslash \{y\} \in \mathcal{A}'_M$. By Lemma 3.4, $\mathcal{A}'_M = \binom{N}{r'}$. Hence $\mathcal{A}_M = \{A \in \mathcal{F}_y^{(r)} \colon A \subset M\}$.

Since $2r \leq \mu(\mathcal{P})$, for any $A \in \mathcal{F}^{(r)} \backslash \mathcal{F}_y^{(r)}$ there exists $B \in \mathcal{A}_M$ such that $A \cap B = \emptyset$. So $\mathcal{A} \subseteq \mathcal{F}_y^{(r)}$. Since $|\mathcal{F}_x^{(r)}| \geq |\mathcal{F}_y^{(r)}|$ (as $|V_{i1}| \geq |V_{ij}|$) and $|\mathcal{A}| = |\Delta_{ij}(\mathcal{A})| = |\mathcal{F}_x^{(r)}|$, it follows that $|\mathcal{A}| = |\mathcal{F}_y^{(r)}| = |\mathcal{F}_x^{(r)}|$, and hence $|V_{ij}| = |V_{i1}|$. □

4 Proof of Theorem 1.5

Let $\mathcal{P}$ be anchored. In the proof that follows, we abbreviate $\mathcal{F}(\mathcal{P})$ to $\mathcal{F}$.

If $r = 1$, there is nothing to prove, so we may assume $r \geq 2$ and thus $\mu(\mathcal{P}) \geq 4$. Moreover, $|V| \geq 5$ since V_1 is non-trivially partitioned. Since a non-centred family of 2-sets must be of size 3, it immediately follows that $\mathcal{F}$ is strictly 2-EKR. We therefore assume $3 \leq r \leq \mu(\mathcal{P})/2$.

Now let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{F}^{(r)}$ such that

$$|\mathcal{A}'| \leq |\mathcal{A}| \text{ for any intersecting family } \mathcal{A}' \subset \mathcal{F}^{(r)}. \tag{3}$$

By Lemmas 3.2 and 3.3, we may assume that $\mathcal{A} = \mathcal{A}^*$ and hence that

$$A \cap B \cap F \neq \emptyset \text{ for any } A, B \in \mathcal{A} \tag{4}$$

(by Lemma 3.2(iv)).

Let x be a fixed element of V_0, and let $\mathcal{J} := \mathcal{F}_x^{(r)}$.

We now develop a notation for partitioning sub-families of $\mathcal{F}^{(r)}$ in accordance with their intersections with the upper sets.

Let $U := \{(i,j)\colon 1 \leq i \leq k, 2 \leq j \leq a_i\}$; that is, U is the set of subscript pairs associated with the upper sets of $\mathcal{P}$. By (4), each set of $\mathcal{A}$ intersects at least one floor set and thus at most $r-1$ upper sets. This is true also of $\mathcal{J}$. Thus, let

$$\mathcal{U} := \{S \subseteq U\colon |S| < r,\ (i,j),(i',j') \in S \text{ and } (i,j) \neq (i',j') \text{ implies } i \neq i'\}$$

(note that $\emptyset \in \mathcal{U}$). Then a family $\mathcal{B}$ of members of $\mathcal{F}^{(r)}$ each intersecting at least one floor set is partitioned as follows: $\mathcal{B} = \bigcup_{S\in\mathcal{U}} \mathcal{B}_S$ where $\mathcal{B}_S$ is the sub-family of $\mathcal{B}$ whose sets intersect all the sets V_{ij}, $(i,j) \in S$, and no other upper sets. For $S \in \mathcal{U}$, let F_S denote the union of those floor sets that are not 'under' any of the upper sets of S: $F_S = F \backslash \bigcup_{(i,j)\in S} V_{i1}$. Then, for $S \neq \emptyset$, a sub-family $\mathcal{B}_S$ is a family of crossing r-sets in which F_S takes the role of X_0 and the upper sets take the role of the X_i for $i \geq 1$ (see Section 2); moreover, for $\mathcal{B}_S = \mathcal{A}_S$, we have $\mathcal{B}_S$ strongly intersecting by (4).

Therefore, by Theorem 2.3(i), $|\mathcal{A}_S| \leq |\mathcal{J}_S|$ for each $S \in \mathcal{U}\backslash\{\emptyset\}$. By Theorem 1.1(i), we also have $|\mathcal{A}_\emptyset| \leq |\mathcal{J}_\emptyset|$. Thus $|\mathcal{A}| \leq |\mathcal{J}|$ (which proves (i)). By (3), $|\mathcal{A}| = |\mathcal{J}|$, and hence $|\mathcal{A}_S| = |\mathcal{J}_S|$ for each $S \in \mathcal{U}$.

For any $S \in \mathcal{U}$, if we can show that $\mathcal{A}_S = (\mathcal{F}_v^{(r)})_S$ for some $v \in F$, then it follows that $\mathcal{A} \subseteq \mathcal{F}_v^{(r)}$, since for all $A \in \mathcal{F}^{(r)}\backslash\mathcal{F}_v^{(r)}$ there exists $B \in (\mathcal{F}_v^{(r)})_S$ such that $A \cap B = \emptyset$, as every maximal set is of size $\geq 2r$. By (3) and the fact that, as we noted in Section 3, $|\mathcal{F}_v^{(r)}|$ is maximised only if $v \in V_0$, we may conclude that $\mathcal{A} = \mathcal{F}_v^{(r)}$ where $v \in V_0$.

If $r < \mu(\mathcal{P})/2$ or $r = \mu(\mathcal{P})/2 < \alpha(\mathcal{P})/2 = |F|/2$, then, since $|\mathcal{A}_S| = |\mathcal{J}_S|$, by taking $S = \emptyset$ and applying Theorem 1.1(ii) we indeed obtain $\mathcal{A}_S = (\mathcal{F}_v^{(r)})_S$ for some $v \in F$.

If $r = \mu(\mathcal{P})/2 = \alpha(\mathcal{P})/2$ and $k > 1$, then we choose S such that $|S| \geq 2$. By Theorem 2.3, $\mathcal{A}_S = (\mathcal{F}_v^{(r)})_S$ for some $v \in F$.

It remains to consider the case $k = 1$. Recall that we are assuming $r \geq 3$. If $|V_0| < 3$ or $|V_0| > r$, then we take $S = \{(1,j)\}$, $j > 1$, and again apply Theorem 2.3. If $3 \leq |V_0| \leq r$ then the non-centred intersecting family $(\mathcal{J}\backslash\{A \in \mathcal{J}\colon A \cap V_0 = \{x\}\}) \cup \{A \in \mathcal{F}^{(r)}\colon A \cap V_0 = V_0\backslash\{x\}\} \subset \mathcal{F}^{(r)}$ has size equal to $|\mathcal{J}|$. Thus the strict EKR property fails only in the cases stated in the theorem. $\square$

5 Proof of Theorem 1.6

Recall that $\mathcal{P}$ is unanchored with $k = 3$, $n_{ij} = c \quad (j = 1, ..., a_i, i = 1, 2, 3)$, and $V_0 = \emptyset$. For simplicity, we assume that $a_1 \le a_2 \le a_3$. As in Section 3, $V_{11} \cup V_{21} \cup V_{13}$ is the floor, denoted by F, and as in Section 4, we abbreviate $\mathcal{F}(\mathcal{P})$ to $\mathcal{F}$.

Let $r \le \mu(\mathcal{P})/2$, and let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{F}^{(r)}$ that is not a star. By Lemma 3.3, $\mathcal{A}^*$ is not a star either. Thus, using Lemma 3.2, we may assume that $\mathcal{A} = \mathcal{A}^*$ and that $A \cap B \cap F \neq \emptyset$ for any $A, B \in \mathcal{A}$.

Let $D_i := \{0, ..., a_i\} \quad (i = 1, 2, 3)$. For any $(d_1, d_2, d_3) \in D_1 \times D_2 \times D_3$, let $\mathcal{A}_{d_1,d_2,d_3}$ be the sub-family of sets $A \in \mathcal{A}$ such that $A \cap V_{id_i} \neq \emptyset$ for all i such that $d_i \neq 0$, and $A \cap V_{ij} = \emptyset$ otherwise. So the families $\mathcal{A}_{d_1,d_2,d_3}$ partition $\mathcal{A}$. Let $\mathcal{J} := \mathcal{F}^{(r)}_{v_{111}}$ and partition it similarly. Note that $\mathcal{J}$ is a star of largest size.

By Lemma 3.2(iv), for any (d_1, d_2, d_3), $(d_1', d_2', d_3') \in D_1 \times D_2 \times D_3$ such that $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$ and $\mathcal{A}_{d_1',d_2',d_3'} \neq \emptyset$, we must have $d_i = d_i' = 1$ for some $i \in [3]$. We now consider two cases.

Case 1: $\{i \in [3] \colon d_i = 1\} = \{i'\}$ for some $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$. Then $d_{i'} = 1$ for any $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$. Thus, let $\mathcal{Q}$ be the double partition obtained from $\mathcal{P}$ by deleting the small sets $V_{12}, \ldots, V_{1a_1}$; then $\mathcal{A}$ is a subfamily of $\mathcal{G} := \mathcal{F}(\mathcal{Q})$. Now $\mathcal{Q}$ is an anchored partition of width 2, and so by Theorem 1.5, $\mathcal{G}$ is strictly r-EKR. So $|\mathcal{A}| < |\mathcal{F}^{(r)}_{v_{i'11}}| \le |\mathcal{J}|$.

Case 2: $|\{i \in [3] \colon d_i = 1\}| > 1$ whenever $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$. So the non-empty classes can only be $\mathcal{A}_{1,1,1}$, $\mathcal{A}_{d_1,1,1}$, $\mathcal{A}_{1,d_2,1}$, and $\mathcal{A}_{1,1,d_3}$, $d_i \in D_i \quad (i = 1, 2, 3)$.

Let $\mathcal{A}_0 := \mathcal{A}_{1,1,1} \cup \mathcal{A}_{0,1,1} \cup \mathcal{A}_{1,0,1} \cup \mathcal{A}_{1,1,0}$ and, similarly, $\mathcal{J}_0 := \mathcal{J}_{1,1,1} \cup \mathcal{J}_{0,1,1} \cup \mathcal{J}_{1,0,1} \cup \mathcal{J}_{1,1,0}$. (These are the subfamilies of $\mathcal{A}$ and $\mathcal{J}$ that consist of r-subsets of F). By Theorem 1.1, $|\mathcal{A}_0| \le |\mathcal{J}_0|$.

Now, for $d_2 > 1$, $\mathcal{A}_{1,d_2,1}$ is a family of crossing r-sets for $\mathcal{Y} := \{V_{11} \cup V_{13}, V_{d_2}\}$, obeying the conditions of Theorem 2.3. Thus, for all $d_2 \in [2, a_2]$, we have $|\mathcal{A}_{1,d_2,1}| \le |\mathcal{C}(\mathcal{Y})^{(r)}_{v_{111}}| = |\mathcal{J}_{1,d_2,1} \cup \mathcal{J}_{1,d_2,0}|$. Similarly, if $d_3 > 1$, then $|\mathcal{A}_{1,1,d_3}| \le |\mathcal{J}_{1,1,d_3} \cup \mathcal{J}_{1,0,d_3}|$. In particular, we note that, if $a_3 > a_1$, then $|\mathcal{A}_{1,1,d_3}| \le |\mathcal{J}_{1,1,d_3} \cup \mathcal{J}_{1,0,d_3}| \quad (a_1 + 1 \le d_3 \le a_3)$.

The remaining subfamilies $\mathcal{A}_{d_1,d_2,d_3}$ that need to be compared with subfamilies of $\mathcal{J}$ are $\{\mathcal{A}_{1,1,d}, \mathcal{A}_{d,1,1} \colon 2 \le d \le a_1\}$. Our strategy is to show that $|\mathcal{A}_{1,1,d}| + |\mathcal{A}_{d,1,1}| < |\mathcal{J}_{1,0,d}| + |\mathcal{J}_{1,1,d}| + |\mathcal{J}_{1,2,d}|$, $d = 2, ..., a_1$, from which the result clearly follows, since we shall have made comparisons linking all the subfamilies of $\mathcal{A}$ with subfamilies of $\mathcal{J}$, and at least one of these comparisons involves a strict inequality.

Let us fix $d \in [2, a_1]$ and define $\mathcal{A}' := \mathcal{A}_{1,1,d} \cup \mathcal{A}_{d,1,1}$. We now define two bijections, $\delta_1\colon V_{31} \to V_{11}$ and $\delta_2\colon V_{1d} \to V_{3d}$, as follows.

$$\delta_1(v_{31p}) = v_{11p} \quad (p = 1, \ldots, c);$$

$$\delta_2(v_{1dp}) = v_{3dp} \quad (p = 1, \ldots, c).$$

For any $X_1 \subseteq V_{31}$, $X_2 \subseteq V_{1d}$, we may denote $\{\delta_1(x)\colon x \in X_1\}$ and $\{\delta_2(x)\colon x \in X_2\}$ by $\delta_1(X_1)$ and $\delta_2(X_2)$ respectively. Now define an injective mapping $\delta\colon \mathcal{A}_{d,1,1} \to \binom{V_{11}\cup V_{21}\cup V_{3d}}{r}$ by

$$\delta(A) = \delta_1(A \cap V_{31}) \cup (A \cap V_{21}) \cup \delta_2(A \cap V_{1d}) \quad (A \in \mathcal{A}_{d,1,1}).$$

Define the compression Δ on $\mathcal{A}'$ by

$$\Delta(\mathcal{A}') = \mathcal{A}_{1,1,d} \cup \{\delta(A)\colon A \in \mathcal{A}_{d,1,1}\} \cup \{A \in \mathcal{A}_{d,1,1}\colon \delta(A) \in \mathcal{A}_{1,1,d}\}.$$

Now let $\mathcal{B} := \Delta(\mathcal{A}')$. Thus, $\mathcal{B} = \mathcal{B}_{1,1,d} \cup \mathcal{B}_{d,1,1}$ where $\mathcal{B}_{1,1,d} = \mathcal{A}_{1,1,d} \cup (\mathcal{B}\backslash\mathcal{A})$ and $\mathcal{B}_{d,1,1} = \mathcal{B}\backslash\mathcal{B}_{1,1,d}$.

Claim 5.1 *(i)* $|\mathcal{B}| = |\mathcal{A}|$.
(ii) $A \cap B \cap (V_{11} \cup V_{21}) \neq \emptyset$ *for any* $A, B \in \mathcal{B}$.

Proof. (i) is straightforward.

We now define $f\colon \mathcal{A}' \to \mathcal{B}$ by: $f(A) = \delta(A)$ if $A \in \mathcal{A}_{d,1,1}$ and $\delta(A) \notin A_{1,1,d}$, and $f(A) = A$ otherwise. So f is a bijection. We prove (ii) by showing that

$$f(A) \cap f(B) \cap (V_{11} \cup V_{21}) \neq \emptyset \text{ for any } A, B \in \mathcal{A}'. \tag{5}$$

We recall that, by Lemma 3.2(iv), $A \cap B \cap F \neq \emptyset$ for any $A, B \in \mathcal{A}$. If $A, B \in \mathcal{A}_{1,1,d}$ then (5) is immediate. If $A \in \mathcal{A}_{1,1,d}$ and $B \in \mathcal{A}_{d,1,1}$ then $f(A) \cap f(B) \cap V_{21} = A \cap B \cap V_{21} \neq \emptyset$, and hence (5). Suppose $A, B \in \mathcal{A}_{d,1,1}$. Since $A \cap B \cap (V_{21} \cup V_{31}) \neq \emptyset$, if $\delta(A), \delta(B) \notin \mathcal{A}_{1,1,d}$ then (5) is straightforward. Suppose $\delta(A) \in \mathcal{A}_{1,1,d}$ and $\delta(B) \notin \mathcal{A}_{1,1,d}$. Since $\delta(A) \cap B \cap V_{21} \neq \emptyset$, we have $A \cap \delta(B) \cap V_{21} \neq \emptyset$, and hence (5). Finally, suppose $\delta(A), \delta(B) \in \mathcal{A}_{1,1,d}$. So $A \cap B \cap V_{21} \neq \emptyset$ because $A \cap \delta(B) \cap V_{21} \neq \emptyset$; hence (5). □

By Theorem 2.3,

$$|\mathcal{B}_{1,1,d}| \leq |\mathcal{J}_{1,0,d}| + |\mathcal{J}_{1,1,d}|. \tag{6}$$

By Claim 5.1(ii), we have $A \cap B \cap V_{21} \neq \emptyset$ for all $A, B \in \mathcal{B}_{d,1,1}$. By Theorem 2.3, $|\mathcal{B}_{d,1,1}| \leq |\mathcal{J}_{1,2,d}|$. If $|\mathcal{B}_{d,1,1}| < |\mathcal{J}_{1,2,d}|$ then we are done.

Suppose $|\mathcal{B}_{d,1,1}| = |\mathcal{J}_{1,2,d}|$. By Theorem 2.3(ii), there exists $v' \in V_{21}$ such that $\mathcal{B}_{d,1,1} = \mathcal{K}_{d,1,1}$ where $\mathcal{K} := \mathcal{F}_{v'}^{(r)}$. Let $\mathcal{E} := \{A \in \mathcal{B}_{d,1,1}\colon A \cap V_{21} = v'\}$.

$\mathcal{E} \neq \emptyset$ since $2r \leq \mu(\mathcal{P}) = 3c$. Let $E \in \mathcal{E}$. If there exists $A \in \mathcal{B}_{1,1,d}$ such that $v' \notin A$, then $A \cap E = \emptyset$, a contradiction. So $\mathcal{B}_{1,1,d} \subseteq \mathcal{K}_{1,1,d}$, and hence $|\mathcal{B}_{1,1,d}| \leq |\mathcal{K}_{1,1,d}| = |\mathcal{J}_{1,1,d}|$. Since $2r \leq 3c$, we have $|\mathcal{J}_{1,0,d}| > 0$, and hence a strict inequality in (6). It follows that $|\mathcal{A}| < |\mathcal{J}|$. □

Acknowledgements. This work is part of the first author's PhD research that was carried out when he was a visitor of the Department of Pure Mathematics and Mathematical Statistics of the University of Cambridge; he is greatly indebted to Professor Imre Leader and Professor Andrew Thomason for making this visit possible and for their kind support. The authors would like to thank Professor John Goldwasser and the referees for useful comments that have considerably improved the presentation.

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